Positroids, knots, and q,t-Catalan numbers.

Joint work with Thomas Lam

[GL20] P. Galashin and T. Lam. Positroids, knots, and q,t-Catalan numbers. arXiv:2012.09745.

[GL21] P. Galashin and T. Lam. Positroid Catalan numbers. arXiv:2104.05701.

Definition. The *Grassmannian* $\mathrm{Gr}(k,n;\mathbb{C})$ is the space of k-dimensional linear subspaces of $\mathbb{C}^n$.

$$\mathrm{Gr}(k,n;\mathbb{C}) \cong \frac{\{M \in \mathrm{Mat}_{k\times n}(\mathbb{C}) \mid \mathrm{rk}(M)=k\}}{\text{row operations}}.$$

Definition (*Top open positroid variety*).

$$\Pi^\circ_{k,n} := \{V \in \mathrm{Gr}(k,n) \mid \Delta_{r+1,\dots,r+k}(V) \neq 0 \quad \text{for all } r \in [n]\}.$$

Example ($k=2, n=4$).

$$\Pi^\circ_{2,4} \cong \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix} \middle| \begin{array}{c} a,b,c,d \in \mathbb{C}: \\ a,d \neq 0, ad-bc \neq 0 \end{array} \right\}.$$

- $\#\Pi^\circ_{k,n}(\mathbb{F}_q) = ?$
- Poincaré polynomial of $\Pi^\circ_{k,n}(\mathbb{C})$?
  ▷ For $\mathrm{Gr}(k,n)$, both are given by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \sum_{\lambda \subseteq k\times(n-k)} q^{|\lambda|}, \quad \text{where}$$

$$[n]_q := 1+q+\dots+q^{n-1}, \quad [n]_q! := [1]_q\cdot[2]_q\cdots[n]_q.$$

Definition (*Rational Catalan numbers*).

- For $a,b \geq 1$ with $\gcd(a,b)=1$, let

$$C_{a,b} := \frac{1}{a+b}\binom{a+b}{a} = \#\,\mathrm{Dyck}_{a,b}, \quad \text{where}$$

$$\mathrm{Dyck}_{a,b} := \{\text{Dyck paths inside an } a\times b \text{ rectangle}\}$$

Theorem ([GL20]). *Let* $\gcd(k,n)=1$.

- Point count *of* $\Pi^\circ_{k,n}$:

$$\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1}\cdot C'_{k,n-k}(q), \quad \textit{where}$$

$$C'_{a,b}(q) = \frac{1}{[a+b]_q}\begin{bmatrix} a+b \\ a \end{bmatrix}_q.$$

- Poincaré polynomial *of* $\Pi^\circ_{k,n}$:

$$\mathcal{P}(\Pi^\circ_{k,n};q) = (q+1)^{n-1}\cdot C''_{k,n-k}(q), \quad \textit{where}$$

$$C''_{a,b}(q) = \sum_{P\in\mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)}.$$

Corollary.

$$\mathrm{Prob}(V \in \Pi^\circ_{k,n}(\mathbb{F}_q)) = \frac{(q-1)^n}{q^n-1}.$$

- Somehow the RHS does not depend on k.

Question. Common generalization?

Left hand side.

- $H^*(\Pi^\circ_{k,n})$ admits a second grading coming from the *Deligne splitting*:

$$H^i(\Pi^\circ_{k,n}) = \bigoplus_{p\in\mathbb{Z}} H^{i,(p,p)}(\Pi^\circ_{k,n}).$$

- The corresponding *mixed Hodge polynomial* $\mathcal{P}(\Pi^\circ_{k,n};q,t) \in \mathbb{N}[q^{\frac12},t^{\frac12}]$ specializes to both $\#\Pi^\circ_{k,n}(\mathbb{F}_q)$ and $\mathcal{P}(\Pi^\circ_{k,n};q)$.

Right hand side.

- *Rational q,t-Catalan numbers*:

$$C_{a,b}(q,t) := \sum_{P\in\mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)}t^{\mathrm{dinv}(P)}.$$

- Specializes to both $C'_{a,b}(q)$ and $C''_{a,b}(q)$.

[GH96] A. M. Garsia and M. Haiman. A remarkable q,t-Catalan sequence and q-Lagrange inversion. *J. Algebraic Combin.*, 5(3):191–244, 1996.

[LW09] N. A. Loehr and G. S. Warrington. A continuous family of partition statistics equidistributed with length. *J. Combin. Theory Ser. A*, 116(2):379–403, 2009.

Theorem ([GL20]). *Let* $\gcd(k,n)=1$. *Then the mixed Hodge polynomial of* $\Pi^\circ_{k,n}$ *is given by*

$$\mathcal{P}(\Pi^\circ_{k,n};q,t) = \left(q^{\frac12}+t^{\frac12}\right)^{n-1} C_{k,n-k}(q,t).$$

Corollary. $C_{a,b}(q,t)$ are q,t-symmetric and q,t-unimodal.

- symmetry is known, unimodality is new.
- Both follow from the *curious Lefschetz property* of cluster varieties:

[LS16] T. Lam and D. E. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. arXiv:1604.06843, 2016.

[GL19] P. Galashin and T. Lam. Positroid varieties and cluster algebras. arXiv:1906.03501, 2019.

Arbitrary positroid varieties.

- Recall:

$$\mathrm{Gr}(k,n;\mathbb{C}) \cong \frac{\{M \in \mathrm{Mat}_{k\times n}(\mathbb{C}) \mid \mathrm{rk}(M)=k\}}{\text{row operations}}.$$

- For a $k\times n$ matrix M with columns $M_1, M_2, \dots, M_n$, let $M_{j+n} := M_j$ for all $j\in\mathbb{Z}$.
- Let $f_M : [n]\to[n]$ be given by

$$f_M(i) \equiv \min\{j \geq i \mid M_i \in \mathrm{span}\,(M_{i+1}, M_{i+2}, \dots, M_j)\}$$

modulo n.

- f_M is a permutation which depends only on the row span of M.
- For a permutation $f\in S_n$, the *open positroid variety* $\Pi^\circ_f \subseteq \mathrm{Gr}(k,n)$ is defined by

$$\Pi^\circ_f := \{V \in \mathrm{Gr}(k,n) \mid f_V = f\}.$$

[KLS13] A. Knutson, T. Lam, and D. E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

- Let $f_{k,n} \in S_n$ be defined as

$$f_{k,n}(i) \equiv i+k \pmod n \quad \text{for all } i\in[n].$$

- Then $\Pi^\circ_{k,n} = \Pi^\circ_{f_{k,n}}$.

Question. $\mathcal{P}(\Pi^\circ_f;q,t) = ?$

Positroid links.

Definition. Permutation $f\in S_n \longrightarrow$ *link* $\hat\beta_f$:

- Connect $i \mapsto f(i)$ by a strand;
- If two strands cross, whichever one has smaller target goes above.

[FPST17] S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston. Morsifications and mutations. arXiv:1711.10598v3, 2017.

[STWZ19] V. Shende, D. Treumann, H. Williams, and E. Zaslow. Cluster varieties from Legendrian knots. *Duke Math. J.*, 168(15):2801–2871, 2019.

- The torus $T\subseteq \mathrm{SL}_n$ of diagonal matrices acts on Π°_f by rescaling columns.

Lemma. *The following are equivalent.*

▷ T*-action on* Π°_f *is free;*
▷ f *is a single cycle* ($c(f)=1$)
▷ $\hat\beta_f$ *is a knot.*

- $f_{k,n}$ is a single cycle iff $\gcd(k,n)=1$.
- If T acts freely, we have

$$\mathcal{P}(\Pi^\circ_f;q,t) = \left(q^{\frac12}+t^{\frac12}\right)^{n-1}\cdot\mathcal{P}(\Pi^\circ_f/T;q,t).$$

- Let $\mathcal{P}_{\mathrm{KR}}(\hat\beta_f;a,q,t)$ denote the triply-graded KR homology polynomial of $\hat\beta_f$.

Theorem ([GL20]). *Assume* $c(f)=1$. *Then*

$$\mathcal{P}(\Pi^\circ_f/T;q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat\beta_f;q,t),$$

where $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat\beta_f;q,t)$ *is the top a-degree term of* $\mathcal{P}_{\mathrm{KR}}(\hat\beta_f;a,q,t)$.

Most general result.

- *Richardson varieties*: $\Pi^\circ_f \cong R^\circ_{v,w}$, where $v\leq w\in S_n$ satisfy $f=wv^{-1}$, and

$$R^\circ_{v,w} := (BwB \cap B_-vB)/B \subseteq G/B.$$

- Richardson braid: $\beta_{v,w} := \beta(w)\cdot\beta(v)^{-1}$, where $\beta(w)$ denotes positive braid lift of $w\in S_n$.

Theorem ([GL20]). *For any* $v\leq w\in W$,

$$HHH^0(F^\bullet_{v,w}) \cong H^*_{T,c}(R^\circ_{v,w}),$$

where $F^\bullet_{v,w} = F^\bullet(w)\otimes F^\bullet(v)^{-1}$ *is the Rouquier complex of* $\beta_{v,w}$.

Proof sketch.

[skipped...]

$q = t = 1$ **specialization.**

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- Let $f \in S_n$ be a single cycle.
- $\Pi_f^\circ \subseteq \mathrm{Gr}(k(f), n(f))$, where

$$n(f) := n, \quad k(f) := \#\{i \in [n] \mid f(i) < i\};$$

$$\gamma(f) := (k(f), n(f) - k(f)).$$

- Extend f to a map $\tilde{f} : \mathbb{Z} \to \mathbb{Z}$ so that $i < \tilde{f}(i) < i + n$ and $\tilde{f}(i+n) = \tilde{f}(i) + n \; \forall i$.
- Let $\ell(\tilde{f})$ be the number of *inversions* of $\tilde{f}$:

$$\ell(\tilde{f}) := \#\{i < j \mid \tilde{f}(i) > \tilde{f}(j), i \in [n]\}.$$

- $\dim \Pi_f^\circ = k(n-k) - \ell(\tilde{f})$.
- Let $i < j$ be an inversion. Swapping $f(i)$ and $f(j)$ splits f into two cycles, $f_1^{(i,j)}$ and $f_2^{(i,j)}$.
- Let $\Gamma(f)$ be the multiset of points $\gamma(f_1^{(i,j)})$ for all inversions of $\tilde{f}$.

Definition. f is *repetition-free* if the multiset $\Gamma(f)$ has no repeated elements.

- Let $\hat{\Gamma}(f) := \Gamma(f) \sqcup \{(0,0), (k, n-k)\}$.
- We say that $\Gamma(f)$ is *convex* if $\hat{\Gamma}(f)$ contains all lattice points of its convex hull.

Theorem ([GL21])**.**

- *If f is repetition-free then $\Gamma(f)$ is convex and centrally symmetric.*
- *Conversely, any convex centrally symmetric subset of $[k-1] \times [n-k-1]$ arises this way.*
- *For f repetition-free, $\mathcal{P}(\Pi_f^\circ/T; q = t = 1)$ counts the number of Dyck paths avoiding $\Gamma(f)$.*

- Conjecturally, $\mathcal{P}(\Pi_f^\circ/T; q, t)$ for f repetition-free coincides with the generalized Catalan numbers of [GHSR20].

[GHSR20] E. Gorsky, G. Hawkes, A. Schilling, and J. Rainbolt. Generalized q,t-Catalan numbers. *Algebr. Comb.*, 3(4):855–886, 2020.

[BHMPS21] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. H. Seelinger. A Shuffle Theorem for Paths Under Any Line. arXiv:2102.07931, 2021.