Connected homology and HLRV conjectures (based on joint with Alex Chandler)

1) HLRV
2) cell decompositions
3) Connected homology
4) Examples, computations.

Symmetric functions package

λ partition

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \ldots$$
$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \ldots$$
$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \ldots$$

$$p_n = \sum x_i^n$$
$$h_n = \sum_{i_1 \leq \ldots \leq i_n} x_{i_1} \ldots x_{i_n}$$
$$e_n = \sum_{i_1 < \ldots < i_n} x_{i_1} \ldots x_{i_n}$$

$m_\lambda = \sum$ permutations of $(x_1^{\lambda_1} x_2^{\lambda_2} \ldots)$ (no repetitions)

Scalar product

$$(p_\lambda, p_\mu) = \delta_{\lambda\mu} z_\lambda$$

(centralizer in S_n)

$$(h_\lambda, m_\mu) = \delta_{\lambda\mu}.$$

$\tilde{H}_\lambda[X;q,t]$ = modified Macdonald polynomial (coefficients are in $R = \mathbb{Q}[q,t]$)

example

$$\tilde{H}_{(2)} = (q+1) m_{1,1} + m_2 = q e_2 + h_2$$

$q \leftrightarrow t$ symmetry $\lambda \leftrightarrow \lambda'$

$$\tilde{H}_{1,1} = (t+1) m_{1,1} + m_2 = t e_2 + h_2$$

Partition function

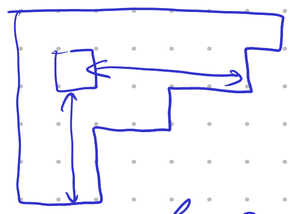
Master partition function & specializations:

fix k, $X_1=(x_{11},x_{12},\dots)$ $X_2=(x_{21},x_{22},\dots)\dots X_k$

$$\Omega_k=\sum_{n=0}^{\infty}\sum_{|\lambda|=n}\frac{\tilde{H}_\lambda[X_1]\,\tilde{H}_\lambda[X_2]\cdots\tilde{H}_\lambda[X_k]}{\prod (q^{a+1}-t^{\ell})(q^{a}-t^{\ell+1})}$$

$n=$ "rank"

arms-legs (product over boxes)



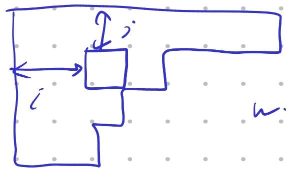
$\ell=3$
$a=4$

Recommended specializations:

Ω_k has k alphabets we can take scalar products with upto k symmetric functions.

$$(\tilde{H}_\lambda, h_n)=1$$ (forgetting alphabet) $\Omega_k\to\Omega_{k-1}$

$$(\tilde{H}_\lambda, h_{n-1,1})=\sum_{\text{weights of }\lambda} q^i t^j$$



weight(□) $=q^i t^j$

$i=2$ $j=1$ get q^2t

$$(\tilde{H}_\lambda, h_k e_m)=e_m(\text{weights}).$$
$k+m=n$

$$(\tilde{H}_\lambda, e_n)=\prod \text{weights}.$$

<u>HLRV</u> plethystic exponential

<u>Def 1</u> $\mathrm{Exp}\left(\sum \text{some monomials } c_i M_i\right)$ $c_i\in\mathbb{Z}$

$$=\prod\frac{1}{(1-M_i)^{c_i}}$$

A better <u>Def 2</u>

$$\prod\frac{1}{(1-M_i)^{c_i}}=\exp\left(\sum c_i\log\frac{1}{1-M_i}\right)=$$
$$=\exp\left(\sum c_i\frac{M_i^k}{k}\right)$$

Def 2 $\quad \mathrm{Exp}\left[f(M_1, M_2, \dots) \right] = \exp\left(\sum_k \frac{1}{k} f(M_1^k, M_2^k, \dots) \right)$

$$\Omega_k = \mathrm{Exp}\left[- \frac{\mathbb{H}(X_1, X_2, \dots, q, t)}{(1-q)(1-t)} \right].$$

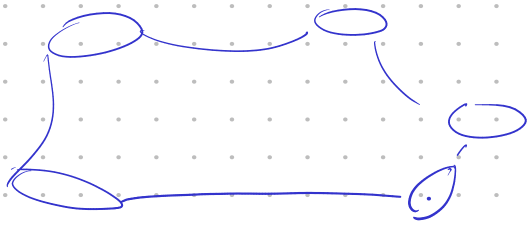
variables are q, t, X_{ij}

Miracle: expanding $\quad \mathbb{H} = \sum_{n=1}^{\infty} \sum_{\substack{\mu^{(1)}, \dots, \mu^{(k)} \\ |\mu^{(i)}| = n}} m_{\mu^{(1)}}(X_1) \dots m_{\mu^{(k)}}(X_k) \times H_{\mu^{(1)} \dots \mu^{(k)}}(q, t)$

where $H_{\mu^{(1)} \dots \mu^{(k)}}$ is a polynomial with positive integer coefficients.

Conjecture HLRV $\quad H_{\mu^{(1)} \dots \mu^{(k)}}$ is the mixed Hodge polynomial of some character variety.

from n, k take sphere with k boundary components. Study local systems on it with prescribed monodromy around boundaries, of rank n



$\mu^{(1)}, \dots, \mu^{(k)}$ partitions of size n $\to$ monodromy around boundaries

example $\quad \mu = (2,1) \quad \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$

$(2,2) \quad \begin{pmatrix} \alpha & & & 0 \\ & \alpha & & \\ & & \beta & \\ 0 & & & \beta \end{pmatrix}.$

fix α, β generic.

Motto: MH polynomials are like polynomials counting Kh-R homologies of links.
(remark: they are symmetric in q, t)

What is known (cell decompositions)

1) restrict to the case $\mu^{(1)} = \underbrace{(1,1,\dots,1)}_{n}$ n arbitrary.

look at $x_{11}, x_{12}, \dots$ make $x_{1i}^2 = 0$

$$\sum x_{1i} = s$$

$$m_{\underbrace{1,1,\dots 1}_{n}}(x_{11}, x_{12}, \dots) = \frac{s^n}{n!}$$

raising x_{1i} to n-th power doesn't do anything

$\Omega_k \rightsquigarrow$ degenerates to $\Omega_k(s, X_2, \dots, X_k, q, t)$

$\mathbb{H} \rightsquigarrow$ degenerates to $\mathbb{H}(s, X_2, \dots X_k, q, t)$

$$\Omega_k = \mathrm{Exp}\left[\frac{\mathbb{H}}{(1-q)(1-t)}\right] \rightsquigarrow$$

$$\Omega_k = \underbrace{\sum \frac{(\tilde{H}_\lambda, h_1)\, \tilde{H}_\lambda\{X_2\} \dots}{\text{arm-legs}}\frac{s^n}{n!}} = \exp\left(\frac{-1}{(1-q)(1-t)} \underbrace{\sum H_{1^n, \mu^{(2)}, \dots, \mu^{(k)}} \cdot m_{\mu^{(2)}}(X_2) \dots m_{\mu^{(k)}}(X_k)\frac{s^n}{n!}}\right)$$

meaning of this (species)

$$\exp\left(\sum \frac{c_n s^n}{n!}\right) = \sum \frac{d_n s^n}{n!}$$

(In fact, introducing S_n-action one gets the plethystic exponential)

$$d_n = \sum_{\substack{\text{set partitions of size } n}} c_{\lambda_1} c_{\lambda_2} \dots c_{\lambda_k}$$

let $\lambda_1, \dots, \lambda_k$ be the sizes of the subsets

Cell decompositions (works for $\mu^{(1)} = (1,1,\ldots 1)$)

Character variety

(1) matrix with distinct e-v's.

idea to use eigenvectors to get a basis.

Character variety is stratified, strata are indexed by $k-1$-tuples of permutations

each stratum for a tuple $\pi_2, \ldots, \pi_k \in S_n$

$\Leftarrow$

(1) ——— (2)

preferred basis

preferred partial flag.

for each point in the character variety corresponds to $S_n / \underbrace{S_{\mu^{(2)}}}_{\text{subgroup}}$,

we choose a minimal representative of such coset.

we write braid $T_{\pi_2} T_{\pi_2^{-1}} T_{\pi_3} T_{\pi_3^{-1}} \ldots T_{\pi_k} T_{\pi_k^{-1}}$,

(*) "Motto": $HHH_{a=0}$(closure of this braid) = MHP of the corresponding stratum.

needs to be corrected. braid is a positive pure braid.

2 problems 1) HHH is a series.

2) braid variety is not quite the stratum.

3) there is exp for gluing functions

Solution

$\exists$ "connected HHH" invariant of a (positive?) braid which is finite dimensional and the formula (*) works.

$$HHH(\underset{\substack{\uparrow \\ \text{braid}}}{\beta}) = \sum_{\substack{\text{set partitions} \\ \text{of } n}} \prod_{\substack{\text{components} \\ C_i}} \left(- \frac{HHH_{\text{connected}}(C_i)}{(1-q)(1-t)} \right).$$

$\hat{\beta}$ has n components

work in progress to construct for positive braids.