Meeting 5/26

Wednesday, May 26, 2021 8:42 AM

<u>Plan:</u> ① Recap

② Bott-Samelson varieties

③ Braid varieties

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① <u>Recap</u> $R = \mathbb{C}[x_1, \dots x_n]$

$$B_i = R \otimes_{R^{(i\,i+1)}} R$$

$T_i = [B_i \to R]$ $\quad T_i^{-1} = [R \to B_i]$

For a braid β, we consider the tensor product of T_i, T_i^{-1} corresponding to the crossings.
r crossings

<u>Expand</u>: big complex with 2^r terms

$BS(\underline{i}) = B_{i_1} \otimes_R B_{i_2} \otimes_R \dots \otimes_R B_{i_s}$ $\quad s \le r$

Bott-Samelson bimodule

labeled by subwords of β

HOMFLY homology: HH(each term) then take homology.

<u>Facts</u> ① $HHH^{a=0}(\beta)$ is invariant under braid relations, conjugation and positive

braid relations, conjugation and <u>positive stabilization</u>

② $HHH^{a=n}(\beta) = HHH^{a=0}(\beta \cdot FT^{-1})$

top

full twist

(G., Hogancamp, Mellit, Nakagane).

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② <u>Bott-Samelson varieties</u>

$$\mathcal{F}\ell_n = \{ 0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathbb{C}^n \}$$

complete flag variety

$$H^*(\mathcal{F}\ell_n) = \frac{\mathbb{C}[x_1 \ldots x_n]}{(\mathbb{C}[x_1 \ldots x_n]^{S_n}_+)}$$

analogue of R

symmetric functions of positive degree

$x_i \sim c_1(\mathcal{L}_i)$

$\mathcal{L}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ natural line bundle

Relations: $\mathcal{F}_n / \mathcal{F}_0 = \mathbb{C}^n$ is filtered by $\mathcal{L}_i$

$\Rightarrow$ sym. fns in x_i compute $c_i(\mathbb{C}^n) = 0$.

$$\mathcal{B}_i = \{ \mathcal{F}, \mathcal{F}' \mid \mathcal{F}_j = \mathcal{F}'_j \text{ for } j \neq i \}$$

$$\begin{tikzcd} & \mathcal{B}_i \arrow[dl, "\mathbb{P}^1"'] \arrow[dr, "\mathbb{P}^1"] & \\ \mathcal{F}\ell & & \mathcal{F}\ell \end{tikzcd}$$

Given $\mathcal{F}$, choices for $\mathcal{F}'$ are

$\mathcal{F}l_n \swarrow \quad \searrow \mathcal{F}l_n$ given $\mathcal{F}$, choices for $\mathcal{F}'$ are parametrized by a line in $\mathcal{F}_{i+1}/\mathcal{F}_{i-1}$ = rank 2 bundle

$$H^*(\mathcal{B}_i) = \frac{\mathbb{C}[x_1 \ldots x_n, x_1' \ldots x_n']}{\left(\begin{array}{l} x_j = x_j' \quad j \neq i, i+1 \\ x_i + x_{i+1} = x_i' + x_{i+1}' \\ x_i x_{i+1} = x_i' x_{i+1}' \\ + \text{ sym. functions vanish} \end{array}\right)} \longleftrightarrow B_i$$

Bott-Samelson variety

$$BS(\underline{i}) = \mathcal{B}_{i_1} \times_{\mathcal{F}l} \mathcal{B}_{i_2} \times \cdots \times_{\mathcal{F}l} \mathcal{B}_{i_s}$$

$= \{(s+1)$-tuples of flags, neighbors $\sim \mathcal{B}_{i_k}\}$

Thm (Bott-Samelson, Soergel, ...) $H^*(\mathcal{BS}(\underline{i})) = \dfrac{BS(\underline{i})}{\mathbb{C}[x_1 \ldots x_n]^{S_n}_+}$

So BS bimodule essentially computes cohomology of BS variety.

What about $T_i = [\underbrace{B_i}_{\pi} \longrightarrow \underbrace{R}_{\pi}]$?

$$\mathcal{B}_i \hookleftarrow \Delta = \{\mathcal{F} = \mathcal{F}'\}$$

$$\widetilde{\mathcal{B}}_i = \{\mathcal{F}, \mathcal{F}' \mid \mathcal{F}_j = \mathcal{F}_j' \ \ j \neq i\} = \mathcal{B}_i \setminus \Delta$$

$\mathcal{F}_i \neq \mathcal{F}_i'$

$$B_i = \{(F, F') \mid F_j = F'_j,\ j \neq i\} = B_i \setminus \Delta$$

$$F_i \neq F'_i$$

<u>Open Bott-Samelson variety</u>

$$\begin{array}{ccc} & \widetilde{B}_{i_1} \times_{\mathcal{F}\ell} \widetilde{B}_{i_2} \times \cdots \times \widetilde{B}_{i_r} = \widetilde{B}(\beta) & \\ \swarrow & & \searrow \\ \mathcal{F}\ell & & \mathcal{F}\ell \end{array}$$

Here β = <u>positive</u> braid

<u>Thm</u> (Broué-Michel, <u>Deligne</u>, ...) $\approx$ Sur l'action de groupe de tresses sur une catégorie

The variety $\widetilde{B}(\beta)$ is invariant under braid moves. $s_i s_{i+1} s_i \longleftrightarrow s_{i+1} s_i s_{i+1}$

Isomorphisms are coherent ...

<u>Rmk</u> $BS(\underline{i})$ is <u>not</u> a braid invariant!

Braid closure $\simeq$ (first flag = last flag).

<u>Thm</u> (Webster-Williamson, ...)

$HHH^{a=n}(\beta)$ can be extracted from the equivariant cohomology of $\widetilde{B}(\beta)$ with weight filtration.

<u>Note</u>: "Inclusion-exclusion" formula for $\widetilde{B}(\beta)$:

<u>Note</u>: "Inclusion-exclusion" formula for $\widetilde{B}(\beta)$:

$\widetilde{B}_i \hookrightarrow B_i$, complement $= \Delta$

$\widetilde{B}(\beta) \hookrightarrow BS(\beta)$, complement $= \bigcup BS(\beta')$

$\beta' = \beta$ with one crossing removed

intersections $= BS(\beta'')$ — β with two crossings removed etc.

⇝ Giant diagram of spaces expressing $\widetilde{B}(\beta)$ in terms of $BS(\beta')$

$$\ldots BS(\beta'') \rightrightarrows BS(\beta') \rightrightarrows BS(\beta)$$

⇝ spectral sequence in cohomology collapses. ⇝ Rouquier complex.

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③ <u>Braid varieties</u> (Mellit, Casals-G.-Gorsky-Simental)

Explicit affine model for $\widetilde{B}(\beta)$

$\beta = \sigma_{i_1} \ldots \sigma_{i_r}$ <u>positive braid</u>

$$B_i(z) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & 1 & \\ & & & 1 & z & \\ & & & & & 1 \\ & & & & & & \ddots \end{pmatrix} \qquad B_\beta(z_1 \ldots z_r) = \\ = B_{i_1}(z_1) \cdot \ldots \cdot B_{i_r}(z_r)$$

Deligne's

$$\begin{pmatrix} \ddots & & & 1 & & \\ & & & \vdots & \ddots & \\ & & & i \; i{+}1 & & 1 \end{pmatrix}$$

Deligne's thm.

$$\boxed{B_i(z_1)\,B_{i+1}(z_2)\,B_i(z_3) = B_{i+1}(z_3)\,B_i(\underbrace{z_2 - z_1 z_3}_{})\,B_{i+1}(z_1)}$$

$\Rightarrow$ different braid words for the same braid $\rightsquigarrow$ same matrix up to change of variables

<u>Ex</u> : Δ = half twist

$$B_\Delta(z_1, \dots, z_{\binom{n}{2}}) = \begin{pmatrix} 0 & & & 1 \\ & & \cdot & z_1 \\ & \cdot & & \vdots \\ 1 & z & \cdots & z_{\binom{n}{2}} \end{pmatrix} = \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{pmatrix} \cdot \begin{pmatrix} \text{upper} \\ \text{triang} \end{pmatrix}$$

actual entries depend on a braid word, related by change of vars

We will consider braids $\gamma = \beta\Delta$ for some positive braid β

$$X(\gamma) = \{ B_\gamma(z_1, \dots z_{r+\binom{n}{2}}) \text{ is upper triangular} \}$$

$$\wr\!\wr$$

$$\left\{ B_\beta(z_1 \dots z_r) \cdot \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{pmatrix} \cdot (\text{upper triang}) \text{ is upper-triangular} \right\}$$

$$= \underbrace{\left\{ B_\beta(z_1 \dots z_r) \cdot \begin{pmatrix} 0 & & 1 \\ & \cdot & \\ 1 & & 0 \end{pmatrix} \text{ is upper-triangular} \right\}}_{X(B;w_0)} \times \mathbb{C}^{\binom{n}{2}}$$

$X(\beta; w_0)$ $\times \mathbb{C}^{\binom{n}{2}}$

<u>Idea</u>: If $\mathcal{F}, \mathcal{F}'$ are flags such that

$$\mathcal{F}_j = \mathcal{F}_j' \quad j \neq i \;, \quad \mathcal{F}_i \neq \mathcal{F}_i'$$

then $\mathcal{F}, \mathcal{F}'$ are related by $B_i(z)$ for some z.

$B_i(z)$ parametrizes Bruhat cell

$$B s_i B = \begin{pmatrix} \text{upper} \\ \text{triang} \end{pmatrix} \cdot B_i(z)$$

Can think of $B_i(z)$ as matrices relating neighboring flags.

<u>Key difference</u>: $X(\gamma)$ and $X(\beta; w_0)$ are by definition <u>affine</u> algebraic varieties, unlike $\mathcal{BS}(\underline{i})$, $\widetilde{\mathcal{B}}(\beta)$

<u>Thm</u> (CGGS) $X(\beta; w_0)$ is nonempty iff β contains w_0 as a subword. In this case it is smooth, and has a smooth compactification

(brick variety, L. Escobar <--> $BS(\underline{i})$)

(brick variety, L. Escobar $\longleftrightarrow BS(\underline{i})$)

Thm (folklore? Mellit?)

this is not a braid invariant

In fact, many different compact.

$$H^*_T(X(\beta,w_0)) = H^*_T(X(\beta\Delta)) =$$

$$= HHH^{a=n}(\beta\Delta) = HHH^{a=0}(\beta\Delta^{-1})$$

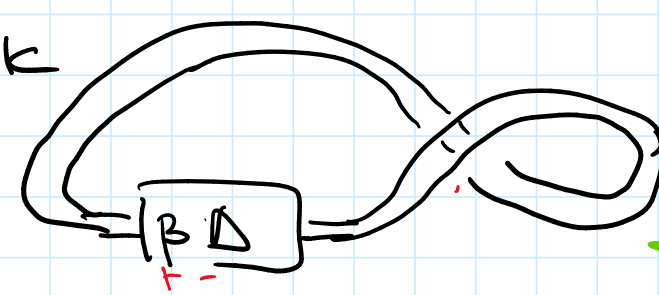
T-equiv. parameters $\longleftrightarrow x_i$

two grading = homological, weight filtration.

Cor $H^*_T(X(\beta;w_0))$ is invariant under braid moves, conjugation and positive stabilizations for $\beta\Delta^{-1}$

Thm (CGGS, to appear) The variety $X(\beta;w_0)$ is invariant under these moves

In fact, the variety $X(\beta';w_0)$ is an invariant of Legendrian link



Idea of proof: • Casals–Ng: this diagram can

Idea of proof: • Casals–Ng: this diagram can be realized by a Legendrian link in $\mathbb{R}^3$

• Chekanov defined a dga $\mathcal{A}_\gamma$ for any Legendrian link

generators = crossings

differential counts some disks...

• Moreover, using the same diagram we can define $\mathcal{A}_\gamma$ for non-positive braids which can be realized as Legendrian links (e.g. non-positive braids equivalent to positive)

Thm (CGGS) (a) $\mathrm{Spec}(H^0(\mathcal{A}_\gamma)) = X / \mathcal{V}$

commutative algebra

invariant under braid moves, conj, stab.

X = some affine alg. variety

$\mathcal{V}$ = collection of commuting vector fields on X (parametrized by negative crossings)

(b) If $\gamma = \beta\Delta$ β positive then crossing)

(b) If $\gamma = \beta \Delta$ β positive then

$$\operatorname{Spec}(H^0(\mathcal{A}_\gamma)) = X(\beta; w_0)$$ [Kálmán]

augmentation variety

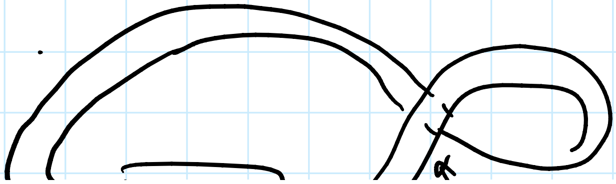
$\mathcal{A}_\gamma$ = "non-commutative" version of the ring of functions on $X(\beta; w_0)$

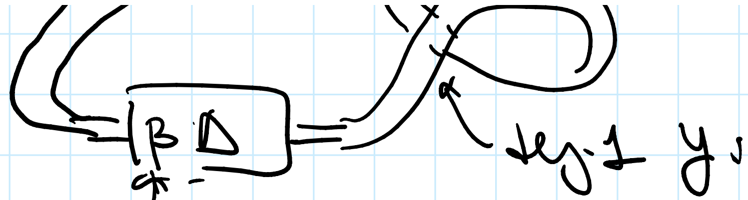
Q: Can we use $\mathcal{A}_\gamma$ to get a nice model for differential forms/alg de Rham complex on $X(\beta; w_0)$.

Q: Understand recursions for link homology geometrically?

Q: Noncomm/matrix-valued dga?

Q: Kitchloo: equivariant spectrum associated to an arbitrary braid



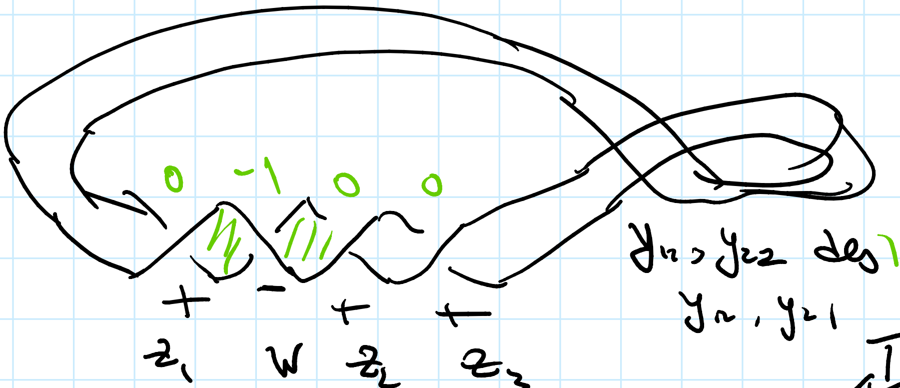


pos. crossing βΔ deg 0 z_i
neg crossing deg -1 w_i

$$\partial(y_i) = \underline{f_i(z_j)}$$

$$\partial(z_i) = \sum w_j \, g_{ij}(z)$$

$$X = \frac{\mathbb{C}[z_i]}{(f_i(z))} \qquad \sum g_{ij}(z)\frac{\partial}{\partial z_j} = \text{derivation}$$



$$\partial(y_{ij}) = \left[\underline{B(z_1)B(0)B(z_2)B(z_3)}\; I\right]_{ij}$$

$$B(z) = \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$$

$$\underbrace{\partial(z_1) = w \qquad \partial(z_2) = -w}$$

$$\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} = \text{vector field}$$

$\partial z_1 \quad - \quad \partial z_2$ = vector field,

$$\partial(\Phi(z_1, z_2, z_3)) = \frac{\partial \Phi}{\partial z_1} - \frac{\partial \Phi}{\partial z_2}$$

$$(z_1, z_2, z_3) \to (z_1 + t, z_2 - t, z_3)$$