Meeting 4/28

Wednesday, April 28, 2021 9:00 AM

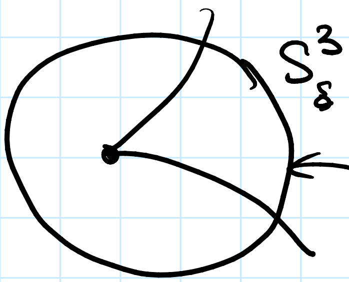
Today: algebraic links

$f(x,y) =$ polynomial

$C = \{f(x,y) = 0\} \subset \mathbb{C}^2$ algebraic curve

Assume it has a singularity at $(0,0)$

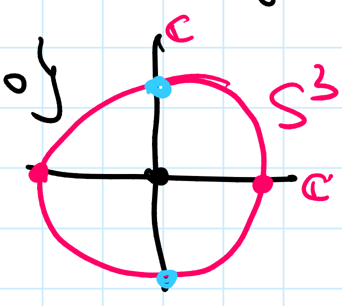
look at the neighborhood of $(0,0)$



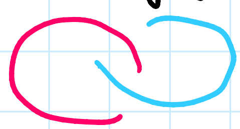
small sphere w. center at $(0,0)$

$C \cap S^3_\delta =$ algebraic link.

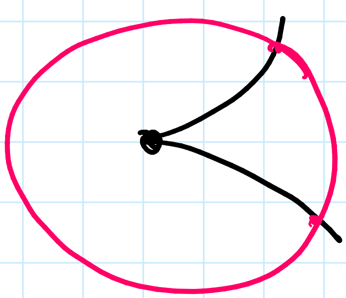
<u>Ex</u> $\{xy = 0\}$



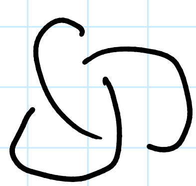
$\rightsquigarrow$ Hopf link



$\{x^2 = y^3\}$



$\rightsquigarrow$ trefoil knot



$\{x^m = y^n\} \longrightarrow (m,n)$ torus link.

Easy fact: # components of the link =

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irreducible components of C
(= # irreducible factors in f)

Alg. links are classified (Eisenbud-Neumann)
In particular, algebraic knots are iterated cables of torus knots.

Conj (Oblomkov, Rasmussen, Shende) $C = \{f=0\}$

$$HHH_{a=0}(\text{link of } C) = H_*(\mathrm{Hilb}^\bullet(C))$$

Hilbert scheme of points on C

Also, a candidate for higher a-degrees.

What is $\mathrm{Hilb}^\bullet(C)$?

$\mathcal{O}_C$ = ring of (alg) functions on C

$$= \frac{\mathbb{C}[x,y]}{(f)}$$

ideal gen. by f

$$\mathcal{O}_{C,0} = \frac{\mathbb{C}[[x,y]]}{(f)}$$

power series

local ring at $(0,0)$

(f) power series local ring at $(0,0)$

$$\mathrm{Hilb}^k(C) = \{\text{ideals } I \subset \mathcal{O}_C : \dim \mathcal{O}_C/I = k\}$$

$$\mathrm{Hilb}^k(C,0) = \{ \text{——————} / \text{——} I \subset \mathcal{O}_{C,0}\}$$

Ex: Unknot $\{y=0\}$

$$\mathcal{O}_{C,0} = \frac{\mathbb{C}[[x,y]]}{(y)} = \mathbb{C}[[x]]$$

$$\mathcal{O}_C = \mathbb{C}[x]$$

need ideals of codim k

Recall: $\mathbb{C}[x]$ is a pid, $\dim \frac{\mathbb{C}[x]}{(f)} = \deg f$

$$\mathrm{Hilb}^k(C) = \{\text{degree } k \text{ monic polynomials } f(x) \text{ in } x\} = \mathbb{C}^k$$

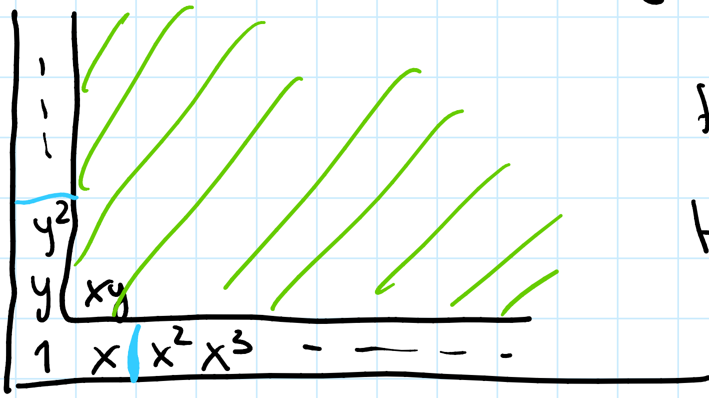
ideal $(f(x)) = I$

$$\mathrm{Hilb}^k(C,0) = \{(x^k)\} = \text{point}$$

any series $x^k + \dots$ is related to x^k by multiplication by an invertible power series

$$\mathrm{Hilb}^{\bullet}(C,0) = \{\underset{1}{pt}, \underset{q}{pt}, \underset{q^2}{pt} \dots\} \qquad q = \text{degree} = k$$

Ex: $\{xy = 0\}$ $\quad \mathcal{O}_{C,0} = \dfrac{\mathbb{C}[[x,y]]}{(xy)}$

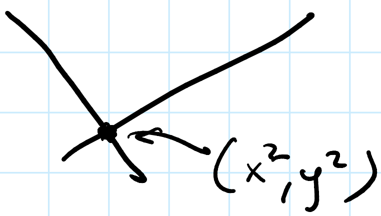


$\mathrm{Hilb}^0 = pt \quad \{\mathcal{O}_{C,0}\}$

$\mathrm{Hilb}^1 = pt \quad \{(x,y)\}$

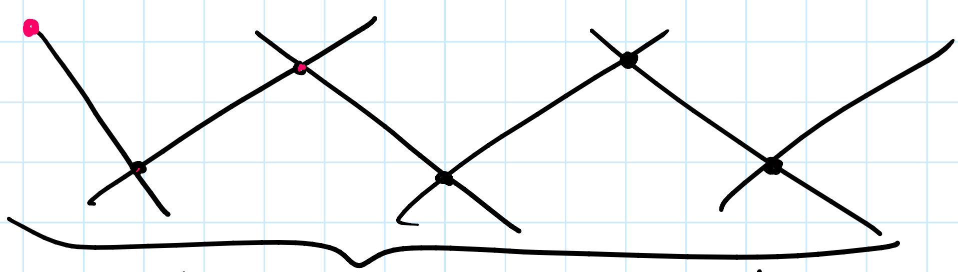
$\mathrm{Hilb}^2 = \mathbb{C}P^1 \quad \{(x^2, y^2, \alpha x + \beta y)\}$ $\alpha, \beta \neq 0$ simultaneous

$\mathrm{Hilb}^3 = \mathbb{C}P^1 \cup \mathbb{C}P^1 \quad \{(x^2, y^3, \alpha x + \beta y^2)\} \cup \{(x^3, y^2, \alpha x^2 + \beta y)\}$



And so on:

$\mathrm{Hilb}^k =$



chain of $(k-1)$ $\mathbb{C}P^1$'s

Exercise: Compute homology & compare with $HHH(\text{Hopf link})$. (only H_0, H_2)

$\bigoplus H_*(\mathrm{Hilb}^k(C))$ two grading = homological & number of points = k

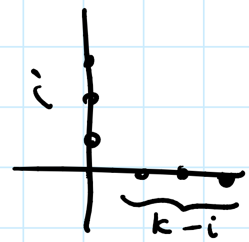
(+) $H_*(\mathrm{Hilb}(C))$ two grading = homological & number of points $= k \mapsto q$

Rmk: There is an affine paving (= cell decomposition).

Ex: (Kivinen) $\mathrm{Hilb}^k(C)$ has $k+1$ components

not local

$$= \mathrm{Bl}_{\mathbb{C}^{i-1}\times\mathbb{C}^{k-i-1}}(\mathbb{C}^i \times \mathbb{C}^{k-i})$$



glued in an interesting way

Ex $\{x^2 = y^3\}$ Can parametrize the curve:

$x = t^3, \ y = t^2 \qquad \mathcal{O}_{C,0} = \mathbb{C}[[t^2, t^3]]$

$\mathrm{Hilb}^0 = \{pt\} = \{\mathcal{O}_{C,0}\}$

$\mathrm{Hilb}^1 = pt = \{(t^2, t^3)\}$

$\mathrm{Hilb}^2 = \mathbb{CP}^1 = \{(t^2+\lambda t^3), (t^3, t^4)\} \quad \lambda \in \mathbb{C}$

In general, $\mathrm{Hilb}^k = \mathbb{CP}^1 = \{(t^k + \lambda t^{k+1}), (t^{k+1}, t^{k+2})\}$

Here, Hilb^k stabilize: $pt, pt, \mathbb{CP}^1, \mathbb{CP}^1, \ldots$

Exercise: compute homology.

Facts: ① The dimension of components of

Facts: ① The dimension of components of $\mathrm{Hilb}^k(C,0)$ are bounded

② If C is irreducible ($\iff$ link(C) is a knot) reduced then $\mathrm{Hilb}^k(C)$ stabilize for $k \gg 0$

③ If $C = \{x^m = y^n\}$ then $\mathrm{Hilb}^\bullet(C)$ has a paving by affine cells

(Goresky-Kottwitz-McPherson, combined with generalized affine Springer fibers. Garner-Kivinen

Oblomkov-Rasmussen-Shende $(m,n)=1$

very explicit formulas for dim of cells)

$\Rightarrow$ explicit combinatorial formulas for $\dim H_*(\mathrm{Hilb}^\bullet C)$.

Big open question: What about other C? If there a paving by affines?

Related space: affine Springer fiber

$\gamma = n \times n$ matrix of $\circlearrowright \mathbb{C}^n((x))$

of

polynomials in x

$$Sp_\gamma = \left\{ V \subset \mathbb{C}^n((x)) : xV \subset V,\ \gamma V \subset V \atop + \text{ some finiteness condition} \right\}$$

$\rightsquigarrow C = \{\det(\gamma(x) - y \cdot Id) = 0\}$

"spectral curve" characteristic polynomial of γ

<u>Fact</u> If C is irreducible then $Hilb^N(C,0)$ stabilizes to Sp_γ such that spectral curve of $\gamma = C$.

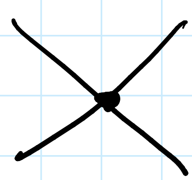
$C = \{x^3 = y^2\}$ $\quad \gamma = \begin{pmatrix} 0 & x^3 \\ 1 & 0 \end{pmatrix}$ $\quad \det(\gamma - y Id) = y^2 - x^3$

$$Sp_\gamma = \mathbb{CP}^1 = Hilb^N(\{x^3 = y^2\})$$

<u>Ex</u> $\gamma = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$

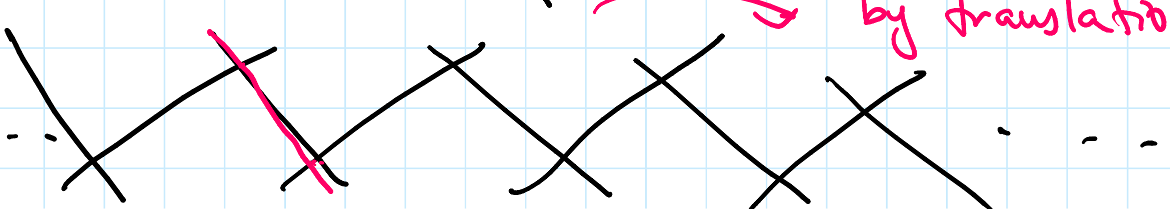
In general, action of $\mathbb{Z}^{\#\text{comp} - 1}$

$C = \{x^2 = y^2\}$

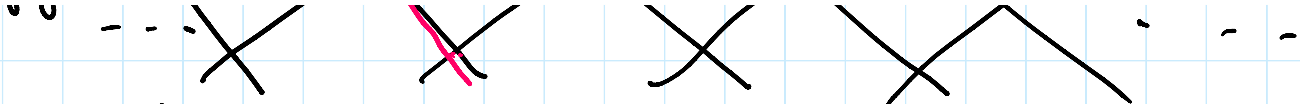


action of $\mathbb{Z}$ by translation

$Sp_\gamma = \ldots$

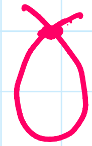


$\ldots$



infinite chain of $\mathbb{CP}^1$ going in both directions

$$= \text{"}\lim_{k\to\infty}\text{"}\ \mathrm{Hilb}^k(C)$$

 $= \mathrm{Sp}_\gamma/\mathbb{Z}$ $\quad H^0=H^1=H^2=\mathbb{Z}$

Note: Hopf link $\rightsquigarrow B \longrightarrow B \longrightarrow R$

$$\mathrm{Hom}(R,-):\ R \xrightarrow{0} R \xrightarrow{x_1-x_2} R$$

Reduced $\doteq$ kill R $\quad \mathbb{C} \quad \mathbb{C} \quad \mathbb{C}$

this matches $H^*(\mathrm{Sp}_\gamma/\mathbb{Z})$

$C=\{x^m=y^n\}$ $\quad x=t^n \quad y=t^m \quad (m,n)=1$

$$\mathbb{C}^n((x)) \xrightarrow{\sim} \mathbb{C}((t))$$

$\mathbb{C}^n$ has basis $1,t,\dots,t^{n-1}$

$$\mathrm{Sp}_\gamma = \left\{ \text{subspaces } V\subset \mathbb{C}((t)) \text{ such that } t^m V\subset V,\ t^n V\subset V \right\}$$

Cells: $\Delta\subset\mathbb{Z}$, $\Delta+m\subset\Delta$, $\Delta+n\subset\Delta$

$$\boxed{\dim(\mathrm{cell}) = \sum \#[a_i, a_i+m[\ \setminus \Delta}$$

where a_i = generators of Δ mod n.

Lusztig-Smelt
Piontkowski
G.-Mazin

relate to q,t-Catalan (dinv...)

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Recursion: Given such Δ, we have two options: (assume $\Delta \subset \mathbb{Z}_{\geq 0}$)

- Δ does not contain 0

  $\Rightarrow$ shift $\Delta \longrightarrow \Delta - 1$, dim does not change

- Δ contains 0, then 0 is a generator mod n

  we can erase $0 \Rightarrow \underline{n}$ is a new generator mod n.

$\Rightarrow$ can control the change in dim if we know $\Delta \cap [0, m+n-1]$

Thm (G.-Mazin-Vazirani) This gives a recursion parametrized by binary sequences of length $m+n \longleftrightarrow \Delta \cap [0, m+n-1]$

This matches Hogancamp-Mellit recursion for HHH

parametrized by pairs of binary sequences of length (m, n).

Ex $(m,n) = (3,4)$ $\qquad \sum \# [a_i, a_i + 4[\setminus \Delta$

a_i = generators

Ex $(m,n) = (3,4)$ [illegible]

a_i = generators mod 3

$\Delta + 3 \subset \Delta \qquad \Delta + 4 \subset \Delta$

normalize such that Δ starts with 0

$0, \_ \_ 3, 4 \_ 6, 7, 8 \ldots$ $\quad 2+1=3$

$0, \_ \_ 3, 4, 5, 6, 7, \ldots$ $\quad 2+0+0=2$

$0, \_ 2\ 3\ 4, 5, 6, 7 \ldots$ $\quad 1+0+0=1$

$0, 1, \_ 3\ 4\ 5\ 6\ 7 \ldots$ $\quad 1+1+0=2$

$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7 \ldots$ $\quad 0+0+0=0$

5 cells $\sim$ cone $(\mathbb{P}^1 \times \mathbb{P}^1)$

Q: $\mathrm{Hilb}^k C \overset{i}{\hookrightarrow} \mathrm{Hilb}^k \mathbb{C}^2$

$i^* H^*(\mathrm{Hilb}^k \mathbb{C}^2)$ = some interesting classes in $\mathrm{Hilb}^k C$

$\mathrm{Sp}_\gamma \overset{i}{\hookrightarrow} \mathrm{Gr}$ = affine Grassmannian

$i^* H^*(\mathrm{Gr})$ = some interesting classes in $H^*(\mathrm{Sp}_\gamma)$

In both cases, we have tautol. rank k bundle

T with fibers $\mathcal{O}/I$

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how to think of $c_i(T)$?

Fact (Oblomkov - Yun) Tautological classes generate $H^*(S_{pq})$ for $\{x^m=y^n\}$

What does it mean for knot homology?

Relations?

Ex $(3,4)$ $\quad c_1, c_2$

$$H^* = \langle 1, \overset{H^2}{c_1}, \overset{H^4}{c_2}, \overset{H^4}{c_1^2}, \overset{H^6}{c_1^3} \rangle$$

all other products $=0$

$c_1 c_2 = 0$