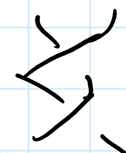
Meeting 6/2

Wednesday, June 2, 2021 8:52 AM

<u>Recap</u> $FT_2 = B \to B \to R$ 

full twist on two strands

$k \geq 0$

$$FT_2^k = \underbrace{B \to B \to B \ldots \to B \to B}_{2k} \to R$$

$$\mathrm{Hom}(\mathbb{1}, FT_2) = R_z \xrightarrow{0} R_y \xrightarrow{x_1 - x_2} R_w \qquad \partial(y) = (x_1 - x_2)w$$

$$\mathrm{Hom}(\mathbb{1}, FT_2^k) = \underset{z^k}{R} \xrightarrow{0} R \xrightarrow{x_1 - x_2} \underset{z^{k-1}w}{\ldots} \to \underset{zw^{k-1}}{R} \xrightarrow{0} R \xrightarrow{x_1 - x_2} R_{w^k}$$

We have multiplicative structure

$$\mathrm{Hom}(\mathbb{1}, \alpha) \otimes \mathrm{Hom}(\mathbb{1}, \beta) \longrightarrow \mathrm{Hom}(\mathbb{1}, \alpha \otimes \beta)$$

$$\mathrm{Hom}(\mathbb{1}, FT_2^k) \otimes \mathrm{Hom}(\mathbb{1}, FT_2^l) \longrightarrow \mathrm{Hom}(\mathbb{1}, FT_2^{k+l})$$

<u>So</u> $\mathcal{A}_2 = \bigoplus_{k=0}^{\infty} \mathrm{Hom}(\mathbb{1}, FT_2^k)$ is a graded algebra

$\mathrm{Hom}(\mathbb{1}, R)$ deg 0 deg 1 $\mathrm{Hom}(\mathbb{1}, FT_2)$

$$\simeq \frac{\mathbb{C}[x_1, x_2, z, w]}{w(x_1 - x_2) = 0}$$

<u>Q</u>: What is $\mathrm{Proj}\, \mathcal{A}_2$?

Subvariety in $\mathbb{C}^2_{x_1, x_2} \times \mathbb{CP}^1_{[z:w]}$

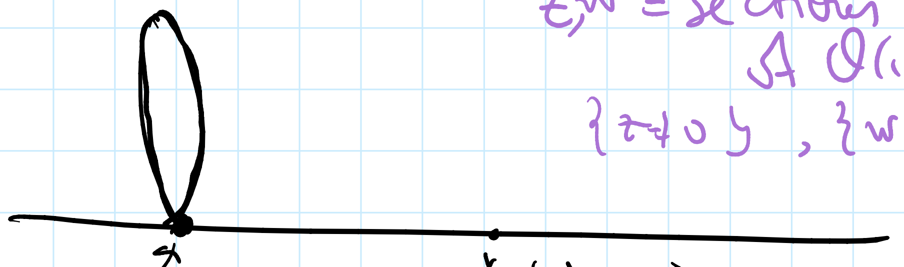
cut out by the equation $w(x_1 - x_2) = 0$

cut out by the equation $w(x_1 - x_2) = 0$

Project to $\mathbb{C}^2$:

z, w = sections of $\mathcal{O}(1)$

$\{z \neq 0\}$, $\{w \neq 0\}$ open charts on Proj A



$x_1 \neq x_2 \Rightarrow w = 0 \Rightarrow z = 1$

one pt

$x_1 = x_2$,

fiber $= \mathbb{CP}^1_{[z:w]}$.

To understand this, we can blow up $\mathbb{C}^2_{x_1,y_1} \times \mathbb{C}^2_{x_2,y_2}$

along the diagonal: $\Delta = \{x_1 = x_2,\ y_1 = y_2\}$

$$Bl_\Delta(\mathbb{C}^2 \times \mathbb{C}^2) \subset \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{CP}^1_{[z:w]}$$

cut out by equation

$$[x_1 - x_2 : y_1 - y_2] = [z : w]$$

$$\underbrace{(x_1 - x_2)\, w = (y_1 - y_2)\, z}$$

our equation if $y_1 = y_2 = 0$

<u>Conclusion:</u> $\operatorname{Proj} A_2 = Bl_\Delta(\mathbb{C}^2 \times \mathbb{C}^2) \cap \{y_1 = y_2 = 0\}$

<u>Rmk</u> $Bl_\Delta(\mathbb{C}^2 \times \mathbb{C}^2)$ is smooth, its intersection

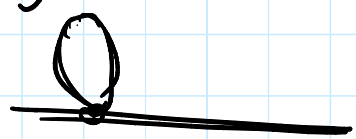
with $\{y_1 = y_2 = 0\}$ is singular with 2 comp with $\{y_1 = y_2 = 0\}$ is singular, with 2 comp.

<u>Why do we care?</u>

β = <u>any</u> braid on 2 strands

$$\widehat{\bigoplus_{k=\infty}} \mathrm{Hom}(\mathbb{1}, \beta \cdot FT_2^k) = \text{graded } A_2\text{-module}$$

$$\Rightarrow \text{sheaf on } \mathrm{Proj}\, A_2$$

$\beta \rightsquigarrow$ Sheaf on 

<u>Which sheaf?</u> $\beta = FT^m = T(2,2m) \rightsquigarrow \mathcal{O}(m)$ line bdle

$$\beta = T(2,2m+1) \rightsquigarrow \mathcal{O}(m)\big|_{Z_2}$$

$$Z_2 = \{x_1 = x_2\} = \mathbb{C}P^1$$

Here m could be positive or negative!

$HHH_{a=0}(\beta) = H^\bullet(\mathrm{Proj}\, A_2, \text{this sheaf})$

$\mathcal{O}(m)$ has H^1 for $m \ll 0 \Leftrightarrow$

odd homology for negative torus links.

<u>Rmk</u> We can y-ify all this

$$A_2^{y\text{-ified}} = \underline{\mathbb{C}[x_1, x_2, y_1, y_2, z, w]}$$

$$A_2^{y\text{-ified}} = \frac{\mathbb{C}[x_1,x_2,y_1,y_2,z,w]}{z(y_1-y_2)=w(x_1-x_2)}$$

$$FT_2^y = B \rightleftarrows B \to R$$

$$HY(FT_2) = \underset{z}{R} \underset{y_1-y_2}{\overset{0}{\rightleftarrows}} R \xrightarrow{x_1-x_2} R_w$$

$$\mathrm{Proj}\, A_2^y = Bl_\Delta(\mathbb{C}^2\times\mathbb{C}^2)$$

<u>Rmk 2</u> This is constant on conjugacy classes:

$$\mathrm{Hom}(\mathbb{1}, \alpha\cdot\beta\cdot FT_2^k) = \mathrm{Hom}(\mathbb{1}, \beta\cdot FT_2^k\cdot\alpha) =$$

$\mathrm{Hom}(\mathbb{1}, XY) = \mathrm{Hom}(\mathbb{1}, YX)$

$$= \mathrm{Hom}(\mathbb{1}, \beta\cdot\alpha\cdot FT_2^k)$$

FT_2 is in the Drinfeld center

$$\mathrm{Sheaf}(\alpha\cdot\beta) = \mathrm{Sheaf}(\beta\cdot\alpha).$$

<u>Rmk 3</u> It would be great to find these graded algebras and modules in other models of knot homology. For affine Springer fibers, this is related to the work of Braverman-Finkelberg-Nakajima, Webster, ...

Braverman-Finkelberg-Nakajima, Webster, ...

What happens in general?

n points on $\mathbb{C}^2$

- $S^n \mathbb{C}^2 = (\mathbb{C}^2)^n / S_n$ affine, very singular

  $\| $

  $\operatorname{Spec} \mathbb{C}[x_1, \dots x_n, y_1, \dots y_n]^{S_n}$ S_n permutes x_i, y_i simultaneously

- $\operatorname{Hilb}^n \mathbb{C}^2 =$ Hilbert scheme of n points in $\mathbb{C}^2$

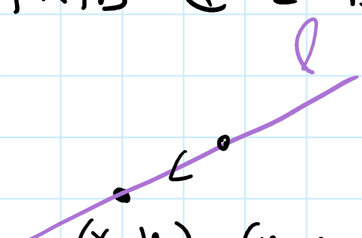
  $= \{ \text{ideals } I \subset \mathbb{C}[x,y] \mid \dim \frac{\mathbb{C}[x,y]}{I} = n \}$

  Smooth!

$\operatorname{Hilb}^n \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^n / S_n$ resolution of singularities

Ex $n = 2$ $(\mathbb{C}^2)^n / S_n$ is singular along the diagonal where 2 points coincide

$\operatorname{Hilb}^2 \mathbb{C}^2 =$ blowup along this diagonal



Ideals supported at this point $p = \{ m_p^2, \ell_q \} = I$

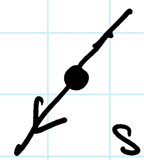
$(x_1, y_1) = (x_2, y_2)$

$\|$

$(x_1,y_1) = (x_2,y_2)$ point $p = \{ m_p , \ell \} = I$
$\quad \overset{\shortparallel}{p}$

maximal ideal at p — equation of arbitrary line through p

$$\dim \frac{\mathbb{C}[x,y]}{m_p^2} = 3 \qquad \dim \frac{\mathbb{C}[x,y]}{I} = 2$$

two points collide along $\ell \Rightarrow$  in the limit, same point w. multiplicity 2 + direction of ℓ

$\{\text{Choice of } \ell\} = \mathbb{CP}^1$ which we saw before.

- $$\begin{tikzcd} X_n \arrow[r, "\rho"] \arrow[d, "\pi"'] & (\mathbb{C}^2)^n \arrow[d] \\ \mathrm{Hilb}^n \mathbb{C}^2 \arrow[r] & S^n \mathbb{C}^2 \end{tikzcd}$$

X_n = isospectral Hilbert scheme (Haiman)

define X_n as (reduced) fiber product

Thm (Haiman) (a) X_n = Blowup of $(\mathbb{C}^2)^n$ along the union of all diagonals Δ

(b) The ideal of $\Delta = \underbrace{\bigcap_{i \neq j} (x_i - x_j, y_i - y_j)}_{\text{ideal of } \{p_i = p_j\}} = J$

Ideal of union = intersection of ideals

(c) $J^k = \bigcap (x_i - x_j, y_i - y_j)^k$

(c) $J^k = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j)^k$

(d) $X_n = \operatorname{Proj} \bigoplus_{k=0}^{\infty} J^k$

$\operatorname{Hilb}^n \mathbb{C}^2 = \operatorname{Proj} \bigoplus_{k=0}^{\infty} (J^k)^{S_n}$ need to be careful with S_n-action.

(e) $\pi_* \mathcal{O}_{X_n} = \mathcal{P} =$ Procesi bundle

this is a vector bundle on $\operatorname{Hilb}^n \mathbb{C}^2$ of rank $n!$.

Thm (from Lec 2) $FT_n =$ full twist on n strands

$$\bigoplus_{k=0}^{\infty} \operatorname{Hom}(\mathbb{1}, FT_n^k) =: A_n = \bigoplus_{k=0}^{\infty} J^k / (y) J^k$$

If we y-ify everything, get $\bigoplus_{k=0}^{\infty} J^k$

This isomorphism preserves the algebra structure!

Cor $\operatorname{Proj} A_n = X_n \cap \{y_1 = \dots = y_n = 0\}$

$\operatorname{Proj} A_n^{y\text{-ified}} = X_n$

Application $\beta =$ arbitrary braid on n strands $\Rightarrow \bigoplus_{k=0}^{\infty} \operatorname{Hom}(\mathbb{1}, FT_n^k \cdot \beta)$

is a graded A_n-module $\Rightarrow$ sheaf on is a graded $\mathbb{C}[\mathcal{O}_n]$-module $\Rightarrow$ sheaf on $\mathrm{Proj}\, A_n$.

<u>Which sheaf?</u>

$$T(n, kn) \longrightarrow \mathcal{O}_{X_n}(k)$$
$$T(n, kn+1) \longrightarrow \mathcal{O}_{Z_n}(k)$$

$$Z_n = \{x_1 = \dots = x_n, y_1 = \dots = y_n = 0\}$$

$$Z_n = \mathrm{Hilb}^n(\mathbb{C}^2, 0)$$

<u>Ex</u> (Haiman) $H^0(Z_n, \mathcal{O}_{Z_n}(1)) =$ q,t-Catalan number.

<u>Conj</u> (G., Negut, Rasmussen)
There is a pair of adjoint (dg) functors

$$K^b(SBim_n) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^b Coh(\mathrm{Hilb}^n \mathbb{C}^2)$$

Such that i^* is monoidal, and we have projection formula

$$i_*(X \otimes i^*(Y)) = i_*(X) \otimes Y$$

$\otimes$ on $\mathrm{Hilb}^n \mathbb{C}^2$ is symmetric!

symmetric!

Furthermore, $i^*(Y)$ is central in $K^b(SBim_n)$, so the order of $\otimes$ does not matter.

Ex $i^*\mathcal{O}(1) = FT_n$ is central.

Conj $D^b Coh(Hilb^n \mathbb{C}^2)$ = categorical (dg) Drinfeld center & cocenter of $K^b(SBim_n)$, so in particular any central object in $K^b(SBim_n)$ appears as i^*(object on $Hilb^n$).

Problem: Consider a bigger algebra

$$\bigoplus_{k_1,\dots,k_n \geq 0} Hom(\mathbb{1}, FT_1^{k_1} \cdot FT_2^{k_2} \cdot \ldots \cdot FT_n^{k_n})$$

It is $\mathbb{Z}^n$ graded, and we can consider an iterative Proj.

Conj The result is the (dg version) of

the flag Hilbert scheme

$$FHilb^n(\mathbb{C}^2) = \{\mathbb{C}[x,y] \supset I_1 \supset \dots \supset I_n\}$$

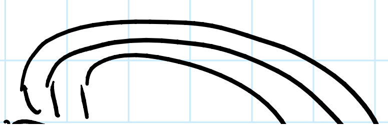
Rmk Oblomkov & Rozansky constructed a different link homology theory using MF over some space related to $Hilb^n \mathbb{C}^2$

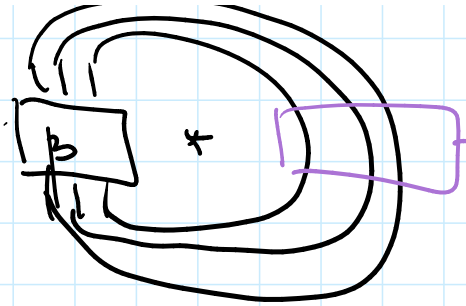
$$\begin{tikzcd} MF(\dots) \arrow[r, "\sim"] \arrow[d] & K^b(SBim_n) \\ Hilb^n & \end{tikzcd}$$

Open problem: relate their construction to ours.

Question: Abstractly, Drinfeld center is braided! Understand the braiding!

Rmk All this is related to annular homology





can add FT_2^k in the annulus.

$$K^b(SBim_n) \longrightarrow \text{Annular category} \longrightarrow Hilb^n$$

rotate by 360°.

$$SBim \hookrightarrow Bim$$

$$K^b(SBim) \to D(Bim)$$

$$\beta \rightsquigarrow \text{object for perm.}$$

$$FT \longrightarrow R$$

$$\mathbb{C}[x_1 \ldots x_n] \rtimes S_n$$

<u>Rmk</u> $FT_2, FT_3, \ldots, FT_n$ all commute

$\mathcal{C}$ = subcategory generated by $FT_2, \ldots, FT_n$ and $\otimes$

$\cap$ $K^b(SBim)$

Expect $\mathcal{C} \simeq$ "$D^bCoh(FHilb_n)$" $\longrightarrow$ Hilb

Expect $\mathcal{C} \simeq {}^{n}D^{b}\mathrm{Coh}(\mathrm{FHilb}_n)^{\mathrm{?}} \rightleftarrows \mathrm{Hilb}$

in dg sense

$$\begin{tikzcd} \mathcal{C} \arrow[d, shift right] \\ K^b(\mathrm{SBim}_n) \arrow[u, shift right, "i_*"'] \end{tikzcd}$$

i_* is adjoint to inclu.