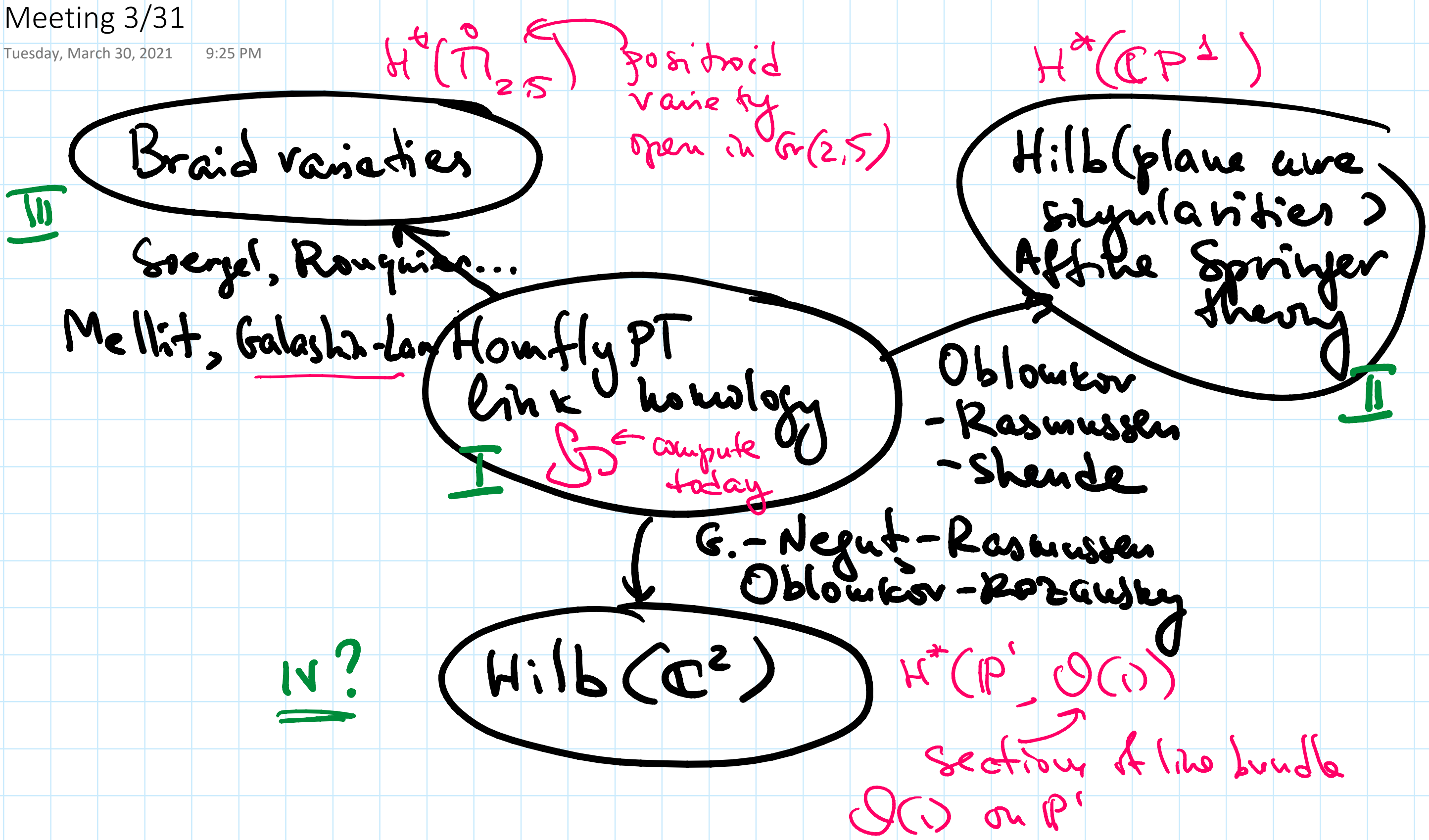
Meeting 3/31

Tuesday, March 30, 2021 9:25 PM



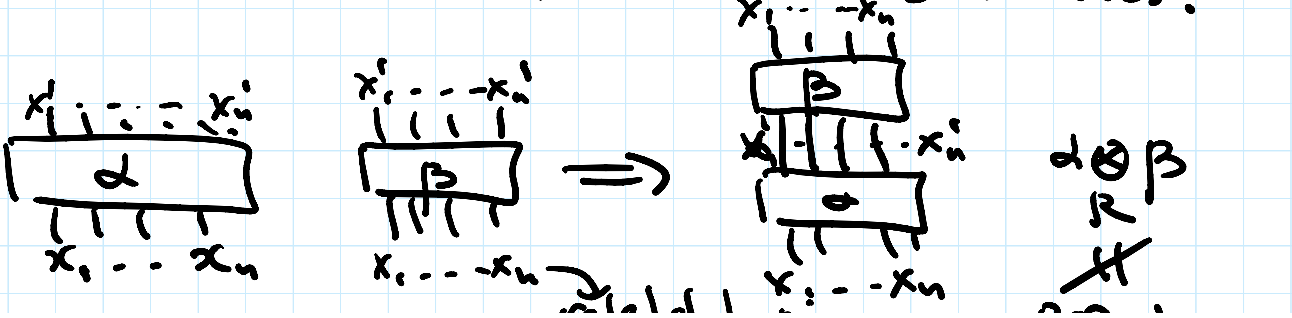
$R = \mathbb{C}[x_1 \dots x_n] \circlearrowleft S_n$

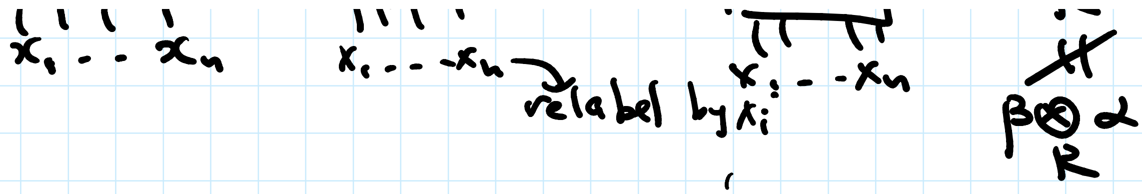
$$B_i = R \underset{R^{(i\,i+1)}}{\otimes} R = \frac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{\begin{pmatrix} x_i + x_{i+1} = x_i' + x_{i+1}' & x_j = x_j' \\ x_i x_{i+1} = x_i' x_{i+1}' & j \neq i, i+1 \end{pmatrix}} \quad (*)$$

← R-R bimodule

acts on the left by x_i on the right by x_i'

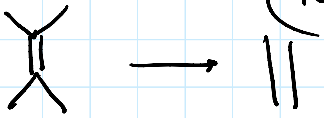
$SBim_n$ = smallest full subcategory of R-R bimod containing R and B_i and closed under $\otimes_R$, $\oplus$ and direct summands.



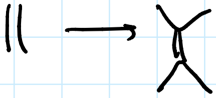


$B_i \longrightarrow R$ projection

$R = \dfrac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{(x_i = x_i')}$



$R \xrightarrow{x_i - x_{i+1}'} B_i$



Well defined:

$(x_i - x_i')(x_i - x_{i+1}') = (\text{symmetric in } x_i', x_{i+1}')$

$= (x_i - x_i)(x_i - x_{i+1}) = 0$

modulo ideal $(*)$

$T_i = [B_i \longrightarrow R]$ $\quad T_i^{-1} = [R \longrightarrow B_i]$

two-term complexes of SBim

Thm (Rouquier) T_i, T_i^{-1} satisfy braid relations up to homotopy equivalence

$$T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1}$$
$$T_i \otimes T_j \simeq T_j \otimes T_i \qquad |i-j| \geq 2$$
$$T_i \otimes T_i^{-1} \simeq T_i^{-1} \otimes T_i \simeq R$$

4-term complexes

Cor β = braid on n strands

$\rightsquigarrow T_\beta$ = Rouquier complex = product of T_i, T_i^{-1} well defined up to homotopy equivalence.

Def $HHH(\beta) = H^*(HH^*(T_\beta))$ **HOMFLY-PT homology**

complex of bimodules

take homology of the result.

term-wise Hochschild cohomology

homology of the result.

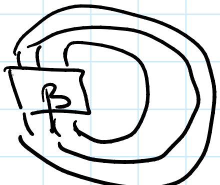
For most of the time: $HH^0(T_\beta) = \mathrm{Hom}_R(R, T_\beta)$

$\mathrm{Hom}(R,-)$ to each term of T_β

Triply graded: • B_i are graded, $\deg(x_i) = 2$
$\deg_q$

• HH^* ← Hochschild degree $= \deg_a$

• homological grading $= \deg_t$



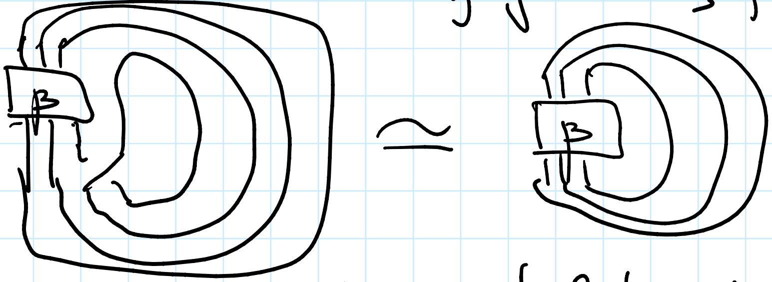
Thm (Khovanov-Rozansky)

$HHH(\beta) =$ (up to shifts in grading)

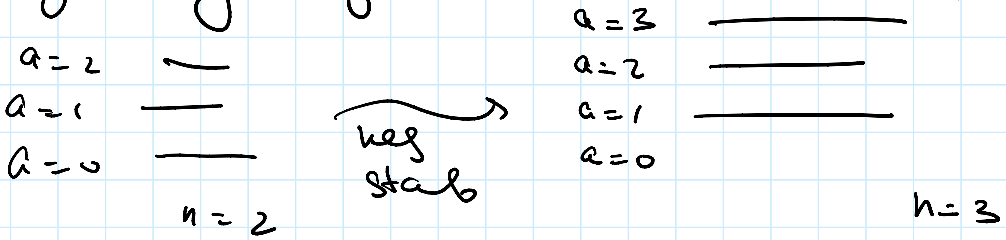
$=$ topological invariant of the closure of β

Rmk ($a=0$) part is not a top. invariant

But: invariant under conjugation, positive stabilization



Destroyed by negative stabilization.



Examples $\boxed{n=2}$ $B = B_1$ $T = [B \to R]$ $T^{-1} = [R \to B]$

Exercise: T^k, $k>0$: $\underbrace{B \to \dots \to \underset{6}{B} \xrightarrow{x_1-x_1'} \underset{4}{B} \xrightarrow{x_1-x_2'} \underset{2}{B} \xrightarrow{\boxed{x_1-x_1'}} \underset{0}{B}}_{k} \to \underset{0}{R}$

$(x_1-x_1')(x_1-x_2') = 0$

a priori

a priori 2^k term complex

after simplification

k

$(x_1-x_1')(x_1-x_2')=0$ mod $(*)$

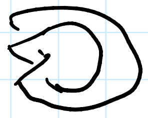
<u>Useful Lemma</u>: $B\otimes B \simeq B \oplus qB$

$\mathrm{Hom}(R, T^k)$: know $\mathrm{Hom}(R,R)=R$ $\mathrm{Hom}(R,B)=R$

$$\begin{tikzcd} \mathrm{Hom}(R,T^k) \arrow[d, phantom, "\parallel"] \\ R \to \cdots \arrow[r] & R \arrow[r, "0"] & R \arrow[r, "x_1-x_2"] & R \arrow[r, "\boxed{0}"] & R \arrow[r, "x_1-x_2"] & R \end{tikzcd}$$

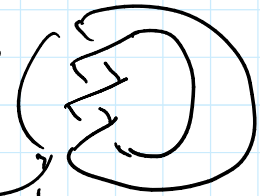
$\mathrm{Hom}(R,-)$ identifies x_i and x_i'

$\boxed{k=2}$ $\underset{z}{R} \xrightarrow{0} R \xrightarrow{x_1-x_2} \underset{w}{R}$

$HHH^0\Big($  $\Big) = \dfrac{R\langle z,w\rangle}{w(x_1-x_2)=0} = R \oplus R/(x_1-x_2)$

Can compute everything with q-gradings.

$\boxed{k=3}$ $R \xrightarrow{x_1-x_2} \underline{R} \xrightarrow{0} R \xrightarrow{x_1-x_2} \underline{R}$

$HHH^0\Big($  $\Big) = R/(x_1-x_2) \oplus R/(x_1-x_2)$

trefoil

$= R/(x_1-x_2)\langle z,w\rangle$

For general $k>0$: $k=2\ell \Rightarrow R \oplus \ell$ copies of $R/(x_1-x_2)$

$k=2\ell+1 \Rightarrow \ell+1$ copies of $R/(x_1-x_2)$

in homological degrees $0,2,4,\ldots 2\ell$

(similar to $H^*(\mathbb{C}P^\ell)$!)

<u>Note</u>: All this homology is supported in <u>even</u> homological degrees!

How about $k<0$?

T^{-k}: $R \to \underbrace{B \to B \to \cdots \to B}$

$$T^{-k} : \underset{\text{homological } 0}{R} \to \underbrace{B \to B \to \dots \to B}_{k}$$

$$T^{-2} : R \to B \to B$$

$\mathrm{Hom}(R, T^{-1}) = R \xrightarrow{1} R$ $\quad HHH^0 = 0$

$$\mathrm{Hom}(R, T^{-2}) : R \xrightarrow{1} R \xrightarrow{0} \underline{\underline{R}}$$

$HHH^*(T^{-2}) \cong R$

free of rank 1

$$\mathrm{Hom}(R, T^{-3}) : \underbrace{R} \xrightarrow{1} R \xrightarrow{0} R \xrightarrow{x_1 - x_2} \underbrace{R}$$

$HHH^* \simeq R/(x_1 - x_2)$ in homological degree 3

odd!

$$\mathrm{Hom}(R, T^{-4}) \quad R \xrightarrow{1} R \xrightarrow{0} R \xrightarrow{x_1 - x_2} R \xrightarrow{0} R$$

$HHH^\bullet = \underset{\text{odd}}{R/(x_1 - x_2)} \oplus \underset{\text{even}}{R}$

$T(2, -4)$ torus link.

Parity breaks down for T^{-k}!

General properties:

- HHH^0 is always a module over R, identify x_i and x_i' by $\mathrm{Hom}(R, -)$
- free over $\mathbb{C}[x_1 + \dots + x_n]$ ← poly in 1 var.

$\Rightarrow$ can quotient by that and get "reduced homology".

Fact Action of x_i on T_β is homotopic to the action of $x'_{w(i)}$, where w is the permutation corresponding to β.

In other words, $\exists\, \xi_i \quad [d, \xi_i] = x_i - x'_{w(i)}$

Cor On HHH, action of x_i and $x'_{w(i)}$ and

Cor On HHH, action of x_i and $x'_{w(i)}$ and $x_{w(i)}$ is the same

$\Rightarrow$ $\boxed{\text{HHH is a module over } \dfrac{R}{(x_i = x_{w(i)})} \text{ for all } i}$ finite rank

If $\bar{\beta}$ is a knot $\Rightarrow$ w is one cycle $\Rightarrow$ identify all $x_i = x_j$

(n=2), K odd $\Rightarrow$ finite dim. module over $R/(x_1 - x_2)$
free

k even $\Rightarrow$ $w = id$ $x_i = x_i$ can and do have summands with full support.

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y-ification: can use homotopies ξ_i as above to deform HHH to "y-ified homology"

$$T_\beta \otimes \mathbb{C}[y_1 \ldots y_n] \circlearrowleft D = d + \sum \xi_i y_i$$

q-degree 2
homol. -1

old differential

$\xi_i \xi_j + \xi_j \xi_i = 0$

$\xi_i^2 = 0$

$$D^2 = \cancel{d^2} + \sum [d, \xi_i] y_i = \sum (x_i - x'_{w(i)}) y_i$$

Close the braid, identify $y_i = y_{w(i)}$

$\Rightarrow$ $D^2 = 0$, get a well defined complex, its homology $= HY(\beta)$.

Ex T^2:

$$\begin{tikzcd} R[y_1, y_2] \arrow[r, "0"'] & R[y_1, y_2] \arrow[r, "x_1 - x_2"] \arrow[l, bend right, "y_1 - y_2"'] & R[y_1, y_2] \end{tikzcd}$$

$D = d +$ green diff.

Fact $HY(\beta)$ = module over $\mathbb{C}[x_1 .. x_c, y_1 -- y_c]$

variables correspond to components of closure of β. = cycles in w.

Big problem: cannot compute HHH for $n>2$ strands by hands $\Rightarrow$ need more tools next time!

$D^2 = \sum (x_i - x'_{w(i)}) y_i$

Close braid $\sum (x_i - x_{w(i)}) y_i$

identify $y_i = y_{w(i)}$

HY is still triply graded

y_i have homol. degree 2 and q-degree -2.

D has q-degree 0

hom. degree 1

Note: If we consider full HHH^* it is a module over $HH^*(R) = \mathbb{C}[x_1 .. x_n]\langle\theta_1 ... \theta_n\rangle$

odd vars

a-degree 1

$HY(\ldots)$ – module over $\mathbb{C}[x_1 .. x_n, y_1 .. y_n]\langle\theta_1 .. \theta_n\rangle$

c components of $\bar{\beta}$

$\mathbb{C}[x_1 ... x_c, y_1 .. y_c]\langle\theta_1 .. \theta_c\rangle$