TANNAKA RECONSTRUCTION FOR FINITE GROUPS

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The goal of this note to address and answer the following

Question 1. *Suppose you are given the category of (complex, finite-dimensional) representations* $\mathbf{Rep}(G)$ *of a finite group G. Can you recover G from the* categorical *data of* $\mathbf{Rep}(G)$*?*

To elucidate, the categorical data of $\mathbf{Rep}(G)$ *a priori* contains information only on the level of morphisms between representations of G, and not any data about the elements of the vector spaces involved. But, for example, because representations of G admit a tensor product, the category $\mathbf{Rep}(G)$ can be equipped with more structure than just the data of morphisms. Thus, there's a question of what is the minimal information about $\mathbf{Rep}(G)$ we need to retain to recover G.

In fact, even if we allow ourselves to know the tensor structure of $\mathbf{Rep}(G)$, we still can not recover G. ~~Indeed, you saw in your homework that the quaternions and the dihedral group of order 8 both have the same character table. Since characters determine representations, and the character of a tensor product is a product of characters, it follows that the structure of $\mathbf{Rep}(G)$ as a tensor category is insufficient to recover G.~~[1] Indeed, examples of distinct groups with equivalent tensor categories of representations are known. See the paper "Isocategorical groups" by Etingof and Gelaki, arXiv:math/0007196.

Thus, a different kind of data is needed to recover G from its tensor category of representations. This comes in the form of a "fiber functor"

$$F : \mathbf{Rep}(G) \to \mathbf{Vect}_{\mathbb{C}},$$

which, in our case, is just a fancy turn of phrase for the forgetful functor that takes a G representation to the underlying vector space (thus forgetting the G action). Having equipped the tensor category $\mathbf{Rep}(G)$ with this extra piece of information, we are ready to reconstruct G.

Theorem 1 (Tannaka reconstruction)**.** *Let* $\mathbf{Rep}(G)$ *denote the tensor category of finite-dimensional, complex representations of a finite group G. Let*

$$F : \mathbf{Rep}(G) \to \mathbf{Vect}_{\mathbb{C}}$$

be the forgetful functor from the category of representations of G to the category of vector spaces. Then

$$\mathrm{Aut}^{\otimes}(F) \simeq G,$$

where $\mathrm{Aut}^{\otimes}(F)$ *denotes the group of tensor-preserving natural automorphisms of the functor* F.

To understand everything I just wrote, the reader is encourage to look at the next section for definitions, and come back to reread the previous discussion, if necessary. I have left a

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[1]This argument only suffices to show that the so-called fusion ring of these groups is the same.

lot of easy verification exercises that should make sure you grasp the definitions. Once the definitions are synthesized (say, by doing these exercises), the proof of Theorem 1 in Section 2 is quite elementary. For our purposes, we do not really need the full definition of a tensor category, but just the definition of a "natural tensor automorphism." For more background on category theory, see Mac Lane's classic reference *Categories for the Working Mathematician*. Note that in the literature, "tensor categories" are often called "monoidal categories."

1. DEFINITIONS AND (NEEDED) EXAMPLES

1.1. **Representation theory.** A *complex, finite-dimensional representation* of a finite group G is a finite-dimensional, left $\mathbb{C}G$-module, where $\mathbb{C}G$ is the group algebra of G. Here the *dimension* of a $\mathbb{C}G$-module is just its dimension as a complex vector space. Thus, it is easy to see that a representation is described by a pair (V,ϕ) where V is a finite-dimensional, complex vector space V, and

$$\phi : G \to \mathrm{GL}(V)$$

is a group homomorphism to the set of invertible linear transformations of V. All of our representations will be finite-dimensional and complex, so I'll stop saying that from now on. Also, everything will be a representation of G, so I'll stop saying that, too.

Now for some examples. The *trivial* representation of G is $(\mathbb{C},\varepsilon)$ where $\varepsilon(g) = 1 \in \mathrm{GL}(\mathbb{C}) = \mathbb{C}^*$ for all g in G.

The *(left) regular* representation of G is $(\mathbb{C}G,\ell)$ where $\ell(g)(x) = gx$. Since, by definition, $\mathbb{C}G$ is spanned by the elements $G \subset \mathbb{C}G$, $\ell(g)$ extends by linearity to a map on all of $\mathbb{C}G$.

Given any two representations (V,ϕ) and (W,ψ), we can define the *tensor product* $(V\otimes W,\phi\otimes\psi)$, by taking $V\otimes W$ to be the tensor product of V and W as complex vector spaces, and letting

$$(\phi\otimes\psi)(g)(\sum_{i=1}^{n} v_i\otimes w_i) = \sum_{i=1}^{n}[\phi(g)(v_i)]\otimes[\psi(g)(w_i)].$$

Let (V,ϕ) and (W,ψ) be two representations. A *morphism* from V to W is a morphism of left $\mathbb{C}G$ modules. In equivalent words, a morphism is a linear map

$$f : V \to W$$

such that for all v in V and all g in G,

$$f(\phi(g)v) = \psi(g)f(v).$$

One example of a morphism is the map ε defined above, extended to all of $\mathbb{C}G$:

$$\varepsilon : \mathbb{C}G \to \mathbb{C}$$
$$\sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g.$$

We call ε the *counit* of $\mathbb{C}G$. Another example of a morphism of representations is the *coproduct*

$$\Delta : \mathbb{C}G \to \mathbb{C}G\otimes\mathbb{C}G$$
$$\sum_{g\in G} c_g g \mapsto \sum_{g\in G} c_g(g\otimes g).$$

You should convince yourself that these are indeed morphisms of representations.[2]

1.2. **Basic category theory.** A *category* $\mathscr{C}$ consists of a collection $\mathrm{Ob}(\mathscr{C})$ of *objects*, and, for each pair of objects A and B, a collection $\mathrm{Mor}(A,B)$ of *morphisms* satisfying the following:

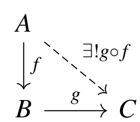
(i) (Law of composition) For every three objects A,B and C in $\mathrm{Ob}(\mathscr{C})$, and every pair of morphisms f in $\mathrm{Mor}(A,B)$ and g in $\mathrm{Mor}(B,C)$, there exists a unique morphism $g \circ f$ in $\mathrm{Mor}(A,B)$.

(ii) (Associativity of composition) Given four objects A,B,C and D, and three maps f in $\mathrm{Mor}(A,B)$, g in $\mathrm{Mor}(B,C)$ and h in $\mathrm{Mor}(C,D)$, the law of composition is such that
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

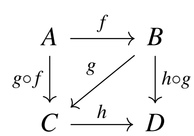
(iii) (Identity maps exist) For every object A, there exists a morphism Id_A such that for every other object B and any morphism $f \in \mathrm{Mor}(A,B)$ and any morphism $g \in \mathrm{Mor}(B,A)$,
$$\mathrm{Id}_A \circ g = g \qquad \text{and} \qquad f \circ \mathrm{Id}_A = f.$$

If we agree to denote a morphism f in $\mathrm{Mor}(A,B)$ by an arrow $A \to B$, we can rewrite the axioms of a category as follows. All of the following diagrams commute.

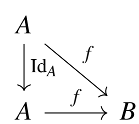
(i) Composition law:



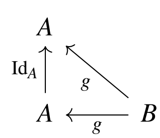
(ii) Associativity:



(iii) Right identity:



Left identity:



Examples of categories are everywhere. For our purposes now, everything we care about is some instance of the category R-**Mod**, where objects are left R-modules over an appropriate ring R, and morphisms are R-module homomorphisms. You should be able to convince yourself that R-**Mod** does actually form a category. We will denote the category of complex vector spaces by
$$\mathbf{Vect} := \mathbb{C}\text{-}\mathbf{Mod}$$

[2]The coproduct and the counit give $\mathbb{C}G$ the structure of a *coalgebra*. Of course, $\mathbb{C}G$ is also an algebra. The algebra and coalgebra structures respect each other, so $\mathbb{C}G$ is an example of a *bialgebra*. We can further upgrade $\mathbb{C}G$ to a *Hopf algebra* by introducing the antipode $g \mapsto g^{-1}$.

and the category of representations of G by

$$\mathbf{Rep}(G) := \mathbb{C}G\text{-}\mathbf{Mod}.$$

Roughly speaking, from a classical perspective, you can think of a category as a branch of mathematics. With this philosophy in mind, a functor is a way to understand one branch of math by relating it to another. More precisely, a *(covariant) functor* $F : \mathscr{C} \to \mathscr{D}$ associates to every object A of $\mathscr{C}$, an object $F(A)$ in $\mathscr{D}$, and to every morphism $f : A \to B$ in $\mathscr{C}$, a morphism $F(f) : F(A) \to F(B)$ in such a way that identities and associativity are preserved. That is, for every object A in $\mathscr{C}$, $F(\mathrm{Id}_A) = \mathrm{Id}_{F(A)}$ and for every pair of maps $f : A \to B$ and $g : B \to C$, $F(g \circ f) = F(g) \circ F(f)$. In retrospect, you might think of a functor as a morphism in the category of all categories.

Again, there are examples of functors all over mathematics. For our purposes there is only one we care about: the forgetful functor

$$F : \mathbf{Rep}(G) \to \mathbf{Vect}.$$

More generally, you might convince yourself that for any ring R and subring S, there is a forgetful functor

$$F : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}.$$

Thinking of functors as relationships between two branches of mathematics, it's inevitable to wonder what the relationships *between* the relationships are. This motivates the following definition. Let $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{C} \to \mathscr{D}$ be two functors. A *natural transformation* $T : F \to G$ consists of a morphism $T_A : F(A) \to G(A)$ in $\mathscr{D}$ for every object A in $\mathscr{C}$, such that for every map $f : A \to B$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(a)} & F(B) \\ \Big\downarrow{\scriptstyle T_A} & & \Big\downarrow{\scriptstyle T_B} \\ G(A) & \xrightarrow{G(a)} & G(D) \end{array}$$

T is called a *natural isomorphism* if T_A is an isomorphism for every object A in $\mathscr{C}$. Finally, T is a *natural automorphism* if every T_A is an automorphism.

A prototypical example of a natural isomorphism occurs in the category of (finite-dimensional) vector spaces. Denote by $** : \mathbf{Vect} \to \mathbf{Vect}$ the functor taking a vector space to its double dual. That is,

$$**(V) := V^{**} := \mathrm{Mor}(\mathrm{Mor}(V, \mathbb{C}), \mathbb{C}).$$

For a map $f : V \to W$, we define

$$\begin{aligned} f^{**} : V^{**} &\to W^{**} \\ e &\mapsto e \circ f^* \end{aligned}$$

where

$$\begin{aligned} f^{*} : W^{*} &\to V^{*} \\ \lambda &\mapsto \lambda \circ f. \end{aligned}$$

As an exercise, you can check that $**$ is naturally isomorphic to the identity functor. [Hint: send V to V^{**} by letting $v \mapsto e_v$, the evaluation map at v.]

1.3. **Tensor categories and functors between them.** Sometimes categories admit more structure than just composition of morphisms. For example, **Vect** and **Rep**(G) both admit tensor products of objects and morphisms. The properties of $\otimes$ in these categories make them prototypical examples of tensor categories. Writing down the full definition of tensor category is more than we need. For now, observe the following: the forgetful functor $F : \mathbf{Rep}(G) \to \mathbf{Vect}$ preserves tensor products, in the sense that

$$F((V,\phi)\otimes(W,\psi)) = F((V,\phi))\otimes F(W,\psi).$$

In fact F does more (namely, respects unit and associativity constraints...), making it a full blown functor between two tensor categories, but we need not bother ourselves with these precise notions here.

However, we do need to define what we mean by a tensor-preserving natural automorphism of F. For our purposes, the following is perfectly sufficient: *a tensor-preserving natural automorphism of F* is a natural automorphism T of F such that for all objects (V,ϕ) and (W,ψ) in **Rep**(G), T satisfies

$$T_{(V,\phi)\otimes(W,\psi)} = T_{(V,\phi)}\otimes T_{(W,\psi)}.$$

We might also refer to such a T as a *natural tensor automorphism of F*.

2. Proof of Tannaka reconstruction

We shall prove Theorem 1 with a series of lemmata. Let $F : \mathbf{Rep}(G) \to \mathbf{Vect}$ be the forgetful functor from the statement of the theorem, with $F(V,\phi) = V$. We first show how to construct a natural tensor automorphism of F from an element of G. The idea is to let g act on $F(V,\phi)$ just like g acts on V according to ϕ.

Lemma 2. *Consider the map*

$$\iota : G \to \mathrm{Aut}^{\otimes}(F)$$
$$g \mapsto T^g$$

where T^g is the natural tensor automorphism of F defined on objects by

$$T^g_{(V,\phi)} : F(V,\phi) \to F(V,\phi)$$
$$v \mapsto \phi(g)(v),$$

and T^g is the identity on all morphisms. Then ι is an injective group homomorphism.

Proof. There are several things to check here: that T^g is a natural automorphism of F, that T^g is tensor-preserving, that ι is a group homomorphism, and that ι is injective. The first three are all completely straightforward from the definitions, so I'll leave them as exercises. The last one follows because for any nontrivial group element g, there exists some representation where g acts nontrivially. For example, the left regular representation $(\mathbb{C}G,\rho)$ is faithful. □

The goal now is to show that ι is surjective. To this end, we will first argue that any natural tensor automorphism T of F is determined by its value $T_{\mathbb{C}G}(1)$, where, in order to make the notation less clumsy, we have written

$$T_{\mathbb{C}G} := T_{(\mathbb{C}G,\rho)},$$

and we will write later

$$T_{\mathbb{C}G\otimes\mathbb{C}G} := T_{(\mathbb{C}G\otimes\mathbb{C}G,\rho\otimes\rho)}.$$

We will then argue that $T_{\mathbb{C}G}(1)$ is an element of G.

Proposition 3. *Let T be any natural tensor automorphism of F. Then $T_{\mathbb{C}G}(1)$ determines T.*

Proof. Let (V,ϕ) be any representation. Then for any $v \in V$, we can construct a uniquely defined morphism of representations

$$f_v : \mathbb{C}G \to V$$
$$1 \mapsto v.$$

Applying T to this morphism, we get the commutative square

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{F(f_v)} & F(V,\phi) \\ \downarrow T_{\mathbb{C}G} & & \downarrow T_{(V,\phi)} \\ \mathbb{C}G & \xrightarrow{F(f_v)} & F(V,\phi) \end{array}$$

Since by definition $F(V,\phi) = V$ and $F(f) = f$ for any morphism f, we can just write

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{f_v} & V \\ \downarrow T_{\mathbb{C}G} & & \downarrow T_{(V,\phi)} \\ \mathbb{C}G & \xrightarrow{f_v} & V \end{array}$$

We'll make these evaluations implicitly from now in on. Anyway, because $f_v(1) = v$, we see from the commuting square that

$$T_{(V,\phi)}(v) = f_v \circ T_{\mathbb{C}G}(1).$$

Since f_v is unique, $T_{(V,\phi)}(v)$ is indeed determined by $T_{\mathbb{C}G}(1)$. Since this is true for all vectors v of all representations (V,ϕ), we conclude that T is determined by $T_{\mathbb{C}G}(1)$. □

Call a nonzero element v of $\mathbb{C}G$ *grouplike* if

$$\Delta(v) = v \otimes v.$$

Lemma 4. *$T_{\mathbb{C}G}(1)$ is grouplike.*

Proof. This lemma is where we will use the fact that T respects the tensor structure of the category $\mathbf{Rep}(G)$. Since the coproduct Δ is a morphism of representations, we can assemble the following commutative diagram

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \\ \downarrow T_{\mathbb{C}G} & & \downarrow T_{\mathbb{C}G \otimes \mathbb{C}G} \\ \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \end{array}$$

Since T is tensor preserving, we can rewrite the arrow on the right as

$$T_{\mathbb{C}G \otimes \mathbb{C}G} = T_{\mathbb{C}G} \otimes T_{\mathbb{C}G}$$

to get a new commutative diagram

$$\begin{array}{ccc} \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \\ \downarrow T_{\mathbb{C}G} & & \downarrow T_{\mathbb{C}G} \otimes T_{\mathbb{C}G} \\ \mathbb{C}G & \xrightarrow{\Delta} & \mathbb{C}G \otimes \mathbb{C}G \end{array}$$

In particular, by applying the maps to 1, this diagram shows

$$\Delta(T_{\mathbb{C}G}(1)) = (T_{\mathbb{C}G} \otimes T_{\mathbb{C}G})(\Delta(1)) = (T_{\mathbb{C}G} \otimes T_{\mathbb{C}G})(1 \otimes 1) = T_{\mathbb{C}G}(1) \otimes T_{\mathbb{C}G}(1),$$

so $T_{\mathbb{C}G}(1)$ is grouplike, as claimed. □

Lemma 5. *Grouplike elements of $\mathbb{C}G$ are precisely the elements of $G \subset \mathbb{C}G$.*

Proof. Let $v = \sum_{g \in G} v_g g$ be grouplike. Since v is grouplike,

$$\Delta(v) = \sum_{g \in G} \sum_{h \in G} v_g v_h g \otimes h.$$

On the other hand, by definition of the coproduct,

$$\Delta(v) = \sum_{g \in G} v_g \Delta(v) = \sum_{g \in G} v_g g \otimes g.$$

Suppose $v_g \neq 0$. Then for all $h \in G$

$$v_g v_h = \delta_{g,h} v_g,$$

where $\delta_{g,h}$ is the Kronecker delta. We conclude that for all $h \in G$

$$v_h = \delta_{g,h}.$$

□

Tannaka reconstruction now follows. Given any T in $\mathrm{Aut}^{\otimes}(F)$, Proposition 3 shows that T depends only on $T_{\mathbb{C}G}(1)$, which Lemmas 4 and 5 show is an element of G. Thus if $T_{\mathbb{C}G}(1) = g$, then $T = T^g$. We conclude that the map ι is surjective and, hence, by Lemma 2, an isomorphism. Thus

$$\mathrm{Aut}^{\otimes}(F) \simeq G.$$