Lectures in Applied Mathematics
Volume 28, (1991)

The Random Vortex Method with Vorticity Creation: Introduction and Guide to Parameter Selection

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Abstract. The random vortex method is a numerical method for approximating solutions of the incompressible Navier-Stokes equations. The method is a grid free particle method and hence does not introduce the types of diffusive errors found in finite difference schemes. This facilitates relatively inexpensive computations at large Reynolds numbers. In the random vortex method the no-slip boundary condition is satisfied by creating particles on the boundary. There are several particle creation algorithms currently in use. The most common of these is the vortex sheet method. The purpose of this paper is to provide the reader with a detailed introduction to the random vortex method and the vortex sheet method.

1. Introduction. The random vortex method is a numerical method for approximating solutions of the incompressible Navier-Stokes equations. Let Ω denote a domain containing a viscous, incompressible fluid. In general, Ω will have a solid boundary $\partial\Omega$ upon which boundary conditions must be satisfied. For example Ω might be the inside of a pipe, or the region surrounding an airplane. In the random vortex method the computational domain is divided into two regions: an interior (or exterior) region Ω_{NS} located away from $\partial\Omega$ and a *sheet layer* Ω_{Pr} located adjacent to $\partial\Omega$. In Ω_{NS} we use the random vortex method [Chorin (1973)] to approximate solutions of the incompressible Navier-Stokes equations, while in Ω_{Pr} we use the vortex sheet method [Chorin (1978)] to approximate solutions of the Prandtl boundary layer equations.

1991 *Mathematics Subject Classification*: Primary 65C20; Secondary 60J15, 76C05, 76D05.

Work performed under the auspices of the U.S. Department of Energy at the Lawrence Livermore National Laboratory under contract number W-7405-ENG-48 and partially supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research under contract number W-7405-Eng-48

This paper is in final form and no version of it will be submitted for publication elsewhere.

0075-8485/91 $1.00 + $.25 per page

We use the term sheet layer to distinguish the *computational* boundary layer from the *physical* boundary layer. While the justification for dividing the computational domain into the two regions Ω_{NS} and Ω_{Pr} is soundly based on the theory of boundary layers [Schlichting], it is sometimes the case that the sheet method is used in regions where the underlying assumptions implicit in the use of the Prandtl equations are in doubt.

Both the random vortex method and the vortex sheet method are particle methods. The particles carry concentrations of vorticity; the velocity field within each of the regions is uniquely determined by the particle positions and their concentrations together with the appropriate boundary conditions. Both methods are fractional step methods. One of the fractional steps consists of evolving the particles and their concentrations in this velocity field. The other step consists of letting the particle positions undergo a random walk to account for the diffusive effects of viscosity.

In Ω_{NS} the particles are called *vortices* or *vortex blobs*. In Ω_{Pr} they are called *vortex sheets*. The no-flow boundary condition is satisfied on $\partial\Omega$ by imposing a potential flow on Ω_{NS} which cancels the normal component of the velocity due to the blobs. The no-slip boundary condition is satisfied by creating vortex sheets on $\partial\Omega$ which subsequently participate in the flow. The two solutions are matched by converting sheets that leave the sheet layer into blobs with the same circulation, converting blobs that enter the sheet layer into sheets with the same circulation, and letting the velocity at infinity in the Prandtl equations be the tangential component of the velocity on $\partial\Omega$ due to the vortices in Ω_{NS}. The sheet creation process and subsequent movement of the sheets into the interior of the flow mimics the physical process of vorticity creation at a boundary and is one of the attractive features of this numerical method.

2. The Navier-Stokes Equations. Let Ω be some domain in R^N where $N = 2$ or 3. We wish to approximate solutions of the incompressible Navier-Stokes equations,

$$\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} = -\nabla p + \nu \Delta \underline{u} \quad , \tag{2.1a}$$

$$\nabla \cdot \underline{u} = 0 \quad , \tag{2.1b}$$

$$\underline{u} = \underline{0} \quad on \quad \partial\Omega \quad , \tag{2.1c}$$

in Ω where $\underline{u}$ is the velocity, p is the pressure, and ν is the kinematic viscosity. We let $\underline{x} = (x, y)$, $\underline{u} = (u, v)$ for $N = 2$ and $\underline{x} = (x, y, z)$, $\underline{u} = (u, v, w)$ for $N = 3$. Note that equation (2.1a) is a vector equation with N components.

The vorticity is defined by

$$\underline{\omega} = \nabla \times \underline{u} \quad . \tag{2.2}$$

In two dimensions the vorticity vector points in the direction perpendicular to the (x, y) plane and hence is usually considered a scalar, $\omega = \partial_x v - \partial_y u$.

Provided that Ω is a simply connected domain, equations (2.1b) and (2.2) imply the existence of some $\underline{\psi}$ such that

$$\underline{u} = \nabla \times \underline{\psi} \quad , \tag{2.3}$$

$$\Delta \underline{\psi} = -\underline{\omega} \quad , \tag{2.4}$$

(see [Marsden and Tromba]). As with the vorticity, in two dimensions $\underline{\psi}$ points in the direction perpendicular to the (x, y) plane and hence is generally thought of as a scalar, ψ.

Let G denote the fundamental solution of the Laplace operator in R^N. Then we have

$$\underline{\psi}(\underline{x}, t) = (G * \underline{\omega})(\underline{x}, t) = \int G(\underline{x} - \underline{x}') \quad \underline{\omega}(\underline{x}', t) d\underline{x}'. \tag{2.5}$$

Let $\underline{K} \equiv \nabla \times G$. Since $\partial_x (G * w) = (\partial_x G) * w$ and similarly for $\partial_y (G * w)$ and $\partial_z (G * w)$ we find from (2.3) and (2.5) that

$$\underline{u}(\underline{x}, t) = (\underline{K} * \omega)(\underline{x}, t) = \int \underline{K}(\underline{x} - \underline{x}') \quad \underline{\omega}(\underline{x}', t) d\underline{x}' \quad . \tag{2.6}$$

Note that in two dimensions $\underline{K}$ is a vector,

$$\underline{K}(\underline{x}) = \frac{1}{2\pi |\underline{x}|^2} (-y, x)$$

which has a singularity of order $O(|\underline{x}|^{-1})$ at $\underline{x} = \underline{0}$. In three dimensions $\underline{K}$ is a matrix,

$$\underline{K}(\underline{x}) = \frac{1}{4\pi |\underline{x}|^3} \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

with a singularity of order $O(|\underline{x}|^{-2})$ at $\underline{x} = \underline{0}$. The singularity in $\underline{K}$ has played a central role in the mathematical theory of vortex methods.

We now take the curl of (2.1a) to obtain the Navier-Stokes equations in vorticity form,

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} \cdot \nabla)\underline{u} + \nu \Delta \underline{\omega} \tag{2.7a}$$

$$\nabla \cdot \underline{u} = 0 \tag{2.7b}$$

$$\underline{u} = \underline{K} * \underline{\omega} \tag{2.7c}$$

$$\underline{u} = \underline{0} \quad on \quad \partial\Omega \tag{2.7d}$$

where $D/Dt \equiv \partial_t + (u \cdot \nabla)$ is the *material derivative*. Note that by taking the curl of (2.1a) we have eliminated the pressure from the set of equations to be solved. This greatly simplifies the task of finding approximations to $\underline{u}$. Furthermore, by (2.7c), we see that the velocity depends linearly on the vorticity. The vortex method takes advantage of this fact by discretizing the vorticity field as a linear superposition of patches of vorticity. (Usually, but not always, these patches have compact support.) Each vortex patch induces a velocity field via (2.7c) which can easily be computed, and the total velocity field is simply the sum of these velocities plus a correction for boundary conditions.

Solutions of (2.7a-d) may be approximated by solving two distinct systems of partial differential equations; using the solution of one of these systems as the initial data for the other. The first of these systems is the incompressible Euler equations,

$$\frac{D\underline{\omega}}{Dt} = (\underline{\omega} \cdot \nabla)\underline{u} \quad , \tag{2.8a}$$

$$\nabla \cdot \underline{u} = 0 \quad , \tag{2.8b}$$

$$\underline{u} = \underline{K} * \underline{\omega} \quad , \tag{2.8c}$$

$$\underline{u} \cdot \underline{\eta} = \underline{0} \quad \text{on} \quad \partial\Omega \quad , \tag{2.8d}$$

where $\underline{\eta}$ is a unit vector normal to $\partial\Omega$. Equations (2.8a-d) describe the flow of an incompressible, inviscid fluid in Ω. The boundary condition (2.8d) is commonly referred to as the *no-flow boundary condition.*

The second system of partial differential equations is the heat equation,

$$\underline{\omega}_t = \nu \Delta \underline{\omega} \quad , \tag{2.9a}$$

with the boundary condition $\underline{u} = \underline{0}$ on $\partial\Omega$. However, the no-flow portion of this boundary condition is usually satisfied by the addition of a potential flow after both fractional steps. Hence, the boundary condition that must to be satisfied during the second fractional step is

$$(\underline{u} \cdot \underline{\tau}) = \underline{0} \quad \text{on} \quad \partial\Omega \quad , \tag{2.9b}$$

where $\underline{\tau}$ is tangent to $\partial\Omega$. Equation (2.9a) describes the rate at which the vorticity is diffusing in the flow. Equation (2.9b) is commonly referred to as the *no-slip boundary condition.*

The idea of solving Euler equations for a small time step Δt and then the heat equation for the same time step and using the result as an approximation to the solution of the Navier-Stokes equations at time Δt is sometimes called *viscous splitting.* This is a special case of a technique known as *operator splitting.* Viscous splitting for the Navier-Stokes equations in the absence of boundaries has been given a rigorous justification by [Beale and Majda (1981)]. The problem is much more difficult when boundaries are

included. Work on this problem appears in [Benfatto and Pulverenti] and [Alessandrini, Douglis, and Fabes].

Numerical methods based on operator splitting are often called *fractional step* methods. Such methods are applicable to a wide variety of partial differential equations. The random vortex method is a fractional step method. The first step consists of a vortex method approximation to the solution of the Euler equations (2.8a-d); the second step is a random walk approximation to the solution of the heat equation (2.9a) and the no-slip boundary condition (2.9b).

In the next section we describe the vortex method for approximating solutions of the incompressible Euler equations when $\Omega = R^N$. This allows us to ignore the issue of satisfying the boundary condition (2.8d). We then discuss the modifications necessary for satisfying this boundary condition. Following this we then introduce the random walk approximation of solutions of the heat equation (2.9a). If Ω contains solid boundaries then this must include a technique for approximately satisfying the no-slip boundary condition (2.9b) on $\partial\Omega$ and in this context we introduce the vortex sheet method.

3. The Vortex Method. The vortex method is a particle method in which fluid particles carrying concentrations of vorticity are followed as their positions and concentrations evolve with the motion of the fluid. Fundamental to all particle methods is the notion of the fluid flow map $\underline{x} : R^N \times [0,T] \to R^N$ defined so that the $\underline{x}(\underline{\alpha},t)$ is the trajectory of the fluid particle which at time $t = 0$ is at the point $\underline{\alpha}$. For fixed $\underline{\alpha}$ this trajectory may be found by solving the ordinary differential equation,

$$\frac{d\underline{x}}{dt}(\underline{\alpha},t) = \underline{u}(\underline{x}(\underline{\alpha},t),t), \qquad \underline{x}(\underline{\alpha},t) = \underline{\alpha}. \tag{3.1}$$

Perhaps the simplest way to explain the vortex method is to begin with the manner in which the vorticity is discretized. As a consequence of the scalar character of the vorticity in two dimensions the two dimensional version of the vortex method is much simpler than its three dimensional counterpart. Therefore we begin by describing the vortex method in R^2.

Two Dimensions. In two dimensions the fluid velocity $\underline{u}$ is perpendicular to the vorticity and hence the right hand side of (2.8a) is zero, $(\underline{\omega}\cdot\nabla)\underline{u} = 0$. Thus (2.8a) becomes

$$\frac{D\underline{\omega}}{Dt} = 0 \quad . \tag{3.2}$$

Equation (3.2) states that the vorticity is passively convected by the flow; i.e. that the vorticity is constant along particle trajectories,

$$\frac{d}{dt}\omega(\underline{x}(\underline{\alpha},t),t) = 0.$$

This is probably the most significant difference between the fluid flow equations in two and three dimensions. In general, when $N = 3$ we have

$(\underline{\omega} \cdot \nabla)\underline{u} \neq 0$. This provides a mechanism for the vorticity to change as it travels in the flow. In particular, the vorticity may now stretch and fold. It is generally believed that this is the mechanism by which turbulence is created and driven [Chorin (1982)]. This explains why the phrase 'two dimensional turbulence' is often considered an oxymoron.

Let $f_\delta : R^2 \to R$ be some scalar function on R^2 and let

$$\tilde{\omega}(\underline{x}, t) = \sum_j f_\delta(\underline{x} - \tilde{\underline{x}}_j(t))\Gamma_j \tag{3.3}$$

be an approximation to the vorticity at some time $t = k\Delta t$. The jth term on the right hand side of (3.3) is referred to as the jth vortex or jth vortex blob, $\tilde{\underline{x}}_j(t)$ is its position at time t and Γ_j its *strength*. For times $t = k\Delta t$ we will often write $\tilde{\underline{x}}_j^k$ for $\tilde{\underline{x}}_j(t)$ and similarly for other computed quantities that depend on time such as the velocity $\tilde{\underline{u}}^k$ and the vorticity $\tilde{\underline{\omega}}^k$. Given the approximate vorticity (3.3) we can use (2.6) to obtain an approximation $\tilde{\underline{u}}$ to the velocity $\underline{u}$,

$$\tilde{\underline{u}}(\underline{x}, t) = \sum_j (\underline{K} * f_\delta)(\underline{x} - \tilde{\underline{x}}_j(t))\Gamma_j = \sum_j \underline{K}_\delta(\underline{x} - \tilde{\underline{x}}_j(t))\Gamma_j \quad , \tag{3.4}$$

where $\underline{K}_\delta \equiv \underline{K} * f_\delta$. Note that the smoothness of $\underline{K}_\delta$ depends on f_δ. In particular, for appropriately chosen f_δ the kernel $\underline{K}_\delta$ will be continuous or even differentiable at $\underline{x} = \underline{0}$ in contrast to $\underline{K}$. For this reason f_δ is known as the *smoothing function.*

If we let f_δ be the Dirac delta function, then the approximate vorticity (3.3) becomes a collection of point masses and the approximate velocity becomes

$$\tilde{\underline{u}}(\underline{x}) = \sum_j \underline{K}(\underline{x} - \underline{x}_j)\Gamma_j \quad .$$

This is the classical *point vortex method* originally considered by [Rosenhead]. The singularity in $\underline{K}$ gives rise to an arbitrarily large velocity when one attempts to evaluate it near a vortex. For this reason it was generally believed that this method was unstable and would not converge to solutions of the incompressible Euler equations. However, in recent work [Goodman, Hou, and Lowengrub] and [Hou and Lowengrub] have shown that the point vortex method is indeed convergent, given the appropriate assumptions.

In [Chorin (1973)] Chorin originally suggested using

$$f_\delta(\underline{x}) = \begin{cases} (2\pi|\underline{x}|\delta)^{-1} & |\underline{x}| < \delta, \\ 0 & |\underline{x}| \geq \delta. \end{cases} \tag{3.5}$$

This yields a velocity of the form (3.4) with

$$\underline{K}_\delta(\underline{x}) = \begin{cases} (-y, x)/2\pi|\underline{x}|\delta & |\underline{x}| < \delta, \\ (-y, x)/2\pi|\underline{x}|^2 & |\underline{x}| \geq \delta. \end{cases} \tag{3.6}$$

This has perhaps been the most widely used velocity kernel in two dimensional implementations of the vortex method.

In general one may choose

$$f_\delta(\underline{x}) = \frac{1}{\delta^N} f(\underline{x}/\delta) \tag{3.7}$$

for some function f. Often f is chosen to be radially symmetric and of compact support. Then the parameter δ determines the support of f_δ. For this reason δ is frequently referred to as the *cutoff radius*. Generally, assuming the exact solution $\underline{u}$ of (2.7a-c) is sufficiently smooth, then the properties of f will determine the accuracy of the vortex method approximation to $\underline{u}$. There is an extensive theory concerning the choice of f and its effect on the convergence of $\tilde{\underline{u}}$ to $\underline{u}$ [Anderson and Greengard] [Beale and Majda (1982a, 1982b, 1985)] [Hald (1979, 1987)] [Hald and del Prete (1978)] [Perlman].

We evolve the computed particle positions $\tilde{\underline{x}}_j(t)$ in time according to (3.1) with $\underline{u}$ replaced by $\tilde{\underline{u}}$ from (3.4). Thus the new particle positions at time $t = (k+1)\Delta t$ are given by

$$\tilde{\underline{x}}_j^{k+1} = \tilde{\underline{x}}_j^k + \Delta t\, \tilde{\underline{u}}(\tilde{\underline{x}}_j^k, k\Delta t) \quad , \tag{3.8}$$

and the vorticity at time $t = (k+1)\Delta t$ is now

$$\tilde{\omega}^{k+1}(\underline{x}) = \sum_j f_\delta(\underline{x} - \tilde{\underline{x}}_j^{k+1})\Gamma_j \quad .$$

This is simply a first order Euler's method solution of (3.1). Higher order time discretizations are possible and have been studied in some detail by [Anderson and Greengard] and [Hald (1987)].

Note that since the vortex strengths Γ_j have not changed we are approximately satisfying (3.2). Since $\tilde{\underline{u}}^{k+1} = \underline{K} * \tilde{\omega}^{k+1}$ explicitly satisfies (2.8b) the new velocity field $\tilde{\underline{u}}^{k+1}$ is a consistent approximation to the exact solution of (2.8a-c) at time $t = (k+1)\Delta t$ with initial data $\tilde{\omega}^k$ at time $t = k\Delta t$.

We have neglected two issues in our discussion of the vortex method in two dimensions. The first issue concerns the question of how to choose the initial particle positions and strengths. We will discuss this at the end of Section 3 below. The other issue is the problem of satisfying the no-flow boundary condition (2.8d), which we discuss in Section 4 immediately afterward. We will now describe the vortex method for approximating solutions of (2.8a-c) in three dimensions.

Three Dimensions. As pointed out previously, in R^3 the vorticity is a vector that may point in any direction. Hence the right hand side of (2.8a) is, in general, nonzero, $(\underline{\omega}\cdot\nabla)\underline{u} \neq 0$. One can think of (2.8a) as an evolution equation for the vorticity in which the vorticity is transported along particle paths by the flow,

$$\frac{D\underline{\omega}}{Dt} = 0 \quad , \tag{3.9a}$$

and then 'stretched' by the flow,

$$\underline{\omega}_t = (\underline{\omega}\cdot\nabla)\underline{u} \quad . \tag{3.9b}$$

Once again we discretize the vorticity as in (3.3), but now the strengths, $\underline{\Gamma}_j(t)$, are vector quantities which depend on the time t. So

$$\underline{\tilde{\omega}}(\underline{x},t) = \sum_j f_\delta(\underline{x} - \underline{\tilde{x}}_j(t))\underline{\Gamma}_j(t)\,, \tag{3.10}$$

where f_δ is again given by (3.7) but with $N = 3$.

As with the two dimensional vortex method we approximate the solution of (3.9a) with initial data (3.10) by advancing the particle positions according to (3.1). And once again the new particle positions are given by (3.8) but with

$$\underline{\tilde{u}}(\underline{x}, k\Delta t) = \sum_j \underline{K}_\delta(\underline{x} - \underline{\tilde{x}}_j^k)\underline{\Gamma}_j^k \quad ,$$

where $\underline{\Gamma}_j^k \equiv \underline{\Gamma}_j(k\Delta t)$. We must also update the vortex strengths $\underline{\Gamma}_j^k$ according to (3.9b). To this end we discretize the right hand side of (3.9b) by writing

$$\underline{\Gamma}_j^{k+1} = \underline{\Gamma}_j^k + \Delta t\ (\underline{\tilde{\omega}}^k(\underline{x}_j^k)\cdot\nabla_x)\ \underline{\tilde{u}}^k(\underline{x}_j^k)\,. \tag{3.11}$$

For many cutoff functions f_δ one can can explicitly write down $\underline{K}_\delta$ and hence its derivatives. Thus, by using

$$\nabla_x \underline{\tilde{u}}^k(\underline{x}) = \sum_j \nabla_x \underline{K}_\delta(\underline{x} - \underline{\tilde{x}}_j^k)\underline{\Gamma}_j(t) \quad .$$

we can explicitly evaluate the right hand side of (3.11).

This is only one of several methods that have been suggested for updating the vortex strengths in three dimensions. In fact, many of these methods have been shown to converge to the exact solution of (2.8a-c) for sufficiently short times. We refer the reader to the literature for further details ([Anderson and Greengard], [Beale and Majda (1982a, 1982b)], [Fishelov], [Greengard], and [Knio and Ghoniem]).

We should also mention that there is a an alternative approach, originally due to Chorin [Chorin (1980,1982)], which is sometimes called the *vortex filament method.* This method essentially discretizes the vorticity vector by tracking two points considered to be at the head and at the tail of the vector with the magnitude of the vorticity being proportional to the distance between the two points. In this method the stretching term is automatically handled as these points move apart from one another. One generally finds it necessary to divide the vortices in two when the distance between these two points becomes sufficiently large. The filament method has been shown to be equivalent to a method of the sort described above, and hence is also convergent under the appropriate conditions [Anderson and Greengard], [Greengard].

Initial Conditions. We have yet to specify how one chooses the initial vortex positions and strengths at time $t = 0$. In practice vortex methods tend to be used in one of two ways. In many applications no initial vorticity field is specified. In other words at time $t = 0$ the domain Ω_{NS} contains no vortices. The vortices at later times arise from vortex sheets which

have entered Ω_{NS} from Ω_{Pr} or some other analogous vorticity creation mechanism. When the vortex sheet method is used the vortex positions are determined by the positions of the sheets as they enter Ω_{NS} and their strengths are a function of the sheet strengths. We will discuss this in more detail below.

In other applications one is given an initial vorticity field $\underline{\omega}(\underline{x})$ with compact support which must be approximated by a sum of the form (3.3). Typically, this is accomplished by creating a grid of spacing h and letting the initial vortex positions $\tilde{\underline{x}}_j^0$ be the centers of the grid cells and letting the initial particle strengths be

$$\underline{\Gamma}_j^0 = \underline{\omega}(\tilde{\underline{x}}_j^0)\, h^N \ ,$$

for $N = 2, 3$. Most, if not all proofs that the vortex method converges to solutions of Euler's equations assume this type of initial condition. In this context the grid spacing h plays a central role in the accuracy of the method. It, together with the cutoff radius δ, determines the accuracy of the initial discretization and the accuracy of the subsequent approximation $\tilde{\underline{u}}$ to the flow field at later times t. Generally these convergence proofs require that h and δ be chosen so that

$$\delta = h^q \quad , \qquad \text{for some } q \in (0, 1)$$

From a standpoint of computational efficiency $q = 1$ would be preferable. However, in general, q depends on the function f and what underlying assumptions have been made about the smoothness of the exact solution $\underline{u}$. For Chorin's kernel (3.5) [Hald (1979)] has suggested choosing $q = 1/2$.

4. The No-Flow Boundary Condition. We now discuss the modifications to the computed flow field which are required to satisfy the no-flow boundary condition (2.8d). Let us begin by assuming that we have a solution $\underline{u}_\omega$ of (2.7a-c) with $\underline{\omega} = \nabla \times \underline{u}_\omega$ but for which (2.8d) does not hold, $\underline{u}_\omega \cdot \underline{\eta} \neq 0$. We seek a scalar function ϕ such that

$$\Delta\phi = 0 \qquad \text{in} \quad \Omega \quad , \tag{4.1a}$$

$$\nabla\phi \cdot \underline{\eta} = -\underline{u}_\omega \cdot \underline{\eta}, \qquad \text{on} \quad \partial\Omega. \tag{4.1b}$$

Now define $\underline{u}_p \equiv \nabla\phi$. It is easy to check that the velocity field given by $\underline{u} \equiv \underline{u}_\omega + \underline{u}_p$ satisfies (2.8a-d) with $\underline{\omega} = \nabla \times \underline{u}_\omega$. The velocity $\underline{u}_p$ is called a *potential flow* [Chorin and Marsden].

Numerically we attempt to perform a discrete analog of the continuous solution just described. In general, at the end of each time step we have some function $\tilde{\underline{u}}_\omega$ which does not satisfy (2.8d). The problem is to find some ϕ such that ϕ satisfies (4.1a,b) with $\underline{u}_\omega$ replaced by $\tilde{\underline{u}}_\omega$. In other words we wish to solve Laplace's equation for ϕ subject to Neumann boundary conditions. There is a vast body of literature devoted to both the theoretical and numerical solution of this problem. Effective numerical techniques for finding $\underline{u}_p$ may often be found there. We describe several simple examples here.

To begin note that the solution of the potential flow problem is dictated in large part by the shape of the domain. For example, let $N = 2$ and suppose that Ω is the upper half plane $\{(x, y) : y > 0\}$. If $\underline{u}_\omega$ is the velocity due to a collection of vortices at (x_j, y_j) with strengths Γ_j, $j = 1, \ldots, M$ then it is a simple matter to check that the potential flow $\underline{u}_p$ we seek is precisely the flow due to a collection of vortices at $(x_j, -y_j)$ with strengths $-\Gamma_j$, $j = 1, \ldots, M$. Here we have employed the well known *method of images* to find a solution of (4.1a,b).

As another example consider flow past a circle of radius r centered at the origin in two dimensions. So $\Omega = \{\underline{x} : |\underline{x}| > r\}$ and $\partial\Omega = \{\underline{x} : |\underline{x}| = r\}$. Suppose $\underline{u}_\omega$ is the flow induced by a collection of vortices at $\underline{x}_j \in \Omega$ with strengths Γ_j, $j = 1, \ldots, M$. Let

$$\underline{x}'_j = \frac{r^2}{|\underline{x}_j|^2} \quad \underline{x}_j$$

be the radial image of the point $\underline{x}_j$. Then a potential flow $\underline{u}_p$ which cancels the normal component of $\underline{u}_\omega$ on $\partial\Omega$ is the flow due to a collection of vortices with positions $\underline{x}'_j$ and strengths $-\Gamma_j$, $j = 1, \ldots, M$. However note that the total circulation about $\partial\Omega$,

$$\Gamma_\Omega = \oint_C \underline{u}_\omega \cdot ds \quad , \tag{4.2}$$

has changed. Here C is any circle of radius $R > \max|\underline{x}_j|$ centered at the origin. Another potential flow which will satisfy

$$\underline{u}_p \cdot \underline{\eta} = -\underline{u}_\omega \cdot \underline{\eta}$$

and yet leave (4.2) unchanged is the flow due to the radial images at positions $\underline{x}'_j$ with strengths $-\Gamma_j$ for $j = 1, \ldots, M$ together with M additional vortices, all located at the origin, and with strengths Γ_j.

This illustrates an important fact about the potential flow. Sometimes it is necessary to consider other properties of the flow field, such as total circulation, when choosing $\underline{u}_p$. Also note that in general ϕ in not unique. Given any ϕ which works, then for any constant c, the function $\phi + c$ also works.

Of course for more general domains the method of images is not always practical, or necessarily even possible. In these cases one can resort to one of several different strategies. Often it is possible to conformally map Ω onto another domain Ω' for which the method of images will work. However, there are potential[1] pitfalls with such a solution. Large numerical errors can occur due to the conformal map; for example in regions where $\partial\Omega$ is not differentiable, such as at a corner. Accurate numerical algorithms for conformal mapping are currently a very active area of research and results in this area will ultimately impact vortex methods. (See e.g. [Howell] and [Howell and Trefethen].)

[1] no pun intended

Another approach to solving the potential flow problem is to discretize the domain with a grid and use a fast Poisson solver on the grid. Here one must be careful of the errors introduced by the grid underlying the Poisson solver. It is important to ensure that these errors do not introduce the very numerical diffusion that the vortex method was designed to eliminate. Other possible approaches include finite element or multigrid solutions of the potential flow problem (4.1a,b).

5. The Random Walk Solution of the Heat Equation. The second fractional step in the random vortex method is the solution of the heat equation (2.9a) subject to the no-slip boundary condition (2.9b). Let $\underline{x}_j^{k+\frac{1}{2}}$ denote the vortex positions after the first of the two fractional steps, i.e. as a result of (3.8). The solution of the diffusion equation (2.9a) with initial data given by vortices at positions $\underline{x}_j^{k+\frac{1}{2}}$ with strengths $\underline{\Gamma}_j^{k+1}$ is approximated by letting the vortices undergo a random walk

$$\underline{x}_j^{k+1} = \underline{x}_j^{k+\frac{1}{2}} + \underline{\eta}_j \tag{5.1}$$

where $\underline{\eta}_j = (\eta_j^1, \ldots, \eta_j^N)$ and the $\{\eta_j^i : i = 1, \ldots, N, \quad \text{all } j\}$ are independent, Gaussian distributed random numbers with mean 0 and variance $2\nu\Delta t$. Any vorticies that end up in the sheet layer Ω_{Pr} or in the image of the sheet layer (with respect to $\partial\Omega$) as a result of the random walk become sheets, and any that end up beyond the image of the sheet layer are discarded. The no-slip boundary condition (2.9b) is approximately satisfied by using the vortex sheet method to cancel the tangential velocity on $\partial\Omega$ induced by the blobs with positions $\underline{x}_j^{k+1}$.

Why do we random walk the particles? Suppose we wish to solve the diffusion equation in all of R^N,

$$\underline{\omega}_t = \nu\Delta\underline{\omega}\,, \tag{5.2}$$

with initial data

$$\underline{\omega}_0(\underline{x}) = \sum_j f_\delta(\underline{x} - \underline{x}_j^{k+\frac{1}{2}})\,\underline{\Gamma}_j^{k+1}\,. \tag{5.3}$$

The exact solution is given by,

$$\begin{aligned}
\underline{\omega}(\underline{x}, t) &= (4\pi\nu t)^{-\frac{N}{2}} \int e^{-(\underline{x}-\underline{\eta})^2/4\nu t}\, \underline{\omega}_0(\underline{\eta}) d\underline{\eta} \\
&= (4\pi\nu t)^{-\frac{N}{2}} \int e^{-\underline{\eta}^2/4\nu t} \sum_j f_\delta(\underline{x} - (\underline{x}_j^{k+\frac{1}{2}} + \underline{\eta}))\,\underline{\Gamma}_j^{k+1}\, d\underline{\eta} \\
&= E[\sum_j f_\delta(\underline{x} - (\underline{x}_j^{k+\frac{1}{2}} + \underline{\eta}))\,\underline{\Gamma}_j^{k+1}]
\end{aligned}$$

where E denotes expectation over Gaussian random variables $\underline{\eta}$ on R^N with mean $\underline{0}$ and variance $2\nu\Delta t$. Thus, the exact solution of (5.2) with initial data (5.3) is *precisely* the expected value of the function obtained from (5.3) after each of the vortex positions has undergone a random walk, as in (5.1).

In other words, the computed solution consisting of the vortices with positions $\underline{x}_j^{k+1}$ given by (5.1) is 'on average' the exact solution to the diffusion equation (5.2) in R^N with initial data (5.3). More generally, when there are boundaries present one must create particles to satisfy the boundary conditions. The reader is referred to [Ghoniem and Sherman], [Hald], and [Puckett (1989)] for an in depth account of particle creation algorithms for satisfying boundary conditions. Here we restrict ourselves to a discussion of the vortex sheet method which is the method most commonly used in conjunction with the random vortex method to satisfy the no-slip boundary condition (2.9b).

6. The Vortex Sheet Method. For simplicity we describe the vortex sheet method in R^2. The generalization to $N = 3$ is straightforward [Chorin (1980)], [Fishelov]. Let (x, y) denote coordinates which are parallel and perpendicular to the boundary respectively. Let (u, v) denote the corresponding velocity components, ω the vorticity, and ν the kinematic viscosity. Assume that the boundary is located at $y = 0$ and let $U_\infty(x, t)$ denote the 'velocity at infinity' which is imposed on the flow from outside the boundary layer. In vorticity formulation the Prandtl equations are

$$\omega_t + u\omega_x + vw_y = \nu\,\omega_{yy} \quad , \tag{6.1a}$$

$$\omega = -u_y \quad , \tag{6.1b}$$

$$u_x + v_y = 0 \quad , \tag{6.1c}$$

$$u(x, 0, t) = 0 \quad , \tag{6.1d}$$

$$v(x, 0, t) = 0 \quad , \tag{6.1e}$$

$$\lim_{y \to \infty} u(x, y, t) = U_\infty(x, t) \quad . \tag{6.1f}$$

Note that in the limiting process ($\nu \to 0$) by which one derives the Prandtl equations from the Navier-Stokes equations (2.1a-c) the vorticity $\omega = v_x - u_y$ has become $\omega = -u_y$ since $v_x = O(\sqrt{\nu} u_y)$.

In the vortex sheet method the vorticity at time $t = k\Delta t$ is approximated by a sum of linear concentrations of vorticity,

$$\tilde{\omega}^k(x, y) = \sum_j \omega_j b_l(x - x_j^k)\delta(y_j^k - y). \tag{6.2}$$

Each term of the sum in (6.2) is referred to as a *vortex sheet*. The jth sheet has center (x_j^k, y_j^k) and *strength* or *weight* ω_j. Here δ is the Dirac delta function, and $b_l = b(x/l)$ is what we refer to as the *smoothing* or *cutoff function* in analogy with the vortex method. The most commonly used cutoff is the *hat* or *tent* function originally proposed by Chorin [Chorin (1978)],

$$b(x) = \begin{cases} 1 - |x| & |x| \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6.3}$$

The parameter l is often referred to as the *sheet length* even though the support of b_l is typically of length nl for some integer $n \geq 2$. Since b_l has finite support and since $\delta(y_j - y)$ is 0 for $y \neq y_j$, we see that the jth sheet is simply a line segment parallel to the boundary which carries a delta function concentration of vorticity. For b_l with b defined by (6.3) each sheet has length $2l$ and the vorticity concentration varies linearly along the length of the sheet - having a value of ω_j at the center and 0 at the ends. We briefly discuss other possible choices for b_l at the end of this section.

We can use (6.1b) and (6.1f) to write the tangential velocity in terms of the vorticity,

$$u(x,y,t) = U_\infty(x,t) + \int_y^\infty \omega(x,s,t)ds. \tag{6.4}$$

Our approximation to u at time $k\Delta t$ is determined by (6.2) and (6.4),

$$\tilde{u}^k(x,y) = U_\infty(x,k\Delta t) + \sum_j \omega_j b_l(x - x_j^k)\mathrm{H}(y_j^k - y), \tag{6.5}$$

where $\mathrm{H}(y)$ is the Heaviside function,

$$\mathrm{H}(y) = \begin{cases} 1 & y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

From (6.5) we see that the jump in $\tilde{u}$ along the jth sheet is $\omega_j b_l(x - x_j)$. This is the motivation for referring to the computational elements as vortex sheets. To find the velocity component normal to the boundary we first use (6.1c) and (6.1e) to write

$$v(x,y,t) = -\int_0^y u_x(x,s,t)ds. \tag{6.6}$$

Then, by approximating u_x with a centered divided difference, we obtain our approximation to v,

$$\begin{aligned} \tilde{v}^k(x,y) &= -\partial_x U_\infty(x,t)y \\ &\quad - \frac{1}{l}\sum_j \omega_j (b_l(x^+ - x_j^k) - b_l(x^- - x_j^k)) \min(y, y_j^k)\,. \end{aligned} \tag{6.7}$$

where $x^+ = x + l/2$ and $x^- = x - l/2$.

Since $\tilde{u}^k$ and $\tilde{v}^k$ were constructed using (6.4) and (6.6) respectively the velocity field $(\tilde{u}^k, \tilde{v}^k)$ automatically satisfies equations (6.1b,c) and the boundary conditions (6.1e,f). Furthermore, given U_∞, this velocity field is completely determined by the sheet positions (x_j^k, y_j^k) and their strengths ω_j.

The vortex sheet method is also a fractional step method. The first step is the numerical solution of the convective part of equation (6.1a)

$$\omega_t + u\omega_x + v\omega_y = 0. \tag{6.8}$$

The second step is the numerical solution of the diffusive part of (6.1a)

$$\omega_t = \nu\omega_{yy} \tag{6.9}$$

subject to the no-slip boundary condition (6.1d). Given an approximation $(\tilde{u}^k, \tilde{v}^k)$ to the velocity field at kth time step the velocity at the next time step is determined as follows.

We first evaluate $(\tilde{u}^k, \tilde{v}^k)$ at the center of each sheet. Denote the velocity at the center of the jth sheet by $(\tilde{u}_j^k, \tilde{v}_j^k)$. Our numerical approximation to (6.8) is found by moving the center of each sheet one time step of length Δt in this direction to obtain

$$(x_j^{k+\frac{1}{2}}, y_j^{k+\frac{1}{2}}) = (x_j^k, y_j^k) + \Delta t(\tilde{u}_j^k, \tilde{v}_j^k). \tag{6.10}$$

In order to approximate (6.9) subject to (6.1d) we first create sheets on the boundary. Let a_r, $r = 1, ..., M$ denote equally spaced grid points at $y = 0$ with grid spacing l: $a_{r+1} - a_r = l$. The sheet positions given by (6.10) in general induce an non-zero tangential velocity on the boundary, which we denote by $\tilde{u}^{k+\frac{1}{2}}(x, 0)$. Let $u_r = \tilde{u}^{k+\frac{1}{2}}(a_r, 0)$ and let ω_{max} denote a computational parameter called the *maximum sheet strength*. Then for each r we create $q_r = [|u_r|/\omega_{max}]$ sheets with centers $(a_r, 0)$ and strengths $-sign(u_r)\ \omega_{max}$ where $[x]$ denotes the greatest integer less than or equal to x.

The numerical solution of the diffusion equation (6.9) is found by letting all sheets, new and old, undergo a random walk in the y direction, reflecting those that go below the boundary. Therefore, the new sheet positions at time $(k+1)\Delta t$ are given by

$$(x_j^{k+1}, y_j^{k+1}) = (x_j^{k+\frac{1}{2}}, |y_j^{k+\frac{1}{2}} + \eta_j|)$$

where the η_j are independent, Gaussian distributed random numbers with mean 0 and variance $2\nu\Delta t$.

We wish to make several comments regarding the sheet creation algorithm here. First, note that in our presentation of the algorithm all sheets have magnitude ω_{max} and that we create no sheets at a_r when $|u_r| < \omega_{max}$. Hence, the no-slip boundary condition is satisfied at a_r only up to order ω_{max}. Originally Chorin [Chorin (1978, 1980)] created sheets at the rth grid point whenever $|u_r| \geq \epsilon$ for some $\epsilon \ll \omega_{max}$ such that $\omega_j \leq \omega_{max}$ for all j and the sum of the strengths of these sheets exactly cancel u_r. For example, ϵ might be chosen to be on the order of the computer's round off error. However, this algorithm creates more sheets than the one described above, and since the work required to compute $(\tilde{u}_j^k, \tilde{v}_j^k)$ at the center of each sheet is, at best, $O(ln^2)$ where n is the number of sheets in the flow, this greatly increases the computational cost of the algorithm. Furthermore, numerical experiments to compare the two sheet creation algorithms [Puckett (1989)], [Zhu] have shown that there is no tangible increase in the accuracy of the numerical approximation when this latter, more expensive, sheet creation algorithm is used.

The second point we would like to make here concerns the manner in which the no-slip boundary condition is satisfied and its relation to the

cutoff function b_l. As noted above, $\tilde{u}^{k+\frac{1}{2}}(x,0)$ is, in general, non-zero. Ideally one would like to add some function to $\tilde{u}^{k+\frac{1}{2}}$ which can be represented by the sum of sheets, and which cancels $\tilde{u}^{k+\frac{1}{2}}(x,0)$ at all points x on the boundary but leaves $\tilde{u}^{k+\frac{1}{2}}(x,y)$ unchanged for $y > 0$. In other words, we wish to find some function of the form $\sum \omega_j b_l(x - x_j)\mathrm{H}(y_j - y)$ such that

$$\tilde{u}^{k+\frac{1}{2}}(x,y) + \sum_j \omega_j \; b_l(x - x_j) \; \mathrm{H}(y_j - y) = \begin{cases} \tilde{u}^{k+\frac{1}{2}}(x,y) & y > 0, \\ 0 & y = 0. \end{cases}$$

In general this is not possible. However, one can find ω_j and (x_j, y_j) so that this holds exactly for $y > 0$ and within $O(l)$ for $y = 0$. For example, when b_l is defined by (6.3) choosing (x_j, y_j) to be the grid points $(a_r, 0)$ reduces the problem to that of finding the coefficients of a piecewise linear interpolant to $-\tilde{u}^{k+\frac{1}{2}}(x,0)$ with node points at the a_r (e.g. see [Schultz]). In other words, we wish to find coefficients c_r such that

$$\sum_r c_r b_l(x - a_r) \approx -\tilde{u}^{k+\frac{1}{2}}(x,0). \tag{6.11}$$

For the piecewise linear basis functions given by (6.3) it turns out that $c_r = -u_r$ is the correct choice, since then the left hand side of (6.11) is the usual piecewise linear interpolant of $-\tilde{u}^{k+\frac{1}{2}}$ at $y \equiv 0$. In actual practice we approximate the left hand side of this expression by creating q_r sheets at each point $x = a_r$ with strengths $\omega_r = \pm\omega_{max}$ such that $q_r\omega_r \approx -u_r$.

This idea can be generalized to make use of higher order interpolation procedures. For example, one can replace b_l with a basis function for cubic splines (e.g. see [Schultz] or [DeBoor]). In [Puckett (1987)] we studied the effect that this type of smoothing function has on the accuracy and rate of convergence of the vortex sheet method. The interested reader is referred there for further details.

Several studies have been made of the accuracy with which the vortex sheet method approximates solutions of the Prandtl equations. The method was used to approximate Blasius flow in both [Chorin] and [Puckett (1989)] and to approximate Falkner-Skan flow in [Summers]. In particular, [Puckett (1989)] contains an extensive tabulation of the error in approximating Blasius flow as a function of the computational parameters Δt, l, and ω_{max} while [Summers] examines the computed solutions for a family of flows, some of which contain stagnation points or separation points.

7. Choosing the Computational Parameters. There are three computational parameters in the vortex sheet method, the time step Δt, the sheet length l, and the maximum sheet strength ω_{max}. The only generally agreed upon constraint that these parameters must satisfy is the so called 'CFL' condition,

$$\Delta t \; U_{max} \leq l \tag{7.1}$$

where $U_{max} = \max U_\infty$. The justification usually given for (7.1) is that one wants to ensure that sheets move downstream at a rate of no more that one grid point per time step. This is an accuracy condition (as opposed

to a stability condition) which ensures that information propagating in the streamwise direction will influence all features which are at least $O(l)$.

We also propose another accuracy condition,

$$\Delta t\, \omega_{max} \leq Cl^2 \tag{7.2}$$

where C is a constant with dimensions $1/L$ and L is a typical length scale. (Usually L is the length of the boundary.) This condition is a consequence of requiring that the degree with which we refine u as a function of y be of the same order as the degree with which we refine features in the streamwise direction, $O(U_{max}/\omega_{max}) = O(L/l)$, and then using (7.1).

Note that since

$$\frac{d}{dx}\omega_j b_l(x - x_j) = O(\omega_{max}/l),$$

sheets induce local (non-physical) streamwise gradients in $\tilde{u}$ which are $O(\omega_{max}/l)$. Condition (7.2) relates the size of these gradients to the ratio $l/\Delta t$.

We wish the circulation about a vortex element to remain constant when a sheet leaves the sheet layer and becomes a vortex or vice-versa. If b_l is the piecewise linear smoothing function with b given by (6.3), then this implies that,

$$|\Gamma_j| = l\,\omega_{max}\,.$$

There are two other computational parameters which remain to be chosen, the cutoff radius δ and the sheet layer thickness ϵ. We would like to relate the cutoff radius δ to the vortex sheet parameters ω_{max} and l. Let us assume that we are using Chorin's cutoff function (3.5), and hence the velocity kernel is given by (3.6). We seek δ so that a vortex at the edge of the boundary layer and its image with opposite sign will induce the same tangential velocity on the boundary as a sheet with the same position and strength. (Note that the sheet does not require an image since, by (6.7), the no-flow boundary condition is satisfied exactly at $y = 0$.) If we set

$$\delta = \frac{l}{\pi}\,,$$

then we find that a vortex at (x, y) and its image at $(x, -y)$ will induce the same tangential velocity on the boundary at $(x, 0)$ as a sheet with center (x, y) provided $|x| < \delta$.

Recall that the random walks have standard deviation $\sqrt{2\nu\Delta t}$. One wishes to avoid having random walks which travel the length of the sheet layer in one time step and this principle is generally taken into account when choosing the sheet layer thickness ϵ. Usually ϵ is taken to be $\epsilon = C\sqrt{\nu\Delta t}$ for some constant C. Typically $C = 2$ or 3. This yields a boundary layer which has the appropriate scale, $O(\sqrt{\nu})$ (see [Schlichting]). We reiterate that ϵ represents the thickness of a numerical boundary layer which should be distinguished from the physical boundary layer. Many aspects of the combined random vortex / vortex sheet algorithm remain to be rigorously justified - such as the use of the vortex sheet method near points of separation - and this may take some time. Nonetheless, many workers have successfully used this hybrid method to model a wide variety of flows. For example see [Baden and Puckett], [Cheer (1983), (1989)],

[Chorin (1980)], [Ghoniem, Chorin and Oppenheim], [Sethian], [Sethian and Ghoniem], [Summers, Hanson, and Wilson], [Tiemroth] and [Zhu].

The interested reader should consult [Baden and Puckett], [Chorin (1980)], [Sethian], [Sethian and Ghoniem], [Tiemroth], [Puckett (1989)], and [Zhu] for a more detailed discussion regarding the relationship between the various parameters.

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