Exercise 1

For each of the following situations, draw a picture, identify the variables, and write an equation for the function of interest.

(a) A square is inscribed in a circle. (That is, the square is entirely contained in the circle, but each corner of the square contacts the perimeter of the circle. There is only one such square for any given circle.) Find the area of the square as a function of the radius of the circle.

\[ A = l^2 \]
\[ \left(\frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2 = r^2 \]
\[ l = \sqrt{2}r \]
\[ A = \left(\sqrt{2}r\right)^2 \]
\[ A = 2r^2 \]

(b) Suppose you have a 10-foot ladder placed against the wall. Find the height of the top of the ladder, where it contacts the wall, as a function of the distance along the floor between the wall and the base of the ladder.

0. Height as a function of distance

2. \( d = \text{floor distance} \), \( h = \text{height} \)

3. \( h^2 + d^2 = 10^2 \)
\[ h^2 = 100 - d^2 \]

4. \( h = \sqrt{100 - d^2} \)
(c) You are teaching a class and you need to order one textbook and one solutions manual for each student. Textbooks are $10 each and solutions manuals are $2 each. When you place the order, you must also pay a $5 flat-rate shipping fee. Find the total price you must pay as a function of the number of students in the class.

\[ P = 10t + 2m + 5 \]
\[ t = s \]
\[ m = s \]

\[ P = 10s + 2s + 5 \]

(d) A rectangle is inscribed in a circle. The diameter of the circle is 1 foot. Express the area of the rectangle as a function of its width.

1. \( r = \frac{1}{2} \)
\[ \left( \frac{l}{2} \right)^2 + \left( \frac{w}{2} \right)^2 = r^2 = \frac{1}{4} \]
\[ A = lw \]
2 Maximum and Minimum Problems

Exercise 2

Suppose that a quadratic function \( Q(x) \) is given by

\[ Q(x) = ax^2 + bx + c. \]

(a) Use the method of completing the square to rewrite the function in the form

\[ Q(x) = c_1(x - c_2)^2 + c_3 \]

where \( c_1, c_2, \) and \( c_3 \) are arithmetic combinations of \( a, b \) and \( c. \)

\[
Q = a \left[ x^2 + \frac{b}{a} x + \frac{c}{a} \right] = a \left[ x^2 + \frac{b}{2a} x + \frac{b^2}{4a^2} + \frac{c}{a} - \frac{b^2}{4a^2} \right] \\
= a \left[ \left(x + \frac{b}{2a} \right)^2 \right] + \left(c - \frac{b^2}{4a} \right) \\
= c_1 \left(x - c_2\right)^2 + c_3
\]

(b) Using your results from part (a), explain why the \( x \)-coordinate of the vertex is \(-\frac{b}{2a}\).

Any number squared is \( \geq 0 \). So if \( a \) is positive, the min. of the range of the function occurs when the squared term is zero, that is, \( x = -\frac{b}{2a} \). Same if \( a \) is negative, but the maximum is at \( x = -\frac{b}{2a} \).

(b) Using a graph, explain in your own words why the \( x \)-coordinate of the vertex does not depend on the number \( c. \)

Exercise 3

Find the maximum or minimum value for each of the problems below.
(a) Two numbers add to 12. What is the largest possible value for their product?

**Variables:** \( x, y, P \) (product)

**Equations:**
\[
x + y = 12
\]
\[
xy = P
\]

We need \( P \) as a function of one variable — say \( x \). So, write \( y \) as a function of \( x \):
\[
y = 12 - x
\]
\[
X(12 - x) = P \rightarrow P = -x^2 + 12x
\]
\[
= - (x - 6)^2 + 36
\]

Max product occurs at \( x = 6 \), so \( P = 36 \).

(b) Among all rectangles whose area is 30 square meters, find the length and width of the one with the smallest perimeter.

**Variables:** \( A, l, w, P \)

**Equations:**
\[
A = lw = 30 \quad \rightarrow \quad w = \frac{30}{l}
\]
\[
P = 2l + 2w
\]
Reduce to one equation:
\[
P = 2l + 2\left(\frac{30}{l}\right) \quad \leftarrow \quad \text{we have not discussed}
\]

**Multiply by \( l \):**
\[
lP = 2l^2 + 60 \quad \leftarrow \quad \text{the vertex of this parabola gives us}
\]
\[
2l^2 - lp + 60 = 0
\]
\[
l = \frac{p + \sqrt{p^2 - 480}}{4} \quad \rightarrow \quad \text{minimum possible} \ P \text{ is where} \ p^2 - 480 = 0,
\]
\[
\frac{p + \sqrt{480}}{4} \quad \rightarrow \quad \text{that is,} \quad P = \frac{480}{4} \quad \rightarrow \quad \frac{p}{l} = \frac{5}{4}, \quad \frac{w}{p} = \frac{3}{5}
\]

(c) While playing goalkeeper for your department's intramural soccer team, you kick the ball into the air, aiming just past your left midfielder. The height of the ball in the air is given by the function \( h(t) = -16t^2 + 12t + 1 \), where \( t \) is the number of seconds for which the ball has been in the air. When does the ball reach its maximum height, and what is the maximum height?

**Variables:** \( t, h \)

**Equations:**
\[
h(t) = -16t^2 + 12t + 1
\]
\[
= -16\left[ t^2 - \frac{3}{4} t - \frac{1}{16} \right]
\]
\[
= -16\left[ (t - \frac{3}{8})^2 - \frac{9}{64} - \frac{1}{16} \right]
\]
\[
= -16\left[ (t - \frac{3}{8})^2 \right] + \frac{9}{4} + 1
\]
\[
= -16\left( t - \frac{3}{8} \right)^2 + \frac{13}{4} \quad \rightarrow \quad \text{max occurs at} \ t = \frac{3}{8} \text{ sec.}
\]
\[
h = \frac{13}{4} = 3.25
\]
(d) Among all cylinders whose volume is $10\pi$ liters, find the height, radius, and surface area of the one with the smallest surface area. (Note: the volume of a cylinder is $\pi r^2h$ and the surface area is $2\pi rh$, where $r$ and $h$ are the radius and the height.)

Variables: $h$, $r$, $A$, $S$

Equations: $\pi r^2h = 10\pi \Rightarrow r^2h = 10$

$S = 2\pi rh$

Reduce to one equation: $h = \frac{10}{r^2}$

$S = 2\pi r\left(\frac{10}{r^2}\right) = \frac{20\pi}{r}$

No minimum! As $r$ increases, $S$ decreases, but never reaches a minimum.

3 Additional Recommended Exercises

4.2 5-30, 35-44, 57-62

4.4 1-6, 8-10, 12-22, 32-37, 39-43, 47-58

4.5 1-23, 26-46, 55-60