Affine Springer fibers & knot homology

§1. Affine Springer fibers (+ generalizations)

§2. Torus links

§3. ?

1.1. $G = GL_n/\mathbb{C}$ (, SL_n)

$K = \mathbb{C}((t))$

$\mathcal{O} = \mathbb{C}[[t]]$

$$Gr_G = G(K)/G(\mathcal{O})$$ "ind-scheme $/\mathbb{C}$"

$$\widetilde{Fl} = G(K)/I$$ $I = ev_0^{-1}(B)$

$B \subset G$

$$G(\mathcal{O}) \xrightarrow{t \mapsto 0} G(\mathbb{C})$$

$$Gr_G = \{ g G(\mathcal{O}) \}$$

$$= \{ \Lambda \subset K^n \mid \Lambda \otimes_{\mathcal{O}} K = K^n \}$$

(b/c $\Lambda = \mathcal{O}^n$ + orbit-stabilizer)

$$Fl_G = \{ \Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_n \subset t^{-1}\Lambda_1 \}$$

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & \mathcal{O} & \mathcal{O} \\ t\mathcal{O} & t\mathcal{O} & \mathcal{O} \end{pmatrix} \quad \text{stabilizes}$$

$$\mathrm{span}_{\mathcal{O}}(e_1, e_2, \dots, e_n)$$
$$\subset \mathrm{span}_{\mathcal{O}}(t^{-1}e_1, e_2, \dots, e_n)$$
$$\subset \cdots \subset \mathrm{span}_{\mathcal{O}}(t^{-1}e_1, \dots, t^{-1}e_{n-1}, e_n)$$

In finite type, Springer fibers
= subsets of G/B consisting of Borel subalgebras
containing fixed $x \in \mathcal{N}ilp \subset \mathrm{Lie}(G)$

1.2. Affine Springer fibers

$\gamma \in \mathrm{Lie}(G)(K)$

<u>Def.</u> $$Sp_\gamma = \{ gG(\mathcal{O}) \mid g^{-1}\gamma g \in \mathrm{Lie}(G)(\mathcal{O}) \} \subset Gr_G$$
$$\| $$
$$\{ \Lambda \subset K^n \mid \gamma\Lambda \subset \Lambda \}$$

$$\widetilde{Sp}_\gamma = \{ gI \mid g^{-1}\gamma g \in \mathrm{Lie}(I) \}$$
$$= \{ \vec{\Lambda} \mid \gamma\Lambda_i \subset \Lambda_i \}$$

Examples $\gamma = 0$, $Sp_0 = Gr_G$

$\widetilde{Sp}_0 = Fl_G$

γ nilpotent, Thm (Sommers)

$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $H_*(Sp_\gamma) \cong H_*(Gr_G)$

$\gamma = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \in gl_2(K) \leftrightarrow \mathrm{Link} = \bigcirc\bigcirc$

$Sp_\gamma \ni t^i\mathcal{O} \oplus t^j\mathcal{O} \quad i,j \in \mathbb{Z}$

?

geom. only depends on conjugacy class

$\mathrm{Stab}_{G(K)}(\gamma) = T(K) \circlearrowright Sp_\gamma$

transitive, factors through $T(K)/T(\mathcal{O}) \cong \mathbb{Z}^2$

$\cong Sp_\gamma$

$\gamma = \begin{pmatrix} t & \\ & -t \end{pmatrix}$

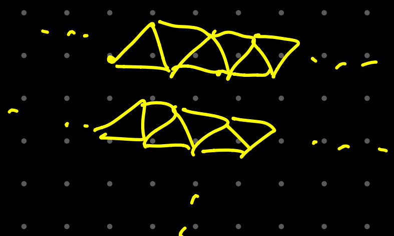
$\mathrm{Link}_0(x^2 - t^2)$

$=$ 

Sp_γ

$\mathbb{Z}^2 \cong X_*(T) \in T(K) = \mathrm{Stab}(\gamma)$

$\gamma = \begin{pmatrix} t^2 & \\ & -t^2 \end{pmatrix}$



Sp_γ

$$\gamma = \begin{pmatrix} & t^3 \\ 1 & \end{pmatrix}$$

$$\mathrm{Hom}_k(\mathbb{G}_m, \mathrm{Stab}(\gamma)) = \{1\}$$

<u>Rmk.</u> $\mathrm{Stab}(\gamma)$ is max. torus for

$$\gamma \in \mathfrak{g}(\mathcal{K})^{r.s.}$$

conjugacy classes of max tori in $G(\mathcal{K})$

$$\longleftrightarrow W/\sim$$

(For $G = GL_n$ $\longleftrightarrow$ splitting type of

$$\det(xI - \gamma)\,)$$

e.g. $x^2 - t^3$ doesn't split $/\mathcal{K}$

$$x^2 - t^{2k} = (x - t^k)(x + t^k)$$

ok to take $\gamma \in G(\mathbb{C}[t^{\pm}])$

$$\begin{pmatrix} s_1 t^{k_1} & & \\ & \ddots & \\ & & s_n t^{k_n} \end{pmatrix}$$

L = link w/ unknot components
linking numbers
$\min(k_i, k_j) = \mathrm{Val}(s_i t^{k_i} - s_j t^{k_j})$

GL_3, GL_4
Lucarelli
Z. Chen

2. (positive) Torus links

$m, n \in \mathbb{Z}_{\geq 0}$

$d = \gcd(m,n)$

If $d = 1$, $\gamma = \overbrace{\begin{pmatrix} & & & t \\ 1 & & & \\ & 1 & & \\ & & \ddots & 1 \end{pmatrix}}^{n}{}^{m}$

in general,
$$\gamma = \begin{pmatrix} s_1\gamma_1 & & \\ & \ddots & \\ & & s_d\gamma_d \end{pmatrix} \quad \text{〃 } \gamma_i \text{ for } \left(\frac{m}{d}, \frac{n}{d}\right)$$

$\mathrm{Stab}(\gamma) \longleftrightarrow$ rectangular partition of n.

E.g. $m = n$ $\begin{pmatrix} s_1 t & & \\ & \ddots & \\ & & s_d t \end{pmatrix}$ $s_i \neq s_j$

Rmk

$\begin{pmatrix} & & t^m \\ 1 & & \\ & \ddots & 1 \end{pmatrix}$ is also OK, for all (m,n)

If $d = 1$, Sp_γ studied by Lusztig-Smelt, Hikita, ...

$$H_*(Sp_\gamma) \overset{ORS}{\longleftrightarrow} HHH(L)$$
$$L = \mathrm{link}(\mathrm{char}(\gamma) = 0)$$

If $m = n+1$, diagonal coinvariants

$$G(K) = \bigsqcup_{w \in W_{aff}} IwI$$

$$IwI/I \cap Sp_\gamma$$

2.2. Full twist.

$$\gamma = \begin{pmatrix} s_1 t^d & & \\ & \ddots & \\ & & s_n t^d \end{pmatrix}$$

$$Sp_\gamma^T = X_*(T) = Gr_G^T$$

<u>Lemma</u> $\bigcup$ r-dim T-orbits on Gr_G

$$= \bigcup_{\substack{M \in \text{Levi subgps} \\ \text{containing } T, \\ \text{semisimple rank } r}} M(K)/M(\mathcal{O})$$

(union of 1d orbits in Sp_γ

$$\longleftrightarrow \quad G_\alpha(K)/G_\alpha(\mathcal{O}) \cap Sp_\gamma$$

ss. rk 1 sg gen. by $U_{\pm\alpha}$, T

Thm Goresky - Kottwitz - Macpherson, ..., Brion

$X/\mathbb{C} \circlearrowleft T$-equivariantly formal

$$H_*^T(X^T) \xrightarrow{\iota_*} H_*^T(X)$$

Borel-Moore

invertible after bc. at countably many characters of T

+ can describe image of ι_*^{-1} using 0d + 1d orbits

many steps

$$\bigcap_\alpha H_*^T(Sp_\gamma^\alpha) = H_*^T(Sp_\gamma)$$

$$\subset H_*^T(X_*(T)) \otimes \mathbb{C}(\mathfrak{t})$$

$$\mathbb{C}[T^* T^\vee]$$

Thm (K.)

$$\left(\prod_{\alpha} \alpha\right)^d H_*^T(Sp_\gamma) \cong \bigcap_{\alpha} \langle y_\alpha, 1 - x_\alpha^\vee \rangle^d$$

$$\subset \mathbb{C}[T^\wedge T^\vee]$$

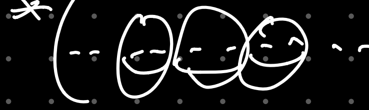
In $G = GL_n$ case

this is $\displaystyle\bigcap_{i<j} \langle x_i - x_j, y_i - y_j \rangle^d$

$$\subset \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}, y_1, \dots, y_n]$$

$$H^*(\widetilde{\mathcal{F}\ell}) \twoheadrightarrow H^*(\widetilde{Sp}_{\frac{n+1}{n}})$$

$$H^*(\mathcal{F}\ell) \twoheadrightarrow H^*(Sp_{\frac{n}{n}})^\wedge$$

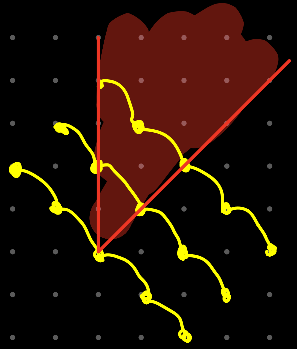
$H^*($  $)$ $\cong \dfrac{k[[\dots, x_i, x_{i+1}, \dots]]}{x_i x_j = 0}$

Andersen-Tubbenhauer

$\cong$

$$Z(T; H(U_q(sl_2)))$$

$\mathrm{Sp}_{\binom{t}{-t}}$



$\mathrm{Hilb}^\bullet(\{x^2 = t^2\})$

$\{\Lambda \subset \mathcal{O}^n \subset K^n\}$

$$\underbrace{H^T_*\left(\mathrm{Sp}_{\begin{pmatrix} t & & \\ & \xi t & \\ & & \xi^2 t\end{pmatrix}} \cap \mathrm{Gr}^t_G\right)}_{?} \cong \bigcap_{i<j} \langle x_i - x_j,\ y_i - y_j \rangle \subset \mathbb{C}[x_1, \dots, y_n]$$

$$\overset{?}{\cong} \mathrm{Hilb}^\bullet(\{x^3 = t^3\})$$

$$\cong \mathrm{Sp}_{\gamma\begin{pmatrix} & t^3 \\ 1 & \end{pmatrix}} \cap \mathrm{Gr}^t_G$$

(follows from Garner-K.)

replace by other C

Y-ified ORS:

$$H^T_*(\mathrm{Hilb}^\bullet(\overbrace{\{x^n = t^n\}})) \cong HY^{a=0}(\mathrm{Link}_0(\{x^n = t^n\}))$$

If C has d components, Sp_γ has an action of a d-torus A,

$$H^A_*(\mathrm{Hilb}^\bullet(C)) \cong HY^{a=0}(\mathrm{Link}_0(C))$$

$$\bigsqcup \mathrm{Hilb}^n =: \mathrm{Hilb}^\bullet$$

$$\bigcap_{i<j} \langle x_i - x_j, y_i - y_j \rangle$$

γ

$$\begin{pmatrix} s_1\gamma_1 & & \\ & \ddots & \\ & & s_d\gamma_d \end{pmatrix} \circlearrowleft A \hookrightarrow T$$

d-torus

$$\frac{\mathfrak{gl}_n(K)}{GL_n(K)} \leftrightarrow \text{char. poly}$$

γ compact if $\widetilde{Sp}_{\gamma} \neq \emptyset$

topological Jordan decomp.

$$Sp_{\gamma_{tn}} \cap Sp_{\gamma_{ss}}$$

top. nilp.

semisimple over $\mathbb{C}$

$\dim_{\mathbb{C}} Sp_{\gamma} < \infty$ iff γ r.s.

$$(x^2-y^3)^2$$

$$\begin{pmatrix} 0 & t^3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$x^4-y^6$$

$$\begin{pmatrix} 0 & -t^3 & & \\ -1 & 0 & & \\ & & 0 & t^3 \\ & & 1 & 0 \end{pmatrix}$$

Pf of Sommers' thm from beginning

γ nilpotent is conjugate to

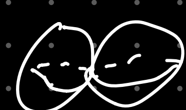
$t^d\gamma$

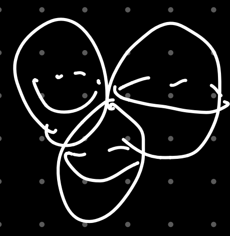
$$Sp_{t^d\gamma} \xrightarrow{d\to\infty} Sp_0 = Gr_G$$

$\mathrm{gr}^P H_*(\widetilde{Sp}_\gamma)$

bigraded S_n-rep

$\rightsquigarrow \mathrm{Sym}[X](q,t)$

$T(2,3)$ 

$T(3,2)$ 

$$\begin{tikzcd} \widetilde{S}_{p_\gamma} \arrow[r] \arrow[d] & {[\tilde{\mathfrak{g}}/G]} \arrow[d] & \widetilde{Fl} \arrow[l] \arrow[d] \\ S_{p_\gamma} \arrow[r] & {[\mathfrak{g}/G]} & Gr_G \arrow[l] \end{tikzcd}$$

$$H_*(\widetilde{S}_{p_\gamma})^{S_n} \cong H_*(S_{p_\gamma})$$