1 Graphing Techniques: Translation, Reflection, and Scaling

1.1 Translation

Recall the equation for a circle with radius 1 centered at the origin:

\[ x^2 + y^2 = 1 \]

Let’s say I want to shift this circle 1 unit to the right. How do I do that? Well, it will still be a circle with radius 1, but now the center will be at (1,0). Therefore, the equation for the new circle is still the general equation for a circle, but now it has radius 1 and center (1,0). We know the equation for this is

\[ (x - 1)^2 + y^2 = 1. \]

Notice that the change was that I replaced the variable \( x \) with the term \( (x - 1) \). Similarly, if I now want to take that shifted circle and shift it up by 1 unit, the new circle will obey the equation for a circle with radius 1 and center (1,1):

\[ (x - 1)^2 + (y - 1)^2 = 1. \]

In general, this is always true, which gives us the following definition.

**Definition:** Given the graph of a function \( y = f(x) \),

Replacing \( x \) by \( (x - c) \), that is, \( y = f(x - c) \), represents a horizontal translation to the right by \( c \).

Replacing \( y \) by \( y - c \) to obtain \( y - c = f(x) \), or \( y = f(x) + c \), represents a vertical translation up by \( c \).

(Similarly, \( y = f(x+c) \) corresponds to translation \( c \) units left, and \( y+c = f(x) \), or \( y = f(x)-c \), corresponds to translation by \( c \) units down.)

**Example:** Consider the graph \( y = 2x \). This is a line with slope 2 and y-intercept 0. To translate this vertically down 1 unit, we draw a line with slope 2 and y-intercept -1. This new line is the graph of the function \( y = 2x - 1 \). Or,

\[ (y + 1) = 2x. \]

In this form, we can see that this is the same function we started with, except that we have replaced \( y \) with \( y + 1 \), corresponding to a vertical shift 1 unit in the negative direction.

Now instead, if I wanted to shift the original graph to the right by 1, I would replace \( x \) with \( x - 1 \):

\[ y = 2(x - 1) = 2x - 2. \]
1.2 Reflection

Recall from last Wednesday, when we discussed symmetry, and Worksheet 2, that any point \((a, b)\) in the \(x, y\)-plane can be reflected in the following ways:

- Reflection across \(x\)-axis: \((a, b) \rightarrow (a, -b)\).
  
  Reflecting the graph of \(y = f(x)\) across the \(x\) axis corresponds to plotting the function 
  \(-y = f(x)\), or, \(y = -f(x)\).

- Reflection across \(y\)-axis: \((a, b) \rightarrow (-a, b)\).
  
  Reflecting the graph of \(y = f(x)\) across the \(y\) axis corresponds to plotting the function 
  \(y = f(-x)\).

- Reflection across origin: \((a, b) \rightarrow (-a, -b)\).
  
  Reflecting the graph of \(y = f(x)\) across the origin corresponds to plotting the function 
  \(-y = f(-x)\), or, \(y = -f(-x)\).

There’s one fourth type of reflection that we will sometimes use: reflection across the line \(y = x\). This corresponds to switching the \(x\) and \(y\) coordinates:

\[(a, b) \rightarrow (b, a).\]

(Draw a picture of this using the point \((3, 2)\).

Reflecting the graph of \(y = f(x)\) across the line \(y = x\) corresponds to plotting the function 
\(x = f(y)\) (or, \(y = f^{-1}(x)\), as a spoiler for inverse functions in a few minutes).

1.3 Scaling

We can also scale a graph: that is, we can shrink or stretch it in the horizontal or the vertical direction. This is exactly like taking a picture on your computer and stretching it vertically or horizontally. Except, the picture is a graph, and we have axes that we can use to measure exactly where the graph goes.

To see how to do this, let’s start with an example: the line \(y = x^2\).

1. Graph the function.

2. How do I stretch this vertically by a factor of 2? Multiply each \(y\) coordinate by 2: \([\text{draw this}]\).

3. Now, the graph for this new function is \(y = 2(x^2) = 2x^2\), since I just doubled every \(y\) coordinate.

4. Notice that this is the same as replacing \(y\) with \(\frac{1}{2} y\). This is a rule:

To scale a graph by a factor of \(c\) vertically, replace \(y\) with \(\frac{1}{c} y\).

5. Now, suppose instead that I want to scale the function by a factor of \(\frac{1}{2}\) in the horizontal direction. That means I take every point and keep its \(y\) coordinate the same, but take one half of its \(x\) coordinate. So in particular the graph will go through \((1, 4)\) and \((1.5, 9)\) instead of \((2, 4)\) and \((3, 9)\). This is now the graph of \(y = (2x)^2 = 4x^2\). That is, to scale the plot by a factor of \(\frac{1}{2}\) horizontally, we replaced \(x\) by \(2x\). This is also a rule:

To scale a graph by a factor of \(c\) horizontally, replace \(x\) with \(\frac{1}{c} x\).
1.4 Multiple Transformations

We can apply multiple transformations at the same time by starting with the graph of a simple function, and then applying each transformation step by step.

- **Example** \( y = \sqrt{-x + 2} \).
  
  Plot \( y = \sqrt{x} \).
  
  Reflect it across the \( y \) axis to get \( y = \sqrt{-x} \).

  Replacing \((-x)\) with \(-x + 2 = -(x + 2)\) shifts the graph of \( \sqrt{-x} \) to the **right** by 2 units.

- **Example** \( y = 2 \left( \frac{1}{x+4} \right) \).
  
  Plot \( \frac{1}{x} \).
  
  Replace \( x \) with \( x + 4 \), in other words, shift to the left by 4 units.

  Multiplying by 2 on the right hand side is the same as multiplying \( y \) by \( \frac{1}{2} \) on the left hand side. So, scale the function by a factor of 2 vertically.

2 Inverse Functions

Recall that a function is a rule for an input-output relationship: it takes each element of the domain and assigns to it one element in the range. Let’s draw two examples of functions:

![Diagram of two functions](image)

**Question:** Can we go backwards? [draw this] In other words, is there a function that takes an element \( y \) of the range and spits out the element in the domain for which applying the original function resulted in the output \( y \)?

**Intuition:** for the first function I just drew, I should be able to. The point \( c \) clearly came from the point \( a \), and the point \( d \) clearly came from the point \( b \). So if we were to go backwards, we would have one arrow from \( c \) to \( a \) and one arrow from \( d \) to \( b \) (draw this). However, for the second function that I drew, since both \( a \) and \( b \) have arrows pointing to \( c \), I don’t know how to go backwards from \( c \).

**Answer:** sometimes. It depends on whether the function is one-to-one.

**Definition** A function is one-to-one if every element in the range comes from **at most one** element in the domain.

Above, we can see that the first function is one-to-one, and the second one is not, since \( c \) comes from two points in the domain.

In general, there are two ways of finding out whether or not a function is one-to-one.

- **Algebraically:** If we set \( f(a) = f(b) \) and can show that \( a = b \), then the function is one-to-one. This is because this shows that if our function takes the same value for two inputs, then those inputs actually have to be the same. For example, let \( f(x) = \sqrt{2x - 1} \).

  \[
  \sqrt{2a - 1} = \sqrt{2b - 1} \implies 2a - 1 = 2b - 1 \implies 2a = 2b \implies a = b.
  \]

  So \( f(x) \) is indeed one-to-one.
Graphically, using the **Horizontal Line Test**: A function is one-to-one if any horizontal line in the plane passes through the graph of the function at most once. (E.g. \( f(x) = x^2 \) is not one-to-one.)

**Definition**: If \( f(x) \) is one-to-one, the domain of \( f \) is \( A \) and the range is \( B \), then \( f^{-1} \) (the inverse function of \( f \)) is the function whose domain is \( B \) and range is \( A \) such that for every \( x \) in \( A \),

\[
f^{-1}(f(x)) = x
\]

and for every \( y \) in \( B \),

\[
f(f^{-1}(y)) = y.
\]

**Remark** (caution): inverse of a number \( x \) is \( x^{-1} = \frac{1}{x} \). Inverse of a function \( f \) is \( f^{-1} \) (looks the same) \( \neq \frac{1}{f(x)} \). Very different!! To denote \( \frac{1}{f(x)} \), you have to just write \( f^{-1}(x) \).

**Remark**: The inverse of the inverse is the original function. That is, \( (f^{-1})^{-1} = f \).

**Question**: Algebraically, how do we actually find the inverse?

**Answer**: Starting with the original function \( f \), say \( y = f(x) \), solve for \( x \) (or the dependent variable) as a function of \( y \) (or the independent variable).

**Example**: Find the inverse of \( f(x) = 2x + 1 \).

**Solution**: Start with

\[
y = 2x + 1
\]

and solve for \( x \):

\[
2x = y - 1 \implies x = \frac{y - 1}{2}.
\]

The formula we end up with for \( x \) is the inverse function for \( y \):

\[
f^{-1}(y) = \frac{y - 1}{2}.
\]

**Example**: Find the inverse of \( f(x) = \sqrt{x - 1} \).

**Solution**:

\[
y = \sqrt{x - 1} \implies y^2 = x - 1 \implies x = y^2 + 1.
\]

Or,

\[
f^{-1}(y) = y^2 + 1.
\]