Notations

link L

a link partition is a set partition on the set of connected comps of L, write $L = L_1 \cup L_2 \ldots \cup L_k$

write $L_1 \sqcup L_2 \sqcup \ldots \sqcup L_k$ for the disjoint union of L_i.

We want invariant, in general depends on L as a link partition

$$H(L, L_1 | L_2 | \ldots | L_k)$$

Example $k=1$ $H(L, L) = HHH(L) = H(L)$ (y-ified homology of L)

$k = \#$ connected components $H(L, L_1 | \ldots L_k) = H_{\text{connective}}(L)$

Definition

if link has k components we work with

$R_k = \mathbb{C}[x_1, x_2, \ldots, x_k, y_1, \ldots, y_k]$ y-ified homology $(a=0)$ HH^0

we have a complex $C_0 \overset{d}{\underset{d}{\rightleftarrows}} C_1$ of free R_k-modules, bigraded so that d has degree qt

x has degree q

y degree t

In fact this complex is an invariant of L (up to homotopy) of complexes.

Reminder it makes sense to speak of exact triangles of complexes $C_\cdot \to C'_\cdot \to C''_\cdot$ if $C''_\cdot = \mathrm{cone}(C_\cdot \to C'_\cdot)$

we will say that

$C'_\cdot$ is filtered by $C_\cdot$, $C''_\cdot$

In general X is filtered by $X_1, X_2, \ldots X_n$ if there exists triangle $X' \to X \to X_n$, where X' is filtered by $X_1, \ldots X_{n-1}$.

$C_\cdot''$ is a cone $\xrightarrow{\text{generalize}}$

X is a twisted complex composed of $X_1, \dots, X_n$ s.t.
$X = \mathrm{cone}(X' \to X_n)$, X' is a twisted complex $X_1, \dots, X_{n-1}$.

joint Chandler

exp $\leftrightarrow$ log exp(Σ) = sum over set partitions.
log(Σ) = alternating sum over pointed set partitions

Def a pointed set partition on $1, \dots, n$ is an ordered set partition $\{1, \dots, n\} = S_1 \cup S_2 \dots \cup S_m$ s.t. $1 \in S_1$.

Theorem given a positive braid β s.t. $\hat{\beta}$ has k components and a link decomposition of $\hat{\beta}$ $L_1, \dots, L_m$ $\exists$

$C(\hat{\beta}, L_1 | L_2 | \dots | L_m)$ such that.

1) $C(\hat{\beta}) =$ is filtered by:

complex for y-ified homology

$\forall$ set partition μ on m
$\mu = (S_1, \dots, S_r)$

$C(L_{S_1} | (L_i)|_{i \in S_1}) \boxtimes C(L_{S_2} | (L_i)|_{i \in S_2}) \boxtimes \dots$

starts with set partition $\{1, \dots, m\}$ goes to $\{1\}, \dots, \{m\}$

$\underbrace{C(L, L_1 | L_2 \dots L_m)}_{\text{module over } R_K}$ $\underbrace{C(L_1)}_{\text{module over } R_{c(L_1)}} \boxtimes \dots \underbrace{C(L_m)}_{\text{module over } R_{c(L_m)}}$

2) $C(L, L_1 | \dots | L_m)$ is a twisted complex build out of
for a pointed set partition $\mu = (S_1, \dots, S_n)$ $C(L_{S_1}) \boxtimes C(L_{S_2}) \boxtimes \dots$

Examples $n=2$ $\quad L = L_1 \cup L_2$

2) $C(L) \to C(L_1) \otimes C(L_2) \simeq C(L, L_1 | L_2)$

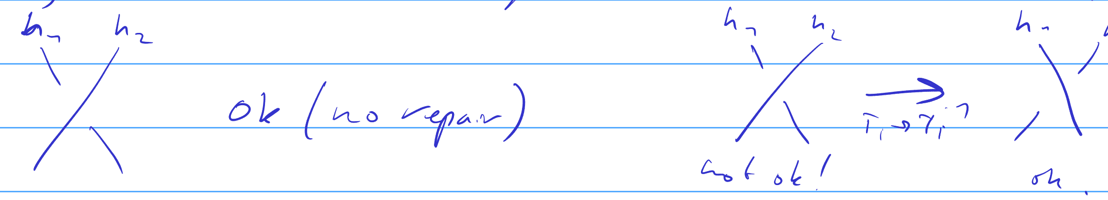
1) $C(L)$ is filtered by $C(L, L_1|L_2)$ ("sub") and $C(L_1)\otimes C(L_2)$ ("quotient").

Idea of proof

how to flip crossings systematically (use maps $T_i \to T_i^{-1}$)

assign heights to components $h_1, \dots, h_k \in \mathbb{R}$ (Khovanov homology is a sheaf over $\mathbb{R}^k$)

if strands of different heights meet we should repair



$h_1 < h_2$ $\qquad\qquad$ $h_1 > h_2$

$\to$ construct complex in 2)

pointed set partition $\rightsquigarrow (h_1, \dots, h_k)$.

Properties: 1) $L = L_1 \sqcup L_2 \Rightarrow C(L, L_1|L_2) = 0$

more generally if

$L_1 \cup \dots L_a$, $L_{a+1} \cup \dots L_m$ don't entangle

$\Rightarrow C(L, L_1 \cup \dots \cup L_m)$,

2) G-H: localizing w.r.t. $(x_i - x_j)$ & $(y_i - y_j)$

where $i \in L_1 \cup \dots L_a$, $j \in L_{a+1} \cup \dots \cup L_m$

produce $C(L)$ isomorphic to $C(L_1 \cup \dots L_a) \otimes C(L_{a+1} \cup \dots L_m)$

$\Rightarrow C_{conn}(L)$ is as a sheaf over $\mathbb{C}^{2k}$

is supported on the diagonal $\mathbb{C}^2 \subset \mathbb{C}^{2k}$.

New stuff (in particular implies that $H_{comm}(L)$ is an invariant of L).

f. generated modules over polynomial rings $\mathbb{C}[A]$ $A = \mathbb{C}^n$.
i.e. sheaves. A.

Let $V \subset A$ be a linear subspace.
a sheaf F on A sits on V nicely
if $F = \pi^* F_0$, where $\pi: A \to A/V$,
F_0 is a sheaf on $A \backslash V$ supported at 0.

Vanishing Lemma suppose F sits on V, G sits on U
then $Ext^i(F, G) = 0$ for $i < \dim U/V \cap U$.

in particular $Hom(F, G) \neq 0 \Rightarrow U = V \cap U \Rightarrow$
$U \subset V$.

if $U, V, U \cap V$ even dimensional
$U \not\subset V \Rightarrow$ also $Ext^1(F, G) = 0$.

$\dim V$ is called "thickness" of F
$Hom(F, G) \neq 0 \Rightarrow F$ is thicker than G.

Proof base change.

$k = 2$ L_1, L_2 knots $L = L_1 \cup L_2$ $R_2 = \mathbb{C}[\mathbb{C}^4]$

$$H^i(L) \to \underbrace{H^i(L_1 \cup L_2)}_{\text{thick}} \to \underbrace{H^i(L, L_1 / L_2)}_{\text{thin}} \to H^{i+1}(L)$$

Example $Y_2 \to \underset{\text{thick}}{R_2} \to \underbrace{R_2/(x_1 - x_2, y_1 - y_2)}_{\text{thin}}$

Solution: take negative links!

Recall that $HH(\beta)$ and $HH(\beta^{-1})$ are connected by duality.

Doing everything for negative links

$k=2$ $\quad L_1, L_2$ knots $\quad L = L_1 \cup L_2 \quad R_2 = \mathbb{C}[\mathbb{C}^4]$

$$\underbrace{H^{i+1}(L\ldots) \leftarrow H^i(L) \leftarrow \underbrace{H^i(L_1 \cup L_2)}_{\text{thick}} \xleftarrow{\text{is zero}} \underbrace{H^i(L, L_1|L_2)}_{\text{thin}}}_{\text{ext vanishes}} \leftarrow H^{i-1}(L)$$

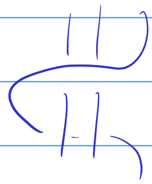
thin

$$\Rightarrow \quad H^i(L) = \underbrace{H^i(L, L_1|L_2) \oplus H^i(L_1 \cup L_2)}$$

in particular this is unique.

$\Rightarrow$ + duality implies that $H_{conn}(L)$ is an invariant of L.

$\mathcal{M}_3^2$ on 3 strands

$\mathcal{M}_3$: 

$\Rightarrow$ hilbert series of H_{conn}

$q^2 + qt + t^2 + 2q + 2t + 1 = P$

$$\frac{1}{(1-q)^3(1-t)^3} - \frac{2(1+q+t)}{(1-q)^2(1-t)^2} + \frac{P(q,t)}{(1-q)(1-t)} = \frac{1}{(1-q)^3(1-t)^3}\left(q^4t^2 \ldots\right.$$

long polynomial)

$\to$ some coefficients are negative)

predicted by Gorsky-Negut.

$H_{conn}($  $) =$ odd

$$\mathcal{Y}_2 \to R \to R/(x_1-x_2,\, y_1-y_2)$$

III

$$\begin{tikzcd}
 & J_2 \boxtimes R_1 \arrow[r] & R_3 \\
J_3 \arrow[ur] \arrow[r] \arrow[dr] & J_2 \boxtimes R_1 \arrow[ur] \arrow[dr] & \\
 & J_2 \boxtimes R_1 \arrow[r] & R_3
\end{tikzcd}$$