Positroids, knots, and q,t-Catalan numbers.

joint w/ Thomas Lam.

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arXiv: 2104.05701

$$Gr(k,n;\mathbb{C}) = \{V \subset \mathbb{C}^n \mid \dim V = k\}$$
$$= \left\{\begin{matrix}\text{full rank}\\ k\times n \text{ matrices}\end{matrix}\right\} \Big/ \left(\begin{matrix}\text{row}\\ \text{operations}\end{matrix}\right)$$

Def Top open positroid variety:

$$\Pi^\circ_{k,n} = \left\{ V \in Gr(k,n) \;\middle|\; \begin{matrix} \Delta_{1,\dots,k}(V) \neq 0 \\ \Delta_{2,\dots,k+1}(V) \neq 0 \\ \vdots \\ \Delta_{n,1,\dots,k-1}(V) \neq 0 \end{matrix} \right\}$$

Ex $(k=2,\ n=4)$

$$\Pi^\circ_{2,4} = \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \;\middle|\; \begin{matrix} a \neq 0 \\ ad - bc \neq 0 \\ d \neq 0 \end{matrix} \right\}$$

Q1. $\# \Pi^\circ_{k,n}(\mathbb{F}_q) = \ ?$

Q2. Poincaré poly of $\Pi^\circ_{k,n}(\mathbb{C})$?

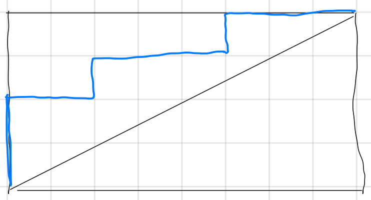
For $Gr(k,n)$, both answers are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}$$

Def. Rational Catalan numbers:

fix $a,b \geq 1$, $\gcd(a,b)=1$

$$C_{a,b} = \frac{1}{a+b}\binom{a+b}{a} = \# \mathrm{Dyck}_{a,b}$$



Thm [GL20] Assume $\gcd(k,n)=1$.

– Point count:

$$\# \Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot C'_{k,n-k}(q), \text{ where}$$

$$C'_{a,b}(q) := \frac{1}{[a+b]_q} \cdot \begin{bmatrix} a+b \\ a \end{bmatrix}_q$$

– Poincaré poly:

$$\mathcal{P}(\Pi^\circ_{k,n}; q) = (q+1)^{n-1} \cdot C''_{k,n-k}(q), \text{ where}$$

$$C''_{a,b}(q) = \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)}$$

Cor.
$$\mathrm{Prob}\left(V \in \Pi^\circ_{k,n}(\mathbb{F}_q)\right) = \boxed{\frac{(q-1)^n}{q^n-1}}$$

$k = \dim V$

weird does not depend on k.

Goal: common generalization?

LHS:

$H^*(\Pi^\circ_{k,n})$ – graded vector space

$$H^i(\Pi^\circ_{k,n}) = \bigoplus_{p\in\mathbb{Z}} H^{i,(p,p)}(\Pi^\circ_{k,n})$$

Deligne splitting.

Mixed Hodge poly: $\mathcal{P}(\Pi^\circ_{k,n}; q,t) \in \mathbb{N}[q^{1/2}, t^{1/2}]$

RHS:

Rat'l q,t-Catalan number
$$C_{a,b}(q,t) = \sum_{P\in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}$$

[GH'96] [LW'09]

$b = a+1$
usual Catalan

Thm [GL20] Assume $\gcd(k,n)=1$.
$$\mathcal{P}(\Pi^\circ_{k,n}; q,t) = (q^{1/2}+t^{1/2})^{n-1} C_{k,n-k}(q,t)$$

Cor. $C_{a,b}(q,t)$ are
- q,t-symmetric — known: Mellit '19
- q,t-unimodal — New

Follow from curious Lefschetz property for cluster varieties.

GSV form

[LS'16] [GL'19]

Scott '06

Arbitrary positroid varieties.

$$M_{i+n} = M_i \quad \forall i\in\mathbb{Z}$$

$f_M : [n] \to [n]$, where $[n] = \{1,2,\dots,n\}$.

$$f_M(i) = \min\{j \geq i \mid M_i \in \mathrm{Span}(M_{i+1},\dots,M_j)\}$$
mod n.

Turns out f_M is a permutation.

Given $f \in S_n$, define [KLS'13]

$$\Pi_f^\circ = \{ V \in Gr(k,n) \mid f_V = f \}.$$

(Assuming f has no fixed pts)

for $f_{k,n} \in S_n$ given by $f_{k,n}(i) \equiv i+k \mod n$,

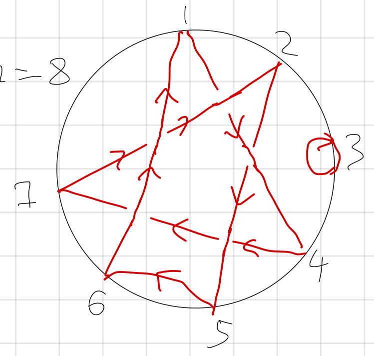
$$\Pi^\circ_{f_{k,n}} = \Pi^\circ_{k,n}.$$

Q: $\mathcal{P}(\Pi_f^\circ; q,t) = ?$

Positroid links.

Def. $f \in S_n \longmapsto$ link $\hat{\beta}_f$

if $a \to b$ crosses $c \to d$ then $a \to b$ is above iff $b < d$.



$T \subseteq SL_n$ – group of diag. matrices

T acts on each Π_f° by column rescaling.

Lemma. TFAE:

(1) T acts freely on Π_f°

(2) $c(f) = 1$ $\quad$ $c(\cdot) = \#$ cycles

(3) $\hat{\beta}_f$ is a knot.

$$c(f_{k,n}) = \gcd(k,n)$$

if T acts freely on Π_f° then

$$\mathcal{P}(\Pi_f^\circ; q,t) = (q^{1/2}+t^{1/2})^{n-1}\,\mathcal{P}(\Pi_f^\circ/T; q,t).$$

Let $P_{KR}(\hat{\beta}_f; a,q,t)$ be the 3-graded Kh-R link invariant.

Th'm [GL'20] Assume $c(f)=1$. Then

$$\mathcal{P}(\Pi_f^\circ/T; q,t) = P_{KR}^{top}(\hat{\beta}_f; q,t),$$

where P_{KR}^{top} is top a-degree coef. of P_{KR}.

Most general result

$v, w \in W$ nonempty iff $\boxed{v \leq w}$

$$R^{\circ}_{v,w} = \left(BwB \cap B_{-}vB \right)/_{B} \subset G/B$$

$$\forall f \in S_n, \ \exists v, w \in S_n : \quad \left.\begin{array}{l} \Pi^{\circ}_f \cong R^{\circ}_{v,w} \\ f = w \cdot v^{-1} \end{array}\right\} (\star)$$

Richardson braid: $\beta_{v,w} := \underbrace{\beta(w)}_{\text{positive braid lift.}} \cdot \beta(v)^{-1}$

Claim: $(\star) \Rightarrow \hat{\beta}_f \cong \hat{\beta}_{v,w}$

Thm [GL20] For $v \leq w \in W$,

$$HH^{0}\left(F_{v,w}\right) \cong H^{*}_{T,c}\left(R^{\circ}_{v,w}\right),$$

where $F_{v,w} = F(w) \otimes F(v)^{-1}$ — Rouquier complexes

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$q = t = 1$ specialization [GL'21]

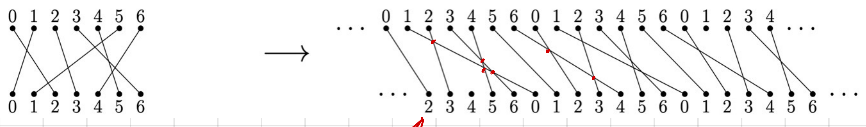
Assume: $f \in S_n$, $c(f) = 1$.

$\rightarrow \mathcal{P}\left(\Pi^{\circ}_f / T ; q, t\right)$ set $q = t = 1$

$f \rightarrow \Pi^{\circ}_f \subset Gr(k, n)$

$n = n(f)$

$k = k(f) = \#\{ i \in [n] \mid f(i) < i \}$



f $\longrightarrow$ $\ell(f) = 6$

$k(f) = 3$

$n(f) = 7$

$\gamma(f) = (3, 4)$.

$\gamma(f) := \left(k(f), n(f) - k(f)\right)$

$\tilde{f} : \mathbb{Z} \to \mathbb{Z}$

$\tilde{f}(i) \equiv f(i) \mod n$

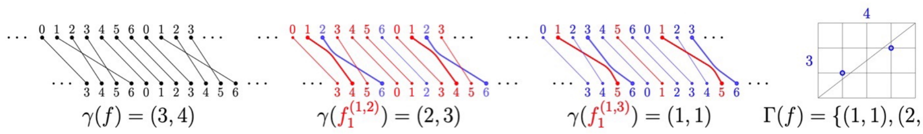
$\tilde{f}(i+n) = f(i) + n$

$i < \tilde{f}(i) < i + n$

$$\dim\left(\Pi^{\circ}_f\right) = k(n-k) - \ell(\tilde{f}).$$

$$\ell(\tilde{f}) = \#\{ i < j \mid \tilde{f}(i) > \tilde{f}(j), \ i \in [n] \}$$

Goal: $f \mapsto$ multiset $\Gamma(f)$ of points inside $k \times (n-k)$ rect.

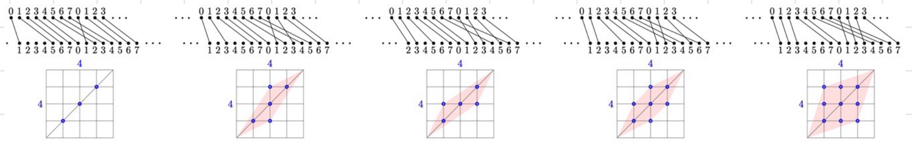


$\gamma(f) = (3,4)$ $\quad$ $\gamma(f_1^{(1,2)}) = (2,3)$ $\quad$ $\gamma(f_1^{(1,3)}) = (1,1)$ $\quad$ $\Gamma(f) = \{(1,1),(2,3)\}$

$f_1^{(i,j)}$ = cycle through i when I swap $f(i)$ and $f(j)$.

Def. f is rep-free if $\Gamma(f)$ has no repeated elts.

Ex. $k=4, n=8$



$\hat{\Gamma} := \Gamma \sqcup \{(0,0), (k, n-k)\}$

Γ is <u>convex</u> if $\hat{\Gamma} = \mathbb{Z}^2 \cap \mathrm{Conv}(\hat{\Gamma})$

Thm [GL21]:

(1) f is rep-free $\Rightarrow$ $\Gamma(f)$ is convex & centrally symm.

(2) Any convex centr symm set Γ arises this way

(3) f is rep-free $\Rightarrow$

$\mathcal{P}(\Pi_f^\circ / T; q=t=1) = \#$ Dyck paths avoiding $\Gamma(f)$.

Conjectural interpr. $\mathcal{P}(\Pi_f^\circ / T; q, t)$ from

[GHSR 20] [BHMPS 21]

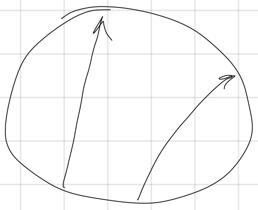
$\uparrow$ Schur pos. iff shape is convex

Thanks!

inversions of $\tilde{f}$

$\updownarrow$

alignments



Fiedler invt

$\# \Pi_f^\circ(\mathbb{F}_q) \Leftarrow$ HOMFLY

$\begin{bmatrix} n \\ k \end{bmatrix}_q$

$q=t=1$, $f = g_1 \cdot g_2 \cdots g_r \leftarrow$ single cycles

$C_f = \prod_{i=1}^{r} C_{g_i}$

$\parallel$

$\mathcal{P}(q=t=1)$