RESEARCH STATEMENT

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My research is mainly focused on algebraic and algebro-geometric aspects of knot theory. A knot is a closed loop in three-dimensional space, a link is a union of several such loops, possibly linked with each other. Besides the immediate mathematical applications, knot theory has implications in physics of quantum systems and in the study of chemical and biological properties of long knotted molecules (such as DNA).

The central questions in knot theory are the classification problem (Can a knot be transformed to another knot without tearing its strands? Can it be pulled apart to look like an ordinary circle?) and the study of the geometric properties of knots or links, such as finding the minimal number of handles of a 2-dimensional surface in 3-dimensional (or 4-dimensional) space with boundary on a given link. Both of these questions can be partially answered with the help of *knot invariants*: two knots are different if their invariants are different; various theorems bound the minimal genus by certain values of knot invariants. Among the classical knot invariants are the Alexander polynomial (defined in 1920's) and further polynomial knot invariants (such as Jones or HOMFLY polynomials) defined in 80's-90's. To a given knot, these invariants associate a polynomial in one or more variables.

More recently, the concept of *knot homology theory* was developed and led to new powerful knot invariants. To a given knot or link, such a theory associates a collection of vector spaces (knot homology groups) whose dimensions are encoded in the coefficients of a given knot polynomial. Some of knot homologies, such as Heegaard–Floer homology [40] (generalizing the Alexander polynomial), are defined in geometric terms and carry deep geometric information about the link. Other invariants, such as Khovanov and Khovanov–Rozansky homology [33, 34] (generalizing the Jones and HOMFLY polynomials), are more combinatorial in nature, but still can be used to check if a given knot is trivial or to provide genus bounds. For example, Rasmussen used Khovanov homology to give an elementary proof [44] of the Milnor conjecture on minimal genus of a surface with boundary on a torus knot.

My research was mostly focused on building and analyzing algebraic and geometric models for various knot homology theories. They turn out to be related to the spaces of algebraic functions on plane curve singularities, coherent sheaves on the Hilbert schemes of points on the plane and representations of rational Cherednik algebras. Further models are related to the combinatorics of matroids, hyperplane arrangements, multi-dimensional semigroups, motivic Poincaré series, Macdonald polynomials and weighted lattice paths in rectangles. Please see below a more detailed description of the research results, organized by topics.

1. Heegaard–Floer homology

Let $C = C_1 \cup \ldots \cup C_n$ be a plane curve singularity in $\mathbb{C}^2$ with n components. Its intersection with a small sphere centered at the origin is a link $L = L_1 \cup \ldots \cup L_n$ with n components, such links are called algebraic. For example, the curve $x^2 = y^2$ corresponds to a pair of linked circles, and the curve $x^2 = y^3$ corresponds to the trefoil knot. In joint project with András Némethi we have computed the Heegaard–Floer homology of

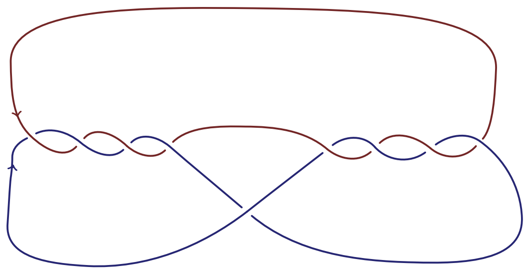

FIGURE 1. The splitting number of this link equals 4.

all algebraic links and related them to certain hyperplane arrangements in the space of algebraic functions on C and to the multi-dimensional semigroup of C.

Theorem 1. [27] *Let $\mathcal{H}(v_1,\ldots,v_n)$ be the space of algebraic functions on C which have order v_1 on the component C_1, order v_2 on the component C_2 etc. Then:*

(a) The space $\mathcal{H}(v_1,\ldots,v_n)$ is either empty or it is a complement to a hyperplane arrangement.

(b) The homology of $\mathcal{H}(v_1,\ldots,v_n)$ is isomorphic to the (minus-version of) Heegaard-Floer homology of the link L in Alexander grading $(v_1,\ldots,v_n)$.

Since the homology of a hyperplane arrangement is determined by its combinatorics, this theorem combined with the results of [10] provides an explicit method of computing the Heegaard-Floer homology of algebraic link.

This work is closely related to the study of L-space links. A 3-manifold is called an L-space if its Heegaard Floer homology has minimal possible rank. A link $L \subset S^3$ is called an L-space link if all Dehn surgeries $S^3_d(L)$ are L-spaces for $d \gg 0$. A remarkable conjecture of Boyer, Gordon and Watson [2] characterizes L-spaces in terms of orders on their fundamental groups. It was proved by Hanselman, Rasmussen, Rasmussen and Watson [31, 30] for graph manifolds. Némethi [39] proved that a negative definite graph manifold is an L-space if and only if it is a link of a rational surface singularity (and hence the graph is rational in the sense of Artin).

In joint works with Jennifer Hom and András Nemethi, we proved the following:

Theorem 2. *a)* [26] *All algebraic links are L-space links.*

b) [15] *A (dn, dm)-cable link of a knot K with d components is an L-space link if and only if K is an L-space knot and $m/n > 2g(K) - 1$.*

In both of these cases, we were able to explicitly compute the Heegaard-Floer homology of L-space links. In particular, this proved a conjecture of Joan Licata (first formulated in 2007) on the structure of this homology for (n, n) torus links. We also studied the structure of the set of L-space surgery coefficients for a given L-space link [25].

In a joint work with Maciej Borodzik [1], we used Heegaard Floer homology to bound the splitting numbers of links (minimal number of crossings that should be changed to separate the components of a link). For L-space links, we computed this bound explicitly and proved that it is sharp in many examples, see Figure 1.

2. KHOVANOV–ROZANSKY HOMOLOGY AND HILBERT SCHEMES

The Hilbert scheme of points on $\mathbb{C}^2$ is a complex manifold which plays a prominent role in modern geometric representation theory, algebraic combinatorics and mathematical physics. I have been working on a large collaborative project focused on understanding the relations between the knot homology and Hilbert scheme. It started with the following conjecture that I first formulated in 2010.

Conjecture 3. [9] *The bigraded dimensions of Khovanov-Rozansky homology of the* $(n, n+1)$ *torus knot are described by the* q, t*-Catalan number* $c_n(q, t)$.

The conjecture was proved by Matthew Hogancamp in 2017 [32]. The q, t-Catalan numbers were introduced by Garsia and Haiman in their work on combinatorics of Macdonald polynomials. They are closely related to the spaces of sections of certain line bundles on the Hilbert scheme of points. More precisely, let T denote the tautological vector bundle of rank n on $\mathrm{Hilb}^n(\mathbb{C}^2)$ and let $\mathcal{O}(1) = \wedge^n(T)$. Let $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ denote the punctual Hilbert scheme of points, then

$$c_n(q, t) = H^0(\mathrm{Hilb}^n(\mathbb{C}^2, 0), \mathcal{O}(1)).$$

In later joint works with Andrei Neguț, Alexei Oblomkov, Jacob Rasmussen, Vivek Shende and others [23, 28, 29, 8, 5, 16, 13], we developed more general algebraic and geometric models for the Khovanov-Rozansky homology of general torus knots. Algebraically, they are related to the representation theory of the Rational Cherednik Algebra and Elliptic Hall Algebra. On decategorified level, we proved the following.

Theorem 4. [5] *Let* $P^{\lambda}_{T(m,n)}(a, q)$ *denote the HOMFLY-PT invariant of the* (m, n) *torus knot colored by a partition* λ*. Then* $P^{\lambda}_{T(m,n)}(-a, q)$ *is equal to the bigraded character of an irreducible representation* $L_{m/n}(n\lambda)$ *of the rational Cherednik algebra and, in particular, all of its coefficients are nonnegative.*

On categorified level, we conjectured the following

Conjecture 5. [29] *The Khovanov-Rozansky homology of the (uncolored)* (m, n) *torus knot is isomorphic to a certain subspace of* $L_{m/n}(n)$*, equipped with an additional filtration.*

Geometrically, these models are related to the spaces of sections of certain line bundles on the flag Hilbert scheme. We proved a direct relation to the "refined Chern-Simons" invariants which attracted a lot of attention in mathematical physics. In joint works with Andrei Neguț, Monica Vazirani and Mikhail Mazin [23, 19, 18, 17, 21, 20], we studied combinatorial models of these invariants and conjectured a generalization of the celebrated "Shuffle conjecture" in combinatorics. This conjecture and our generalization were recently proved by Erik Carlsson and Anton Mellit [3, 36]. The connection between all these algebraic, geometric and combinatorial models to the actual knot invariants remained conjectural until 2017, when most of these conjectures were proven by Ben Elias, Matthew Hogancamp and Anton Mellit [4, 32, 37]. In summer 2016 University of Oregon organized a graduate school focused exclusively on this topic:

`http://pages.uoregon.edu/belias/WARTHOG/torus/index.html`.

Finally, in 2016 in joint work with Andrei Neguț and Jacob Rasmussen, we have developed a blueprint for a more general framework which includes all of the above results as special cases. In short, our main conjecture reads as follows:

Conjecture 6. [24] *Given a braid* β *on* n *strands, there exists a vector bundle (or a coherent sheaf)* $\mathcal{F}_\beta$ *on the Hilbert scheme of* n *points such that the Khovanov-Rozansky homology* $\mathrm{HHH}(\beta)$ *of the closure of* β *is isomorphic to*

$$\mathrm{HHH}(\beta) = H^*(\mathrm{Hilb}^n(\mathbb{C}^2), \mathcal{F}_\beta \otimes \wedge^\bullet T^*),$$

where T*, as above, denotes the tautological bundle on the Hilbert scheme. Furthermore, adding a full twist to* β *corresponds to the tensor product of* $\mathcal{F}_\beta$ *with* $\mathcal{O}(1)$.

If true, this conjecture provides a concrete and very explicit way of computing knot homology using algebraic geometry of Hilbert schemes. It clarifies the structure of the categorified Jones-Wenzl projectors in the Hecke algebra, and related them to the fixed points of torus action on the Hilbert scheme. We proved this conjecture in lots of special cases, and outlined the construction of $\mathcal{F}_\beta$ and a detailed strategy for the proof in general. We are actively working on completing this proof.

Furthermore, Oblomkov and Rozansky [41, 42, 43] proposed yet another construction of $\mathcal{F}_\beta$ using completely different ideas. They proved that their approach yields a knot invariant, but it is currently unknown if it agrees with the original definition of the Khovanov-Rozansky homology. We plan to compare our construction of $\mathcal{F}_\beta$ with theirs.

3. Further results

1) I have obtained an explicit formula for the generating function of S_n–equivariant Euler characteristics of the moduli spaces $\mathcal{M}_{g,n}$ of genus g curves with n marked points [6]. I have also obtained a similar formula for the moduli spaces of hyperelliptic curves [11, 12].

2) In a joint work with Andrei Neguț [22] we studied K-theoretic stable bases (in the sense of Maulik-Okounkov [38]) for the Hilbert schemes of points, and conjectured a precise relation between the wall-crossing matrices for these bases and the involutions on the q-Fock space studied by Leclerc and Thibon [35].

3) In a joint work [14] with my advisor Sabir M. Gusein-Zade, we have proposed a construction of local characteristic numbers of singular varieties using homological algebra.

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