Derived traces of Soergel categories

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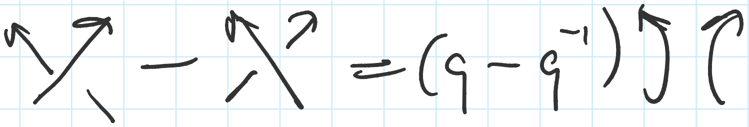
① Motivation: skein theory



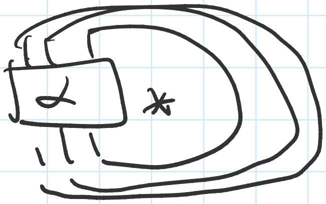
Turaev

$$\frac{\{\text{braids on } n \text{ strands}\}}{\{\text{skein relation}\}} = H_n \quad \text{Hecke algebra} \qquad \frac{\{\text{closed braids in the annulus}\}}{\{\text{skein relation}\}} = \Lambda_n$$

degree n symmetric functions

Explicit trace map

$$H_n \xrightarrow{Tr} \Lambda_n$$



Jones, Ocneanu: $\beta \longrightarrow \sum_\lambda Tr(\beta|_{V_\lambda}) \cdot S_\lambda$

V_λ = irrep of H_n

Facts: ① There is a basis of Λ_n given by Schur functions

② Homfly-pt polynomial factors through Λ_n

$$\beta \xrightarrow{Tr} Tr(\beta) \in \Lambda_n \xrightarrow{ev} \mathbb{C}(a,q)$$

evaluation of symm. fn. at a special point

③ The center of H_n acts on $\Lambda_n = \frac{H_n}{\quad}$

③ The center of H_n acts on $\Lambda_n = \frac{H_n}{[H_n, H_n]}$

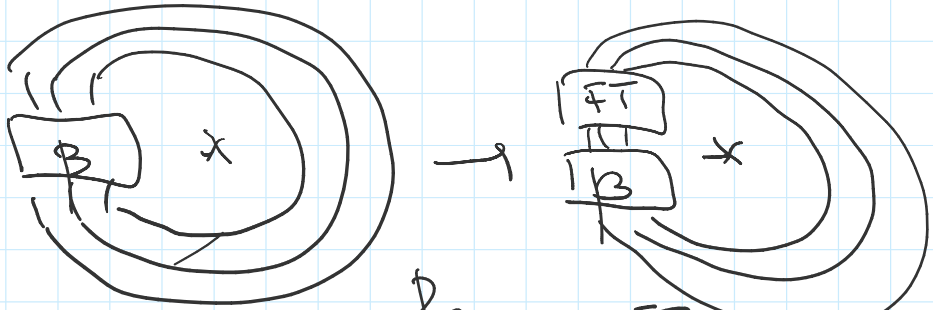
In particular, full twist braid acts on Λ_n

Goal: Categorify all this.

Partially done by: Grigsby-Licata-Wehrli
Queffelec-Rose
Robert-Wagner
annular link homology

Beliakova-Putyra-Wehrli: categorical traces

Rmk Λ_n = skein module for the solid torus ↻ Dehn twist



Because FT is central, can cut anywhere

$f \in \Lambda_n$ $\quad ev(f) = f\left[p_k = \frac{1-a^k}{1-q^k}\right]$

expand f in power sums $\quad a = q^N$

$ev_N(f) = f(1, q, \dots q^{N-1})$

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(2) Categorical traces:

② Categorical traces:

$\mathcal{C}$ - (dg) category ⇝ $HH_*(\mathcal{C})$ Hochschild homology

$$\begin{tikzcd}
\bigoplus_{z} \mathrm{Hom}(z,z) & \bigoplus_{z_1,z_2} \overset{f}{\mathrm{Hom}(z_1,z_2)} \otimes \overset{g}{\mathrm{Hom}(z_2,z_1)} \arrow[l, "d"'] \\
 & \bigoplus_{z_1,z_2,z_3} \mathrm{Hom}(z_1,z_2) \otimes \mathrm{Hom}(z_2,z_3) \otimes \mathrm{Hom}(z_3,z_1) \arrow[u] \\
 & \cdots \arrow[u]
\end{tikzcd}$$

$d(f \otimes g) = fg - gf$

Fact • $\mathcal{C} = A\text{-mod} \Rightarrow HH_*(\mathcal{C}) = HH_*(A)$

• $\mathcal{C}$ monoidal (has $\otimes$) $\Rightarrow$ natural algebra structure on $HH_*(\mathcal{C})$ (and cyclic bar complex is a dg algebra)

$\mathcal{C}$ - monoidal category $\mapsto Tr(\mathcal{C})$ derived horizontal trace dg category

Objects = $\mathrm{Obj}(\mathcal{C})$

Morphisms:

$$\begin{tikzcd}
\mathrm{Hom}_{Tr}(X,Y) = \Big| \bigoplus_{z} \mathrm{Hom}(Xz, zY) & \bigoplus_{z_1,z_2} \overset{f}{\mathrm{Hom}(Xz_1, z_2Y)} \otimes \mathrm{Hom}(z_2,z_1)\, g \arrow[l, "d"'] \\
 & \bigoplus_{z_1,z,z_3} \mathrm{Hom}(Xz_1, z_2Y) \otimes \mathrm{Hom}(z_2,z_3) \otimes \arrow[u]
\end{tikzcd}$$

$d(f \otimes g)$ = difference of

$$Xz_1 \xrightarrow{f} z_2Y \xrightarrow{g} z_1Y$$

$XZ_1 \xrightarrow{f} Z_2 Y \xrightarrow{g} Z_1 Y$

$XZ_2 \xrightarrow{g} XZ_1 \xrightarrow{f} Z_2 Y$

Z_1, Z_2, Z_3 $\quad$ $\mathrm{Hom}(Z_2, Z_3) \otimes \mathrm{Hom}(Z_3, Z_1)]$

Thm (GHW)
- This is well defined, $d^2 = 0$
- Composition of morphisms is well defined
- There is a natural dg functor $\mathrm{Tr}: \mathcal{C} \to \mathrm{Tr}(\mathcal{C})$
- $\mathrm{End}_{\mathrm{Tr}(\mathcal{C})}(\mathrm{Tr}(\mathbb{1})) = HH_*(\mathcal{C})$ as an algebra

if $X = Y = \underline{\mathbb{1}}$, then $\mathrm{Hom}_{\mathrm{Tr}}(X, Y)$ is given by cyclic bar complex.

Rmk (Underived) horizontal trace (Beliakova et al.)

$$\mathrm{Hom}_{\mathrm{Tr}^0}(X, Y) = \bigoplus_{Z} \mathrm{Hom}(XZ, ZY) \big/ \sim$$

Also: • If $\mathcal{C}$ has duals

then $\boxed{\mathrm{Tr}(XY) \simeq \mathrm{Tr}(YX)}$

(note: in the above definition, can take $Z = X$

Id on $\underbrace{XY}X \longrightarrow X\underbrace{YX}$ defines

a morphism in $\mathrm{Hom}_{\mathrm{Tr}}(XY, YX)$

X^* defines the inverse map.

Rmk (derived) Drinfeld center of $\mathcal{C}$

acts on $\mathrm{Tr}(\mathcal{C})$

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③ $\mathcal{C}$ = Soergel bimodules

$R = \mathbb{C}[x_1, \ldots x_n] \circlearrowleft S_n$

$B_i = R \otimes_{R^{s_i}} R \qquad s_i = (i \ i{+}1)$

↑ R-R bimodule

$SBim_n$ = subcategory in R-R bimod generated by R, B_i under $\oplus$, $\otimes$, grading shifts & direct summands

Thm (Soergel) $K_0(SBim_n) = H_n$

Rmk Can also define for any Coxeter group + realization

$T_i = [B_i \longrightarrow R] \qquad T_i^{-1} = [R \longrightarrow B_i]$

Thm (Rouquier) T_i, T_i^{-1} satisfy braid relations up to homotopy

In particular, $T_i \otimes T_i^{-1} \simeq R$

$T_i T_{i+1} T_i \cong T_{i+1} T_i T_{i+1}, \quad T_i T_j \cong T_j T_i$
$|i-j| \geq 2$

Cor $\underset{\text{braid}}{\beta} \rightsquigarrow T(\beta)$ complex of Soergel bimodules, well defined up to homotopy equivalence

Rmk $SBim_n$ is not abelian, just additive.

Our categorification of $H_n = K^b(SBim_n)$ bounded homotopy category.

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④ **Results** $\mathcal{C} = K^b(SBim_n)$

Thm 1 (GHW). $HH_*(\mathcal{C}) = HH_*(SBim_n) =$

$$= \mathbb{C}[x_1 \ldots x_n] \otimes \Lambda(\theta_1 \ldots \theta_n) \rtimes S_n \quad \Big| \quad \text{in char } 0$$

even HH_* degree 0

odd HH_* degree -1

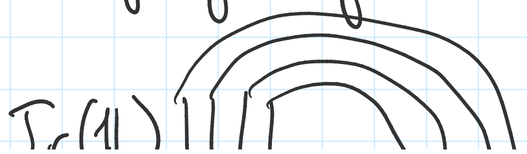
$$= HH_*(R) \rtimes S_n$$

The same holds for any Coxeter group

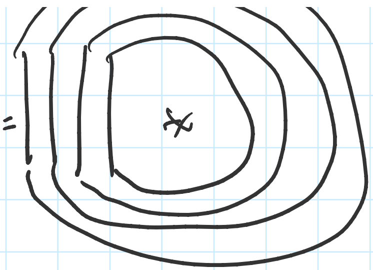
- This is formal as dg algebra

Cor $\mathrm{End}(Tr(\mathbb{1})) = HH_*(\mathcal{C})$, in particular, there is an action of S_n on $Tr(\mathbb{1})$.

Topologically:



action of S_n permutes

$\mathrm{Tr}(\mathbb{1}) =$ 

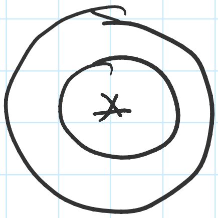
action of S_n permutes these circles.

x_i = variables on circles

θ_i = "monodromy" of x_i around the annulus.

This defines new objects $\mathbb{S}^\lambda(\mathbb{1})$ labeled by Young diagrams in the <u>idempotent completion of $\mathrm{Tr}(\mathcal{B})$</u>

<u>Rmk</u> $\mathrm{Tr}(\mathcal{B})$ is usually not idempotent complete nor triangulated → need to formally complete.

<u>Ex</u> 

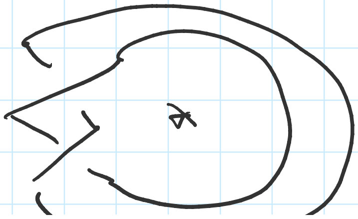
$s = (12)$ swaps circles $s^2 \sim 1$

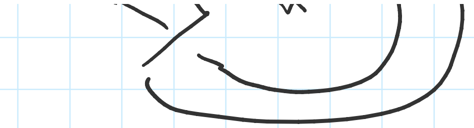
$\frac{1+s}{2}, \frac{1-s}{2}$ homotopy idempotents

→ S^2, Λ^2 are their formal images.

$\mathrm{Tr}(\mathbb{1}) = S^2 \oplus \Lambda^2$

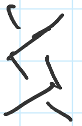
<u>Thm 2</u> (GHW) In type A, (completion of) $\mathrm{Tr}(\mathcal{B})$ is generated by the summands of $\mathrm{Tr}(\mathbb{1})$ that is, by these Schur functors $\mathbb{S}^\lambda$.

<u>Ex</u>  = ? full twist on two strands.



two strands.

We can check

 $= B \to B \to R \overset{Tr}{\rightsquigarrow}$

$$\begin{tikzcd} \Lambda^2 & \cancel{\Lambda^2} \arrow[r, "1"] & \cancel{\Lambda^2} \\ \oplus & \oplus & \oplus \\ \Lambda^2 \arrow[rr, bend right, "\theta_1-\theta_2"'] & \Lambda^2 \arrow[r, "x_1-x_2"'] & S^2 \end{tikzcd}$$

$Tr(B) = \Lambda^2 \oplus \Lambda^2$

$Tr(R) = Tr(\mathbb{1}) = \Lambda^2 \oplus S^2$

Simplify:

$$\begin{tikzcd} \Lambda^2 & & \\ \oplus & \Lambda^2 \arrow[r, "x_1-x_2"] & S^2 \\ \Lambda^2 \arrow[rru, bend right, "\theta_1-\theta_2"'] & & \end{tikzcd}$$

Note: $\deg(\theta_1-\theta_2) = -1$

$\Rightarrow$ degree of differential is 1

Note: $B \to B \to R = \text{Cone}[B \to \text{Cone}[B \to R]]$

$$\big\downarrow Tr$$

$$\text{Cone}[Tr(B) \longrightarrow \text{Cone}(Tr(B) \to Tr(R))]$$

In general, the trace of some complex is $SBim_n$ is a twisted complex with long arrows carrying θ_i

Rmk In underived trace, we do not have these long arrows. In particular,

$Tr(FT_2)$ has 2 indecomposable summands

$Tr_0(FT_2)$ has 3 indecomposable summands

forget θ's

forget Θ's.

If FT_2 defines an invertible endofunctor of $Tr(\mathcal{C})$, then $Tr(FT_2) = FT_2 \cdot Tr(\mathbb{1})$ should have 2 summands.

Thm3 Homfly-PT homology factors through Tr (in fact, factors through Tr_0):

Evaluation: evaluate $Tr(\mathbb{1})$ at some $V^{\otimes n}$ where V is a vector space with action of X

$\Rightarrow$ evaluation functor $Tr(\mathcal{C}) \longrightarrow Vect$

$$\begin{tikzcd}
\Lambda^2 & & & \Lambda^2 & \\
\oplus & \Lambda^2 \arrow[r, "x_1-x_2"] & S^2 \arrow[r, Rightarrow, "\text{forget low arrow}"'] & \oplus & \Lambda^2 \arrow[r, "x_1-x_2"] \arrow[d, Rightarrow, "\text{evaluate at } V"] & S^2 \\
\Lambda^2 \arrow[urr, bend right, red, "\theta_1-\theta_2"'] & & & \Lambda^2 & \Lambda^2 V \arrow[r, "x_1-x_2"] & S^2 V
\end{tikzcd}$$

$$\begin{matrix} \Lambda^2 V \\ \oplus \\ \Lambda^2 V \end{matrix}$$

$V = \mathbb{C}^N \cong \frac{\mathbb{C}[x]}{x^N}$

$\Rightarrow sl(N)$ Khovanov–Rozansky homology

Idea of proof: $K^b(SBim_n)$ has a semiorthogonal decomposition by T_w $w \in S_n$

decomposition by T_w $w \in S_n$

positive braid lifts of permutations.

$\mathrm{Hom}(T_v, T_w) = 0$ unless $v \leq w$ in Bruhat order.

$$HH_*(SBim_n) = HH_*(K^b(SBim_n)) = \text{(semiorth dec.)}$$

$$= \bigoplus_w HH_*(\mathrm{End}(T_w)) =$$

$$= \bigoplus_w HH_*(\mathrm{End}(\mathbb{1})) \leftarrow T_w \text{ invertible}$$

$$= \bigoplus_w HH_*(R) = \bigoplus_w \mathbb{C}[x_1 \ldots x_n] \otimes \Lambda(\theta_1 \ldots \theta_n)$$

This gives HH_* as a vector space

in particular $w \in S_n \longleftrightarrow [\mathrm{Id}(T_w)] \in HH_*$

Rmk This should be related to Lusztig's character sheaves

$\mathrm{Tr}(\mathbb{1}) \longleftrightarrow$ Springer sheaf.

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \mathrm{Tr}(\mathcal{C}) \arrow[r] & \mathrm{Hom}_{\mathrm{Tr}}(\mathrm{Tr}(\mathbb{1}), -) \arrow[d] \\ & & \text{(dg) modules over } \mathbb{C}[x_1 \ldots x_n]\langle\theta_1 \ldots \theta_n\rangle \rtimes S_n \end{tikzcd}$$

$\mathbb{C}[x_1 \ldots x_n]\langle\theta_1 \ldots \theta_n\rangle \rtimes S_n$

$T = [B \to R]$

$\mathrm{Hom}(\mathbb{1}, T) \neq 0$

$\mathrm{Hom}(T, \mathbb{1}) = 0$ in $\mathcal{C}$

$\Lambda^2 \xrightarrow{x_1 - x_2} S^2$

have maps both ways

$\mathrm{Tr}(\mathbb{1}) \rightleftarrows \mathrm{Tr}(T)$

$\mathrm{Hom}_{\mathrm{Tr}}(\mathrm{Tr}(T), \mathrm{Tr}(\mathbb{1}))$

$= \bigoplus \mathrm{Hom}(T \cdot z, z \cdot \mathbb{1}) / \sim$

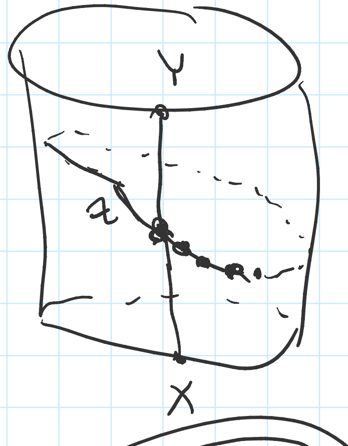
Rmk $\mathrm{Hilb}^n \mathbb{C}^2$ P = Procesi bundle

$\mathrm{End}(P) = \mathbb{C}[x_1 \ldots x_n, y_1 \ldots y_n] \rtimes S_n$

and summands of P generate $D^b\mathrm{Coh}(\mathrm{Hilb})$.

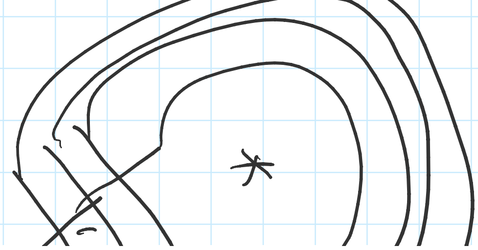
Expect

$$\begin{tikzcd} D^b\mathrm{Coh}(\mathrm{Hilb}) \arrow[r, leftrightarrow] & \mathrm{Tr}(\mathrm{SBim}_n) \\ P \arrow[r, leftrightarrow] & \mathrm{Tr}(\mathbb{1}) \\ & \mathrm{SBim} \arrow[ul] \arrow[uu, bend right, "\mathrm{Tr}"'] \end{tikzcd}$$



$\mathrm{Hom}(X z_1, z_2 Y) \otimes$

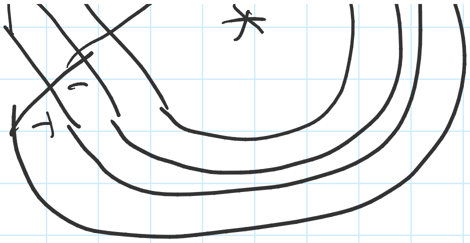
$\mathrm{Hom}(z_2, z_3) \otimes \ldots$



can choose

$+ - + - \ldots$

Lifted Circles



lifts of Coxeters
$\Rightarrow$ ribbon Schur
functions.