Consider the triangle drawn below.

Exercise 1

1. Suppose \( a = 5 \) and \( b = 12 \). Find \( c \), and then find \( \sin(\angle A) \), \( \cos(\angle A) \), \( \tan(\angle A) \), \( \sec(\angle A) \), \( \csc(\angle A) \), and \( \cot(\angle A) \).

\[
c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13.
\]

\[
\sin(\angle A) = \frac{5}{13}; \quad \cos(\angle A) = \frac{12}{13}; \quad \tan(\angle A) = \frac{5}{12};
\]

\[
\sec(\angle A) = \frac{13}{12}; \quad \csc(\angle A) = \frac{13}{5}; \quad \cot(\angle A) = \frac{12}{5}.
\]

2. Now suppose \( a = 10 \) and \( b = 24 \). As before, find \( c \), \( \sin(\angle A) \), \( \cos(\angle A) \), \( \tan(\angle A) \), \( \sec(\angle A) \), \( \csc(\angle A) \), and \( \cot(\angle A) \). What do you notice? Can you explain your observations?

\[
c = \sqrt{10^2 + 24^2} = \sqrt{100 + 576} = 26.
\]

\[
\sin A = \frac{10}{26} = \frac{5}{13}; \quad \cos A = \frac{24}{26} = \frac{12}{13}; \quad \tan A = \frac{10}{24} = \frac{5}{12};
\]

\[
\sec A = \frac{26}{24} = \frac{13}{12}; \quad \csc A = \frac{10}{26} = \frac{5}{13}; \quad \cot A = \frac{24}{26} = \frac{12}{5}.
\]

They are the same; similar triangles have the same angles and the trig functions are well-defined functions of these angles.

Exercise 2

Consider the triangle drawn below.

Given that the two shorter sides of this right triangle have the same length, we know \( \angle A = 45^\circ \) and \( \angle B = 45^\circ \).
1. What are the length of the hypotenuse and the height of the triangle?

Let \( h = \text{height}, \ y = \text{hypotenuse}. \)

- Find \( y \): Pythagorean theorem. \( x^2 + x^2 = y^2 \)
  \[ 2x^2 = y^2 \rightarrow y = \sqrt{2}x \]
- Find \( h \): Use half-triangle. \( \left(\frac{y}{2}\right)^2 + h^2 = x^2 \)
  \[ \frac{y^2}{2} + h^2 = x^2 \Rightarrow h = \frac{x}{\sqrt{2}} \]

2. Use the information you have to find \( \sin(45^\circ) \), \( \cos(45^\circ) \) and \( \tan(45^\circ) \).

\[
\sin(45^\circ) = \frac{x}{y} \quad \text{or} \quad \frac{h}{x} \quad \tan(45^\circ) = \frac{x}{y} = 1
\]
\[
\cos(45^\circ) = \frac{y}{2} \quad \text{or} \quad \frac{x}{y} = \frac{1}{\sqrt{2}} \quad \frac{\sqrt{2}}{2}
\]

Exercise 3

1. Draw a diagram of the unit circle. Label the points on the unit circle corresponding to 0°, 90°, 180°, 135°, 225°, 150°, 270° based on standard position.

2. Use the identities you know about trigonometric functions of angles to fill in the following table.
Exercise 4

Consider the triangle below.

1. What is the length of the base of the triangle $ACD$?

$$\sqrt{5^2 - 3^2} = \sqrt{25 - 9} = \sqrt{16} = 4.$$

2. What is the length of the hypotenuse $x$ of the triangle $BCD$?

$$x = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}.$$

3. What is $\sin \angle DBC$?

$$\sin(\angle DBC) = \frac{3}{x} = \frac{3}{\sqrt{13}}.$$
4. Using the fact that \( \sin(\theta) = \sin(180° - \theta) \), what is \( \sin \angle ABC \)?

\[
\sin(\angle ABC) = \sin(180° - \angle ABC) = \sin(\angle DBC) = \frac{3}{\sqrt{13}}.
\]

**Exercise 5**

You place the end of a ladder on the ground 6 feet away from the wall of your house. The ladder points up at a 20° angle from the ground. Unfortunately, with the ladder at this angle and distance from the wall, its top only reaches 3/4 of the way up the wall.

1. Draw a picture of this situation.

2. How tall is the wall? (You don’t need to simplify your answers by evaluating any trigonometric functions.)

\[
\tan(20°) = \frac{(3/4)h}{6} \quad \Rightarrow \quad h = \frac{6 \cdot \tan(20°)}{3/4} = \frac{2 \tan(20°)}{1/4} = 8 \tan(20°) \text{ ft.}
\]

3. How long is the ladder? (Again, no need to simplify.)

\[
L = \sqrt{6^2 + (6 \tan(20°))^2} = \sqrt{6^2 + (6 \tan(20°))^2} = 6 \sqrt{\tan^2(20°) + 1} = 6 \sec(20°) \text{ ft.}
\]
4. At what distance from the wall do you need to place the bottom of the ladder so that the top exactly reaches the top of the wall? Estimate (a very rough estimate is fine here) the angle the ladder makes with the ground in this situation.

\[
d = \sqrt{(6 \sec(20°))^2 - (8 \tan(20°))^2}
\]

\[
\theta = \text{maybe } 30° \text{? Slightly bigger than } 20°.
\]

**Exercise 6**

Consider the triangle below, which is not a right triangle.

1. Find the height \( h \) in terms of the angle \( \theta \) and the side lengths \( a \) and \( b \).

\[ \sin \theta = \frac{h}{a} \implies h = a \sin \theta \]

2. Use the formula for the area of a triangle \( (A = \frac{\text{base} \cdot \text{height}}{2}) \) to find the area of this triangle.

\[ A = \frac{b \cdot a \sin \theta}{2} = \frac{ab \sin \theta}{2} \]

3. Suppose \( \theta = 30° \). Use what you know about 30-60-90 triangles to find a simpler expression for the area of this triangle.

\[ \sin \theta = \frac{1}{2} \implies A = \frac{ab \left( \frac{1}{2} \right)}{2} = \frac{ab}{4}. \]
Exercise 7

This exercise asks you to consider polygons inscribed in the unit circle.

1. What is the area of the unit circle? (That is, the circle of radius 1 with center at the origin (0, 0).)
   \[ \pi r^2 = \pi \]
   \[ r = 1 \]

2. Draw a picture of a square inscribed in the unit circle. That is, the square is contained inside the circle, with the four corners of the square touching the perimeter of the circle.

3. Notice that the length of half of the diagonal of the square is the same as the radius of the circle. Using this, what is the length of each side of the square?
   \[ \left( \frac{\sqrt{2}}{2} \right)^2 + \left( \frac{\sqrt{2}}{2} \right)^2 = 1 \rightarrow \frac{2l^2}{4} = 1 \]
   \[ \frac{l^2}{2} = 1 \Rightarrow l^2 = 2 \]
   \[ \Rightarrow l = \sqrt{2} \]

4. What is the area of the square? What percentage of the area of the circle is occupied by the square?
   \[ \text{area of square} = l^2 = (\sqrt{2})^2 = 2 \]
   \[ \text{percentage: } \frac{2}{\pi} \approx 0.64 \text{ or } 64\% \]
5. Now, draw a picture of a regular hexagon inscribed inside the unit circle.

6. (Bonus) What percentage of the area of the circle is taken up by the hexagon? (Hint: split the hexagon into triangles, and use the formula you obtained in the previous exercise.)

Note: we can divide the hexagon into six identical triangles (drawn above). The internal angle must be \( \frac{360^\circ}{6} \) since the angle all the way around is \( 360^\circ \) and we are dividing it into six parts.

\[
\frac{360}{6} = 60^\circ.
\]

Zoom in to one triangle:

\[
\text{From previous exercise: area of each triangle is } A = \frac{1}{2} \cdot \frac{1}{2} \cdot \sin(60^\circ) = \frac{\sqrt{3}/2}{2} = \frac{\sqrt{3}}{4}.
\]

\[
\text{Total area of hexagon: } A \cdot 6 = \frac{3 \sqrt{3}}{2} = \frac{3 \sqrt{3}}{2}.
\]

\[
\text{Percentage: } \frac{3 \sqrt{3}}{2} = 0.83 = 83\%.
\]

1 Additional Recommended Exercises

6.1 1-12, 25-48, 53-56, 58
6.2 1-33
6.3 1-13, 51-59, 61-68 is 180 of each are divided