# Knot invariants, Hilbert schemes and Macdonald polynomials (joint with A. Neguț, J. Rasmussen)

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# Outline

Knot invariants

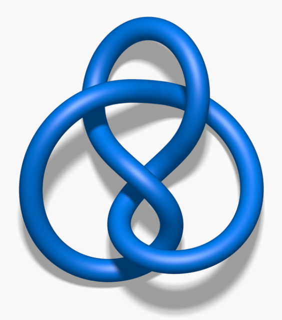
HOMFLY homology

Hilbert schemes

# Knot invariants

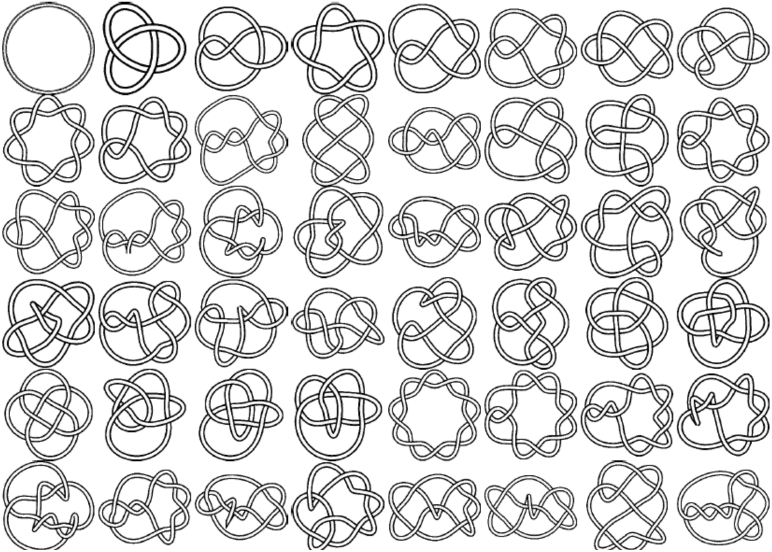
A link is a closed 1-dimensional submanifold in the sphere $S^3$. If it has one connected component, it is called a knot. Using stereographic projection, one can place knots and links in $\mathbb{R}^3$.

Two knots are equivalent if one can be continuously deformed into another without tearing their strands.



# Knot invariants

## Examples of knots[1]



[1] Rolfsen's knot table, `www.katlas.org`

# Knot invariants

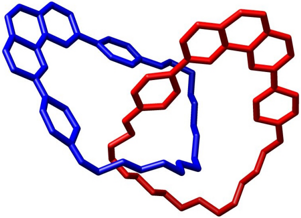
Sir William Thomson (Lord Kelvin): atoms are knots of swirling vortices in the aether. Chemical elements would thus correspond to knots and links.

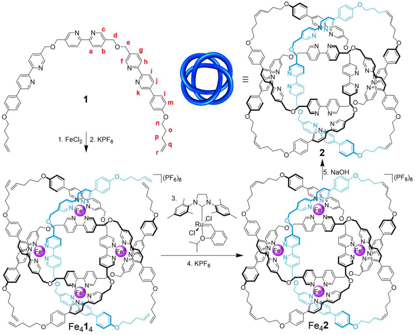
Now we know that periodic table looks different, but knots are still important for:

- Biochemistry: long molecules (such as DNA) are knotted, and the geometry affects their chemical properties
- Quantum computers: quantum entanglement of states can be described by knots
- Physics: Chern-Simons theory, string theory

# Knot invariants

The 2016 Nobel Prize in Chemistry was awarded to researchers who constructed a new type of molecules called "catenanes" consisting of several linked loops. Chemists also constructed complicated molecular knots which could lead to materials with new properties.






Catenane (left), synthesis of a molecular $8_{19}$ knot (right, Science).

# Knot invariants

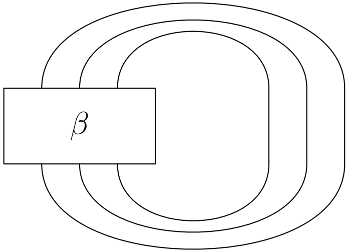
## Knots and braids

A possible algebraic approach to knot invariants uses the braid group $\mathrm{Br}_n$. It has generators $s_1, \ldots, s_{n-1}$ and relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \; s_i s_j = s_j s_i \text{ if } |i-j| > 1.$$

Every knot is a closure of a braid, two braids close to the same knot if they are related by a sequence of moves:

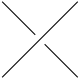
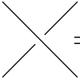
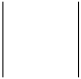
$$\alpha \sim \beta\alpha\beta^{-1}, \; \alpha \sim \alpha s_{n-1} \sim \alpha s_{n-1}^{-1} \text{ for } \alpha \in \mathrm{Br}_{n-1}.$$

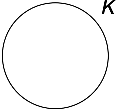
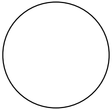


# HOMFLY homology

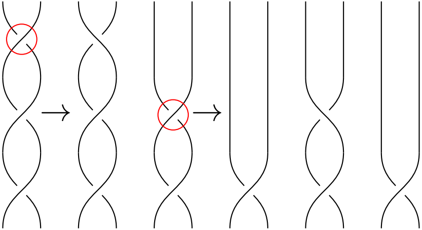
## HOMFLY polynomial

HOMFLY-PT polynomial was discovered by Jim Hoste, Adrian Ocneanu, Kenneth Millett, Peter J. Freyd, W. B. R. Lickorish, and David N. Yetter, and independently by Jozef H. Przytycki and PawełTraczyk. It is closely related to representation theory of quantum groups and Hecke algebras. It is a rational function in $a$ and $q$ defined by the following *skein relation*:

$a^{-1}$  $-a$  $= (q - q^{-1})$ 

 $k$ $= \left(\frac{a^{-1}-a}{q-q^{-1}}\right)^{k-1}$,  $= 1$

# HOMFLY homology



$$P(s^3) = a^2P(s) + a(q - q^{-1})P(s^2) =$$
$$a^2P(s) + a^3(q - q^1)P(\mathbf{1}) + a^2(q - q^{-1})^2P(s) =$$
$$a^2 + a^3(a^{-1} - a) + a^2(q - q^{-1})^2 = a^2(q^2 + q^{-2}) - a^4.$$

## Theorem (G., Oblomkov, Rasmussen, Shende)

*For torus knots, all coefficients of* $\pm P(a\sqrt{-1}, q)$ *are nonnegative.*

The proof uses representation theory of *rational Cherednik algebras*.

# HOMFLY homology



Khovanov and Rozansky developed a knot homology theory, which assigns a collection of homology groups to each knot. The Euler characteristic of this homology coincides with the HOMFLY polynomial.

It is known that Khovanov-Rozansky homology carry nontrivial geometric information: for example, they can be used to estimate the genus of a surface with boundary at the knot. However, their definition uses heavy commutative algebra (Hochschild homology of complexes of Soergel bimodules), and explicit computations are very hard.

# HOMFLY homology

## Main conjecture

### Conjecture

*For every braid $\beta$ there exists a vector bundle (or a coherent sheaf, or a complex of sheaves) $\mathcal{F}_\beta$ on the Hilbert scheme of points on the plane such that the space of sections of $\mathcal{F}_\beta$ (or sheaf cohomology) is isomorphic to the HOMFLY homology of $\overline{\beta}$.*

If true, the conjecture would yield very explicit computations of HOMFLY homology in many cases.

### Theorem (G., Neguț)

*If $\beta$ is a torus knot, then the invariants of $\mathcal{F}_\beta$ agree with the* refined Chern-Simons invariants, *defined by Aganagic-Shakirov and Cherednik using Macdonald polynomials and double affine Hecke algebras.*

The *symmetric power* $S^n\mathbb{C}^2$ is the moduli space of unordered $n$-tuples of points on $\mathbb{C}^2$.

The *Hilbert scheme* $\mathrm{Hilb}^n\,\mathbb{C}^2$ is the moduli space of codimension $n$ ideals in $\mathbb{C}[x,y]$. Such an ideal is supported on a finite subset of $n$ points in $\mathbb{C}^2$ (with multiplicities), this defines the Hilbert-Chow morphism:

$$HC : \mathrm{Hilb}^n\,\mathbb{C}^2 \to S^n\mathbb{C}^2.$$

## Theorem (Fogarty)

$\mathrm{Hilb}^n\,\mathbb{C}^2$ *is a smooth manifold of dimension* $2n$*.*

# Hilbert schemes

## ADHM description

Hilbert scheme of points has an alternative (and sometimes more useful) description as follows:

$$\mathrm{Hilb}^n(\mathbb{C}^2) = \{(X, Y, v) \ : \ [X, Y] = 0, \text{stability condition}\}/G,$$

where $X$ and $Y$ are two commuting $n \times n$ matrices, $v$ is a vector in $\mathbb{C}^n$ and $G = GL(n)$ acts by

$$g.(X, Y, v) = (gXg^{-1}, gYg^{-1}, gv).$$

Stability condition: $X^a Y^b v$ span $\mathbb{C}^n$.

# Hilbert schemes

## Torus action

The natural scaling action of $(\mathbb{C}^*)^2$ lifts to an action on $S^n\mathbb{C}^2$ and on $\mathrm{Hilb}^n\,\mathbb{C}^2$. It has a finite number of fixed points corresponding to monomial ideals.

**Example:**

| | | | | | |
|---|---|---|---|---|---|
| $y^3$ | $xy^3$ | $x^2y^3$ | $x^3y^3$ | $x^4y^3$ | $x^5y^3$ |
| $y^2$ | $xy^2$ | $x^2y^2$ | $x^3y^2$ | $x^4y^2$ | $x^5y^2$ |
| $y$ | $xy$ | $x^2y$ | $x^3y$ | $x^4y$ | $x^5y$ |
| $1$ | $x$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ |

The ideal is generated by $y^3, xy^2, x^3y, x^4$

The *punctual Hilbert scheme* $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ is the scheme-theoretic fiber of the Hilbert-Chow morphism over $\{n \cdot 0\}$.

### Theorem (Briançon,Haiman)

$\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ *is reduced, irreducible and Cohen-Macaulay. Its dimension equals* $n - 1$.

### Example

$\mathrm{Hilb}^2(\mathbb{C}^2, 0) = \mathbb{P}^1$; $\mathrm{Hilb}^3(\mathbb{C}^2, 0)$ is a (projective) cone over twisted cubic in $\mathbb{P}^3$. The vertex of the cone is the monomial ideal $(x^2, xy, y^2)$.

Consider the moduli space of flags

$$\mathrm{FHilb}^n(\mathbb{C}^2) := \{\mathbb{C}[x,y] \supset J_1 \supset J_2 \supset \ldots \supset J_n\},$$

where $J_i$ is an ideal in $\mathbb{C}[x,y]$ of codimension $i$. There are $n$ line bundles $\mathcal{L}_i := J_i/J_{i+1}$ on $\mathrm{FHilb}^n$.

## Example

$\mathrm{FHilb}^2(\mathbb{C}^2,0) = \mathrm{FHilb}^2(\mathbb{C}^2,0) = \mathbb{P}^1$; $\mathrm{FHilb}^3(\mathbb{C}^2,0)$ is isomorphic to the Hirzebruch surface $\mathbb{P}(\mathcal{O}+\mathcal{O}(-3)) \to \mathbb{P}^1$. It is a blowup of the singular cone $\mathrm{Hilb}^3(\mathbb{C}^2,0)$.

The flag Hilbert scheme also has a description in terms of commuting *triangular* matrices.

$$\mathrm{FHilb}^n(\mathbb{C}^2) = \{(X, Y, v) \ : \ [X, Y] = 0, \text{stability condition}\}/B,$$

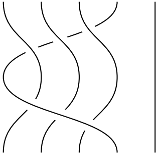
where $X$ and $Y$ are two commuting *lower triangular* $n \times n$ matrices, and $B$ is the group of invertible lower triangular matrices. We will need the following subvarieties in it:

$$\mathrm{FHilb}^n(\mathbb{C}^2, \mathbb{C}) : \ \ Y \text{ nilpotent},$$

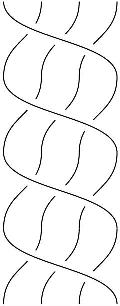
$$\mathrm{FHilb}^n(\mathbb{C}^2, 0) : \ \ \text{both } X \text{ and } Y \text{ are nilpotent.}$$

# Hilbert schemes

We will need an important family of braids:

$L_i =$ 

One could check that $L_i L_j = L_j L_i$ for all $i$ and $j$. The full twist (also known as $(n, n)$ torus braid) can be written as:

 $= L_1 \cdots L_n.$

# Hilbert schemes

## Main conjecture, more precise form

### Conjecture

*Given an n-strand braid $\beta$, there exists a sheaf $\mathcal{F}_\beta$ on* $\mathrm{FHilb}^n(\mathbb{C}^2,\mathbb{C})$ *such that*

$$\mathcal{F}(\beta \cdot L_1^{a_1} \cdots L_n^{a_n}) = \mathcal{F}(\beta) \otimes \mathcal{L}_1^{a_1} \otimes \cdots \otimes \mathcal{L}_n^{a_n};$$

$$\mathcal{F}(\mathbf{1}) = \mathcal{O},\ \mathcal{F}(s_1 \cdots s_{n-1}) = \mathcal{O}_{\mathrm{FHilb}^n(\mathbb{C}^2,0)}.$$

*In general, the support of $\mathcal{F}_\beta$ can be determined from the equation:*

$$\mathrm{supp}\,\mathcal{F}_\beta = \{x_i = x_{\beta(i)} \textit{ for all } i\}.$$

Finally,

$$H^*_{\mathbb{C}^* \times \mathbb{C}^*}(\mathrm{FHilb}^n(\mathbb{C}^2,\mathbb{C}), \mathcal{F}_\beta) = \text{HOMFLY homology of } \overline{\beta}$$

# Hilbert schemes

## Example: $(2,2k+1)$ torus knots

Consider the $(2,2k+1)$ torus braid:

$$\beta = s^{2k+1} = s(s^2)^k = s \cdot L_2^k.$$

According to the conjecture,

$$\mathcal{F}_\beta = \mathcal{F}(s) \otimes \mathcal{L}_2^k = \mathcal{O}_{\mathrm{FHilb}^2(\mathbb{C}^2,0)} \otimes \mathcal{L}_2^k.$$

Recall that $\mathrm{FHilb}^2(\mathbb{C}^2,0) = \mathbb{P}^1$, and one can check that $\mathcal{L}_2 = \mathcal{O}(1)$. Therefore the ($a=0$ part of) HOMFLY-PT homology of $(2,2k+1)$ torus knot equals $H^*(\mathbb{P}^1, \mathcal{O}(k))$ and has dimension $k+1$ (for $k \geq 0$). This can be easily verified from the definition.

# Hilbert schemes

## Example: $(n, n)$ torus links

In 2016, Elias and Hogancamp proved a combinatorial formula for the HOMFLY homology of the $(n, n)$ torus links. Our conjecture yields a *different* formula, and the comparison between the two leads to interesting identities.

Let $A \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the space of antisymmetric polynomials with respect to the diagonal action of $S_n$. Let $J$ be the ideal generated by $A$. It is bigraded by $x$- and $y$-degree.

### Conjecture

*HOMFLY homology of the $(n, n)$ torus link is isomorphic to $J/\mathbf{y}J$.*

# Hilbert schemes



## Theorem (Haiman)

*a) The bigraded Hilbert series of $J$ equals* $\frac{1}{(1-q)^n(1-t)^n}(\nabla p_1^n, e_n)$*, where $p_n$ are power sums, $e_n$ are elementary symmetric functions and $\nabla$ is a certain operator defined in terms of Macdonald polynomials:*

$$\nabla \widetilde{H}_\lambda = \left(\prod_{\square \in \lambda} \square\right) \cdot \widetilde{H}_\lambda.$$

*b) $J$ is a free module over $\mathbb{C}[y_1, \ldots, y_n]$. As a consequence, the Hilbert series of $J/\mathbf{y}J$ equals* $\frac{1}{(1-q)^n}(\nabla p_1^n, e_n)$.

# Hilbert schemes

Example: $(n, n+1)$ torus knots

## Conjecture

*HOMFLY homology of the $(n, n+1)$ torus knot is isomorphic to $J/(\mathbf{x}J, \mathbf{y}J)$.*

## Theorem (Garsia, Haiman, Haglund)

*a) The Hilbert series of $J/(\mathbf{x}J, \mathbf{y}J)$ equals $(\nabla e_n, e_n)$.*
*b) The dimension of $J/(\mathbf{x}J, \mathbf{y}J)$ equals to the Catalan number $\frac{1}{n+1}\binom{2n}{n}$. The Hilbert series can be written as an explicit combinatorial sum over Dyck paths.*

This result is closely related to so-called *Shuffle conjecture* in combinatorics, recently proved by Carlsson and Mellit. The study of $\mathcal{F}_\beta$ for more general torus knots led us to *rational Shuffle conjecture*, recently proved by Mellit. We expect that this circle of ideas will yield a lot of other interesting combinatorial identities.

# Hilbert schemes

## Example: $(n, \infty)$ torus knots

Another interesting test of the conjecture comes from the study of the asymptotic behavior of the HOMFLY-PT homology of $(n, m)$ torus knots at $m \to \infty$. By a theorem of Stosic, there is a well defined limit of this homology.

### Theorem (Hogancamp)

*The limit of the ($a = 0$ part of) HOMFLY-PT homology of $T(n, m)$ is isomorphic to the free polynomial algebra in $n$ variables.*

Different "pieces" of the homology converge to this limit at a different rate for large $m$.

### Conjecture

*These "pieces" correspond to the local coordinate algebras of* $\mathrm{FHilb}^n$ *at the fixed points of the action of* $\mathbb{C}^* \times \mathbb{C}^*$.

Thank you