Chapter 1

The Field of Reals and Beyond

Our goal with this section is to develop (review) the basic structure that characterizes the set of real numbers. Much of the material in the first section is a review of properties that were studied in MAT108; however, there are a few slight differences in the definitions for some of the terms. Rather than prove that we can get from the presentation given by the author of our MAT127A textbook to the previous set of properties, with one exception, we will base our discussion and derivations on the new set. As a general rule the definitions offered in this set of *Companion Notes* will be stated in symbolic form; this is done to reinforce the language of mathematics and to give the statements in a form that clarifies how one might prove satisfaction or lack of satisfaction of the properties. YOUR GLOSSARIES ALWAYS SHOULD CONTAIN THE (IN SYMBOLIC FORM) DEFINITION AS GIVEN IN OUR NOTES because that is the form that will be required for successful completion of literacy quizzes and exams where such statements may be requested.

1.1 Fields

Recall the following **DEFINITIONS:**

- The **Cartesian product** of two sets A and B, denoted by $A \times B$, is

$$\{(a, b) : a \in A \wedge b \in B\}.$$

- A **function** h from A into B is a subset of $A \times B$ such that

  (i) $(\forall a)\,[a \in A \Rightarrow (\exists b)\,(b \in B \wedge (a, b) \in h)]$; i.e., $\operatorname{dom} h = A$, and

  (ii) $(\forall a)\,(\forall b)\,(\forall c)\,[(a, b) \in h \wedge (a, c) \in h \Rightarrow b = c]$; i.e., h is single-valued.

- A **binary operation** on a set A is a function from $A \times A$ into A.

- A **field** is an algebraic structure, denoted by $(\mathbb{F}, +, \cdot, e, f)$, that includes a set of objects, $\mathbb{F}$, and two binary operations, addition $(+)$ and multiplication $(\cdot)$, that satisfy the Axioms of Addition, Axioms of Multiplication, and the Distributive Law as described in the following list.

  (A) **Axioms of Addition** ($(\mathbb{F}, +, e)$ is a commutative group under the binary operation of addition $(+)$ with the additive identity denoted by e);

  (A1) $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

  (A2) $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow (x + y = y + x))$ (commutative with respect to addition)

  (A3) $(\forall x)\,(\forall y)\,(\forall z)\,\big(x, y, z \in \mathbb{F} \Rightarrow \big[(x + y) + z = x + (y + z)\big]\big)$ (associative with respect to addition)

  (A4) $(\exists e)\,[e \in \mathbb{F} \wedge (\forall x)\,(x \in \mathbb{F} \Rightarrow x + e = e + x = x)]$ (additive identity property)

  (A5) $(\forall x)\,(x \in \mathbb{F} \Rightarrow (\exists\,(-x))\,[(-x) \in \mathbb{F} \wedge (x + (-x) = (-x) + x = e)])$ (additive inverse property)

  (M) **Axioms of Multiplication** ($(\mathbb{F}, \cdot, f)$ is a commutative group under the binary operation of multiplication $(\cdot)$ with the multiplicative identity denoted by f);

  (M1) $\cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$

  (M2) $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow (x \cdot y = y \cdot x))$ (commutative with respect to multiplication)

  (M3) $(\forall x)\,(\forall y)\,(\forall z)\,\big(x, y, z \in \mathbb{F} \Rightarrow \big[(x \cdot y) \cdot z = x \cdot (y \cdot z)\big]\big)$ (associative with respect to multiplication)

  (M4) $(\exists f)\,\big[f \in \mathbb{F} \wedge f \neq e \wedge (\forall x)\,(x \in \mathbb{F} \Rightarrow x \cdot f = f \cdot x = x)\big]$ (multiplicative identity property)

  (M5) $(\forall x)\,(x \in \mathbb{F} - \{e\} \Rightarrow$
  $$\big[\big(\exists\,\big(x^{-1}\big)\big)\,\big(x^{-1} \in \mathbb{F} \wedge (x \cdot (x^{-1}) = (x^{-1}) \cdot x = f\big)\big])$$
  (multiplicative inverse property)

(D) **The Distributive Law**

$$(\forall x)\,(\forall y)\,(\forall z)\,(x, y, z \in \mathbb{F} \Rightarrow [x \cdot (y + z) = (x \cdot y) + (x \cdot z)])$$

Remark 1.1.1 *Properties (A1) and (M1) tell us that $\mathbb{F}$ is closed under addition and closed under multiplication, respectively.*

Remark 1.1.2 *The additive identity and multiplicative identity properties tell us that a field has at least two elements; namely, two distinct identities. To see that two elements is enough, note that, for $\mathbb{F} = \{0, 1\}$, the algebraic structure $(\mathbb{F}, \oplus, \otimes, 0, 1)$ where $\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and $\otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ are defined by the following tables:*

$\oplus$	0	1
0	0	1
1	1	0

$\otimes$	0	1
0	0	0
1	0	1

,

is a field.

Remark 1.1.3 *The fields with which you are probably the most comfortable are the rationals $(\mathbb{Q}, +, \cdot, 0, 1)$ and the reals $(\mathbb{R}, +, \cdot, 0, 1)$. A field that we will discuss shortly is the complex numbers $(\mathbb{C}, +, \cdot, (0, 0), (1, 0))$ Since each of these distinctly different sets satisfy the same list of field properties, we will expand our list of properties in search of ones that will give us distinguishing features.*

When discussing fields, we should distinguish that which can be claimed as a basic field property ((A),(M), and (D)) from properties that can (and must) be proved from the basic field properties. For example, given that $(\mathbb{F}, +, \cdot)$ is a field, we can claim that $(\forall x)\,(\forall y)\,(x, y \in \mathbb{F} \Rightarrow x + y \in \mathbb{F})$ as an alternative description of property (A1) while we can not claim that additive inverses are unique. The latter observation is important because it explains why we can't claim $e = w$ from $(\mathbb{F}, +, \cdot, e, f)$ being a field and $x + w = x + e = x$; we don't have anything that allows us to "subtract from both sides of an equation". The relatively small number of properties that are offered in the definition of a field motivates our search for additional properties of fields that can be proved using only the basic field properties and elementary logic. In general, we don't claim as axioms that which can be proved from the "minimal" set of axioms that comprise the definition of a field. We will list some properties that require proof and offer some proofs to illustrate an approach to doing such proofs. A slightly different listing of properties with proofs of the properties is offered in Rudin.

Proposition 1.1.4 *Properties for the Additive Identity of a field* $(\mathbb{F}, +, \cdot, e, f)$

1. $(\forall x)\,(x \in \mathbb{F} \wedge x + x = x \Rightarrow x = e)$
2. $(\forall x)\,(x \in \mathbb{F} \Rightarrow x \cdot e = e \cdot x = e)$
3. $(\forall x)\,(\forall y)\left[(x, y \in \mathbb{F} \wedge x \cdot y = e) \Rightarrow (x = e \vee y = e)\right]$

Proof. (of #1) Suppose that $x \in \mathbb{F}$ satisfies $x + x = x$. Since $x \in \mathbb{F}$, by the additive inverse property, $-x \in \mathbb{F}$ is such that $x + -x = -x + x = e$. Now by substitution and the associativity of addition,

$$e = x + (-x) = (x + x) + (-x) = x + (x + -x) = x + e = x.$$

(of #3) Suppose that $x, y \in \mathbb{F}$ are such that $x \cdot y = e$ and $x \neq e$. Then, by the multiplicative inverse property, $x^{-1} \in F$ satisfies $x \cdot x^{-1} = x^{-1} \cdot x = f$. Then substitution, the associativity of multiplication, and #2 yields that

$$y = f \cdot y = \left(x^{-1} \cdot x\right) \cdot y = x^{-1} \cdot (x \cdot y) = x^{-1} \cdot e = e.$$

Hence, for $x, y \in \mathbb{F}$, $x \cdot y = e \wedge x = e$ implies that $y = e$. The claim now follows immediately upon noting that, for any propositions P, Q, and M, $[P \Rightarrow (Q \vee M)]$ is logically equivalent to $[(P \wedge \neg Q) \Rightarrow M]$. ■

Excursion 1.1.5 *Use #1 to prove #2.*

The key here was to work from $x \cdot e = x\,(e + e)$.

Proposition 1.1.6 *Uniqueness of Identities and Inverses for a field* $(\mathbb{F}, +, \cdot, e, f)$

1. *The additive identity of a field is unique.*

2. *The multiplicative identity of a field is unique.*

3. *The additive inverse of any element in* $\mathbb{F}$ *is unique.*

4. *The multiplicative inverse of any element in* $\mathbb{F} - \{e\}$ *is unique.*

Proof. (of #1) Suppose that $\omega \in \mathbb{F}$ is such that

$$(\forall x)\,(x \in \mathbb{F} \Rightarrow x + \omega = \omega + x = x)\,.$$

In particular, since $e \in \mathbb{F}$, we have that $e = e + \omega$. Since e is given as an additive identity and $\omega \in \mathbb{F}$, $e + \omega = \omega$. From the transitivity of equals, we conclude that $e = \omega$. Therefore, the additive identity of a field is unique.

(of #3) Suppose that $a \in \mathbb{F}$ is such that there exists $w \in \mathbb{F}$ and $x \in \mathbb{F}$ satisfying

$$a + w = w + a = e \qquad and \qquad a + x = x + a = e.$$

From the additive identity and associative properties

$$\begin{aligned} w = w + e &= w + (a + x) \\ &= (w + a) + x \\ &= e + x \\ &= x. \end{aligned}$$

Since a was arbitrary, we conclude that the additive inverse of each element in a field is unique. ■

Excursion 1.1.7 *Prove #4.*

*******Completing this excursion required only appropriate modification of the proof that was offered for #3. You needed to remember to take you arbitrary element in F to not be the additive identity and then simply change the operation to multiplication. Hopefully, you remembered to start with one of the inverses of your arbitrary element and work to get it equal to the other one.*******

Proposition 1.1.8 *Sums and Products Involving Inverses for a field* $(\mathbb{F}, +, \cdot, e, f)$

1. $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow -(a+b) = (-a) + (-b))$
2. $(\forall a)(a \in F \Rightarrow -(-a) = a)$
3. $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow a \cdot (-b) = -(a \cdot b))$
4. $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot b = -(a \cdot b))$
5. $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot (-b) = a \cdot b)$
6. $(\forall a)\left(a \in \mathbb{F} - \{e\} \Rightarrow \left(a^{-1} \neq e \wedge \left[a^{-1}\right]^{-1} = a \wedge -\left(a^{-1}\right) = (-a)^{-1}\right)\right)$
7. $(\forall a)(\forall b)\left(a, b \in \mathbb{F} - \{e\} \Rightarrow (a \cdot b)^{-1} = \left(a^{-1}\right)\left(b^{-1}\right)\right)$

Proof. *(of #2) Suppose that $a \in \mathbb{F}$. By the additive inverse property $-a \in \mathbb{F}$ and $-(-a) \in \mathbb{F}$ is the additive inverse of $-a$; i.e., $-(-a) + (-a) = e$. Since $-a$ is the additive inverse of a, $(-a) + a = a + (-a) = e$ which also justifies that a is an additive inverse of $-a$. From the uniqueness of additive inverses (Proposition 1.1.6), we conclude that $-(-a) = a$.* ■

Excursion 1.1.9 *Fill in what is missing in order to complete the following proof of #6.*

Proof. *Suppose that $a \in \mathbb{F} - \{e\}$. From the multiplicative inverse property, $a^{-1} \in \mathbb{F}$ satisfies $\underset{(1)}{\underline{\hspace{6cm}}}$. If $a^{-1} = e$, then, by Proposition 1.1.4(#2), $a^{-1} \cdot a = e$. Since multiplication is single-valued, this would imply that $\underset{(2)}{\underline{\hspace{3cm}}}$ which contradicts part of the $\underset{(3)}{\underline{\hspace{6cm}}}$ property. Thus, $a^{-1} \neq e$.*

Since $a^{-1} \in \mathbb{F} - \{e\}$, by the $\underset{(4)}{\underline{\hspace{5cm}}}$ property, $\left(a^{-1}\right)^{-1} \in \mathbb{F}$ and satisfies $\left(a^{-1}\right)^{-1} \cdot a^{-1} = a^{-1} \cdot \left(a^{-1}\right)^{-1} = f$; but this equation also justifies that $\left(a^{-1}\right)^{-1}$ is a multiplicative inverse for a^{-1}. From Proposition $\underset{(5)}{\underline{\hspace{3cm}}}$, we conclude that $\left(a^{-1}\right)^{-1} = a$.

From (#5), $\left(-\left(a^{-1}\right)\right)\cdot(-a) = a^{-1}\cdot a = f$ *from which we conclude that* $-\left(a^{-1}\right)$ *is a* $\underset{(6)}{\underline{\qquad\qquad\qquad\qquad}}$ *for* $-a$*. Since* $(-a)^{-1}$ *is a multiplicative inverse for* $(-a)$ *and multiplicative inverses are unique, we have that* $-\left(a^{-1}\right) = (-a)^{-1}$ *as claimed.* ■

Acceptable responses are: (1) $a\cdot a^{-1} = f$, (2) $e = f$, (3) multiplicative identity, (4) multiplicative inverse, (5) 1.1.6(#4), and (6) multiplicative inverse.

Proposition 1.1.10 *Solutions to Linear Equations. Given a field* $(\mathbb{F},+,\cdot,0,1)$*,*

1. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow(\exists!x)\,(x\in\mathbb{F}\wedge a+x=b))$
2. $(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\wedge a\neq 0\Rightarrow(\exists!x)\,(x\in\mathbb{F}\wedge a\cdot x=b))$

Proof. *(of #1) Suppose that* $a,b\in\mathbb{F}$ *and* $a\neq 0$*. Since* $a\in\mathbb{F}-\{0\}$ *there exists* $a^{-1}\in\mathbb{F}$ *such that* $a\cdot a^{-1} = a^{-1}\cdot a = 1$*. Because* $a^{-1}\in\mathbb{F}$ *and* $b\in\mathbb{F}$*,* $x \underset{def}{=} a^{-1}\cdot b\in\mathbb{F}$ *from (M1). Substitution and the associativity of multiplication yield that*

$$a\cdot x = a\cdot\left(a^{-1}\cdot b\right) = \left(a\cdot a^{-1}\right)\cdot b = 1\cdot b = b.$$

Hence, x *satisfies* $a\cdot x=b$*. Now, suppose that* $w\in\mathbb{F}$ *also satisfies* $a\cdot w=b$*. Then*

$$w = 1\cdot w = \left(a^{-1}\cdot a\right)\cdot w = a^{-1}\cdot(a\cdot w) = a^{-1}\cdot b = x.$$

Since a *and* b *were arbitrary,*

$$(\forall a)\,(\forall b)\,(a,b\in\mathbb{F}\Rightarrow(\exists!x)\,(x\in\mathbb{F}\wedge a+x=b)).$$

■

Remark 1.1.11 *As a consequence of Proposition 1.1.10, we now can claim that, if* $x,w,z\in\mathbb{F}$ *and* $x+w=x+z$*, then* $w=z$ *and; if* $w,z\in\mathbb{F}$*,* $x\in\mathbb{F}-\{0\}$ *and* $w\cdot x=z\cdot x$*, then* $w=z$*. The justification is the uniqueness of solutions to linear equations in a field. In terms of your previous experience with elementary algebraic manipulations used to solve equations, the proposition justifies what is commonly referred to as "adding a real number to both sides of an equation" and "dividing both sides of an equation by a nonzero real number."*

Proposition 1.1.12 *Addition and Multiplication Over Fields Containing Three or More Elements. Suppose that* $(\mathbb{F}, +, \cdot)$ *is a field and* $a, b, c, d \in \mathbb{F}$*. Then*

1. $a + b + c = a + c + b = \cdots = c + b + a$
2. $a \cdot b \cdot c = a \cdot c \cdot b = \cdots = c \cdot b \cdot a$
3. $(a + c) + (b + d) = (a + b) + (c + d)$
4. $(a \cdot c) \cdot (b \cdot d) = (a \cdot b) \cdot (c \cdot d)$

Proposition 1.1.13 *Multiplicative Inverses in a field* $(\mathbb{F}, +, \cdot, 0, 1)$

1. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\left[\begin{array}{l} (a, b, c, d \in \mathbb{F} \wedge b \neq 0 \wedge d \neq 0) \\ \Rightarrow b \cdot d \neq 0 \wedge \left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right) = (a \cdot c) \cdot (b \cdot d)^{-1} \end{array}\right]$
2. $(\forall a)\,(\forall b)\,(\forall c)\left[(a, b, c \in \mathbb{F} \wedge c \neq 0) \Rightarrow \left(a \cdot c^{-1}\right) + \left(b \cdot c^{-1}\right) = (a + b) \cdot c^{-1}\right]$
3. $(\forall a)\,(\forall b)\,[(a, b \in \mathbb{F} \wedge b \neq 0) \Rightarrow \left((-a) \cdot b^{-1}\right) = \left(a \cdot (-b)^{-1}\right) = -\left(a \cdot b^{-1}\right)]$
4. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a \in \mathbb{F} \wedge b, c, d \in \mathbb{F} - \{0\}) \Rightarrow c \cdot d^{-1} \neq 0$
$\wedge \left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right)^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1} = \left(a \cdot b^{-1}\right) \cdot \left(d \cdot c^{-1}\right)]$
5. $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a, c \in \mathbb{F} \wedge b, d \in \mathbb{F} - \{0\}) \Rightarrow b \cdot d \neq 0 \wedge$
$\left(a \cdot b^{-1}\right) + \left(c \cdot d^{-1}\right) = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1}]$

Proof. *(of #3) Suppose* $a, b \in \mathbb{F}$ *and* $b \neq 0$*. Since* $b \neq 0$*, the zero of the field is its own additive inverse, and additive inverses are unique, we have that* $-b \neq 0$*. Since* $a \in \mathbb{F}$ *and* $b \in \mathbb{F} - \{0\}$ *implies that* $-a \in \mathbb{F}$ *and* $b^{-1} \in \mathbb{F} - \{0\}$*, by Proposition 1.1.8(#4),* $(-a) \cdot b^{-1} = -\left(a \cdot b^{-1}\right)$*. From Proposition 1.1.8(#6), we know that* $-\left(b^{-1}\right) = (-b)^{-1}$*. From the distributive law and Proposition 1.1.8(#2),*

$$a \cdot (-b)^{-1} + a \cdot b^{-1} = a \cdot \left((-b)^{-1} + b^{-1}\right) = a \cdot \left(-\left(b^{-1}\right) + b^{-1}\right) = a \cdot 0 = 0$$

from which we conclude that $a \cdot (-b)^{-1}$ *is an additive inverse for* $a \cdot b^{-1}$*. Since additive inverses are unique, it follows that* $a \cdot (-b)^{-1} = -\left(a \cdot b^{-1}\right)$*. Combining our results yields that*

$$(-a) \cdot b^{-1} = -\left(a \cdot b^{-1}\right) = a \cdot (-b)^{-1}$$

as claimed. ■

Excursion 1.1.14 *Fill in what is missing in order to complete the following proof of #4.*

Proof. *(of #4) Suppose that $a \in \mathbb{F}$ and $b, c, d \in \mathbb{F} - \{0\}$. Since $d \in \mathbb{F} - \{0\}$, by Proposition $\underset{(1)}{\underline{\qquad\quad}}$, $d^{-1} \neq 0$. From the contrapositive of Proposition 1.1.4(#3), $c \neq 0$ and $d^{-1} \neq 0$ implies that $\underset{(2)}{\underline{\qquad\qquad\qquad}}$. In the following, the justifications for the step taken is provided on the line segment to the right of the change that has been made.*

$$
\begin{aligned}
\left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right)^{-1} &= \left(a \cdot b^{-1}\right) \cdot \left(c^{-1} \cdot \left(d^{-1}\right)^{-1}\right) && \underset{(3)}{\underline{\qquad\qquad}} \\
&= \left(a \cdot b^{-1}\right) \cdot \left(c^{-1} \cdot d\right) && \underset{(4)}{\underline{\qquad\qquad}} \\
&= a \cdot \left(b^{-1} \cdot \left(c^{-1} \cdot d\right)\right) && \underset{(5)}{\underline{\qquad\qquad}} \\
&= a \cdot \left(\left(b^{-1} \cdot c^{-1}\right) \cdot d\right) && \underset{(6)}{\underline{\qquad\qquad}} \\
&= a \cdot \left(d \cdot \left(b^{-1} \cdot c^{-1}\right)\right) && \underset{(7)}{\underline{\qquad\qquad}} \\
&= a \cdot \left(d \cdot (b \cdot c)^{-1}\right) && \underset{(8)}{\underline{\qquad\qquad}} \\
&= (a \cdot d) \cdot (b \cdot c)^{-1}. && \underset{(9)}{\underline{\qquad\qquad}}
\end{aligned}
$$

From Proposition 1.1.8(#7) combined with the associative and commutative properties of addition we also have that

$$
\begin{aligned}
(a \cdot d) \cdot (b \cdot c)^{-1} &= (a \cdot d) \cdot \left(b^{-1} \cdot c^{-1}\right) \\
&= \left((a \cdot d) \cdot b^{-1}\right) \cdot c^{-1} \\
&= \left(a \cdot \left(d \cdot b^{-1}\right)\right) \cdot c^{-1} \\
&= \underset{(10)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}} \\
&= \left(\left(a \cdot b^{-1}\right) \cdot d\right) \cdot c^{-1} \\
&= \left(a \cdot b^{-1}\right) \cdot \left(d \cdot c^{-1}\right).
\end{aligned}
$$

Consequently, $\left(a \cdot b^{-1}\right) \cdot \left(c \cdot d^{-1}\right)^{-1} = (a \cdot d) \cdot (b \cdot c)^{-1} = \left(a \cdot b^{-1}\right) \cdot \left(d \cdot c^{-1}\right)$ as claimed. ■

*******Acceptable responses are: (1) 1.1.8(#6), (2) $c \cdot d^{-1} \neq 0$, (3) Proposition 1.1.8(#7), (4) Proposition 1.1.8(#6), (5) associativity of multiplication, (6) associativity of multiplication, (7) commutativity of multiplication, (8) Proposition 1.1.8(#7), (9) associativity of multiplication, (10) $\left(a\cdot\left(b^{-1}\cdot d\right)\right)\cdot c^{-1}$.***

The list of properties given in the propositions is, by no means, exhaustive. The propositions illustrate the kinds of things that can be concluded (proved) from the core set of basic field axioms.

Notation 1.1.15 *We have listed the properties without making use of some **notational conventions** that can make things look simpler. The two that you might find particularly helpful are that*

- the expression $a+(-b)$ may be written as $a-b$; $(-a)+(-b)$ may be written as $-a-b$; and
- the expression $a\cdot b^{-1}$ may be written as $\dfrac{a}{b}$. (Note that applying this notational convention to the Properties of Multiplicative Inverses stated in the last proposition can make it easier for you to remember those properties.)

Excursion 1.1.16 *On the line segments provided, fill in appropriate justifications for the steps given in the following outline of a proof that for a, b, c, d in a field, $(a+b)-(c-d)=(a-c)+(b+d)$.*

Observation		*Justification*
$(a+b)-(c-d)=(a+b)+(-(c+(-d)))$		*notational convention*
$(a+b)+(-(c+(-d)))=(a+b)+((-c)+(-(-d)))$		________ (1)
$(a+b)+((-c)+(-(-d)))=(a+b)+((-c)+d)$		________ (2)
$(a+b)+((-c)+d)=a+(b+((-c)+d))$		________ (3)
$a+(b+((-c)+d))=a+((b+(-c))+d)$		________ (4)
$a+((b+(-c))+d)=a+(((-c)+b)+d)$		________ (5)
$a+(((-c)+b)+d)=a+((-c)+(b+d))$		________ (6)
$a+((-c)+(b+d))=(a+(-c))+(b+d)$		________ (7)
$(a+(-c))+(b+d)=(a-c)+(b+d)$		________ (8)

Acceptable responses are: (1) Proposition 1.1.8(#1), (2) Proposition 1.1.8(#2), (3) and (4) associativity of addition, (5) commutativity of addition, (6) and (7) associativity of addition, and (8) notational convention.

1.2 Ordered Fields

Our basic field properties and their consequences tell us how the binary operations function and interact. The set of basic field properties doesn't give us any means of comparison of elements; more structure is needed in order to formalize ideas such as "positive elements in a field" or "listing elements in a field in increasing order." To do this we will introduce the concept of an ordered field.

Recall that, for any set S, a **relation on** S is any subset of $S \times S$

Definition 1.2.1 *An **order**, denoted by $<$, **on a set** S is a relation on S that satisfies the following two properties:*

1. ***The Trichotomy Law**: If $x \in S$ and $y \in S$, then one and only one of*

$$(x < y) \text{ or } (x = y) \text{ or } (y < x)$$

is true.

2. ***The Transitive Law**: $(\forall x)\,(\forall y)\,(\forall z)\left[x, y, z \in S \wedge x < y \wedge y < z \Rightarrow x < z\right]$.*

Remark 1.2.2 *Satisfaction of the Trichotomy Law requires that*

$$(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow (x = y) \vee (x < y) \vee (y < x))$$

be true and that each of

$$\begin{aligned}
&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((x = y) \Rightarrow \neg\,(x < y) \wedge \neg\,(y < x)))\,,\\
&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((x < y) \Rightarrow \neg\,(x = y) \wedge \neg\,(y < x)))\,, \text{ and}\\
&(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow ((y < x) \Rightarrow \neg\,(x = y) \wedge \neg\,(x < y)))
\end{aligned}$$

be true. The first statement, $(\forall x)\,(\forall y)\,(x, y \in S \Rightarrow (x = y) \vee (x < y) \vee (y < x))$ is not equivalent to the Trichotomy Law because the disjunction is not mutually exclusive.

Example 1.2.3 *For $S = \{a, b, c\}$ with a, b, and c distinct, $< = \{(a, b), (b, c), (a, c)\}$ is an order on S. The notational convention for $(a, b) \in <$ is $a < b$. The given ordering has the minimum and maximum number of ordered pairs that is needed to meet the definition. This is because, given any two distinct elements of S, x and y, we must have one and only one of $(x, y) \in <$ or $(y, x) \in <$. After making free choices of two ordered pairs to go into an acceptable ordering for S, the choice of the third ordered pair for inclusion will be determined by the need to have the Transitive Law satisfied.*

Remark 1.2.4 *The definition of a particular order on a set S is, to a point, up to the definer. You can choose elements of $S \times S$ almost by preference until you start having enough elements to force the choice of additional ordered pairs in order to meet the required properties. In practice, orders are defined by some kind of formula or equation.*

Example 1.2.5 *For $\mathbb{Q}$, the set of rationals, let $< \subset \mathbb{Q} \times \mathbb{Q}$ be defined by $(r, s) \in < \Leftrightarrow (s + (-r))$ is a positive rational. Then $(\mathbb{Q}, <)$ is an ordered set.*

Remark 1.2.6 *The treatment of ordered sets that you saw in MAT108 derived the Trichotomy Law from a set of properties that defined a linear order on a set. Given an order $<$ on a set, we write $x \leq y$ for $(x < y) \vee x = y$. With this notation, the two linear ordering properties that could have been introduced and used to prove the Trichotomy Law are the Antisymmetric law,*

$$(\forall x)(\forall y)((x, y \in S \wedge (x, y) \in \leq \wedge (y, x) \in \leq) \Rightarrow x = y),$$

and the Comparability Law,

$$(\forall x)(\forall y)(x, y \in S \Rightarrow ((x, y) \in \leq \vee (y, x) \in \leq)).$$

Now, because we have made satisfaction of the Trichotomy Law part of the definition of an order on a set, we can claim that the Antisymmetric Law and the Comparability Law are satisfied for an ordered set.

Definition 1.2.7 *An* ***ordered field*** $(\mathbb{F}, +, \cdot, 0, 1, <)$ *is an ordered set that satisfies the following two properties.*

(OF1) $(\forall x)(\forall y)(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \Rightarrow x + z < y + z\right]$

(OF2) $(\forall x)(\forall y)(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge 0 < z \Rightarrow x \cdot z < y \cdot z\right]$

Remark 1.2.8 *In the definition of ordered field offered here, we have deviated from one of the statements that is given in our text. The second condition given in the text is that*

$$(\forall x)(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right];$$

let's denote this proposition by $(altOF2)$. *We will show that satisfaction of* $(OF1)$ *and* $(altOF2)$ *is, in fact, equivalent to satisfaction of* $(OF1)$ *and* $(OF2)$. *Suppose that* $(OF1)$ *and* $(OF2)$ *are satisfied and let* $x, y \in \mathbb{F}$ *be such that* $0 < x$ *and* $0 < y$. *From* $(OF2)$ *and Proposition 1.1.4(#2),* $0 = 0 \cdot y < x \cdot y$. *Since x and y were arbitrary, we conclude that* $(\forall x)(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right]$. *Hence,* $(OF2) \Rightarrow (altOF2)$ *from which we have that* $(OF1) \wedge (OF2) \Rightarrow (OF1) \wedge (altOF2)$. *Suppose that* $(OF1)$ *and* $(altOF2)$ *are satisfied and let* $x, y, z \in \mathbb{F}$ *be such that* $x < y$ *and* $0 < z$. *From the additive inverse property* $(-x) \in \mathbb{F}$ *is such that* $[x + (-x) = (-x) + x = 0]$. *From* $(OF1)$ *we have that*

$$0 = x + (-x) < y + (-x).$$

From $(altOF2)$, *the Distributive Law and Proposition 1.1.8 (#4),* $0 < y + (-x)$ *and* $0 < z$ *implies that*

$$0 < (y + (-x)) \cdot z = (y \cdot z) + ((-x) \cdot z) = (y \cdot z) + (-(x \cdot z)).$$

Because $+$ *and* $\cdot$ *are binary operations on* $\mathbb{F}$, $x \cdot z \in \mathbb{F}$ *and* $(y \cdot z) + (-(x \cdot z)) \in \mathbb{F}$. *It now follows from* $(OF1)$ *and the associative property of addition that*

$$0 + x \cdot z < ((y \cdot z) + (-(x \cdot z))) + x \cdot z = (y \cdot z) + (-(x \cdot z) + x \cdot z) = y \cdot z + 0.$$

Hence, $x \cdot z < y \cdot z$. *Since* x, y, *and* z *were arbitrary, we have shown that*

$$(\forall x)(\forall y)(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge 0 < z \Rightarrow x \cdot z < y \cdot z\right]$$

which is $(OF2)$. *Therefore,* $(OF1) \wedge (altOF2) \Rightarrow (OF1) \wedge (OF2)$. *Combining the implications yields that*

$$(OF1) \wedge (OF2) \Leftrightarrow (OF1) \wedge (altOF2) \text{ as claimed.}$$

To get from the requirements for a field to the requirements for an ordered field we added a binary relation (a description of how the elements of the field are ordered or comparable) and four properties that describe how the order and the binary operations "interact." The following proposition offers a short list of other order properties that follow from the basic set.

Proposition 1.2.9 *Comparison Properties Over Ordered Fields.*
For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$ *we have each of the following.*

1. $0 < 1$
2. $(\forall x)(\forall y)\left[x, y \in \mathbb{F} \wedge x > 0 \wedge y > 0 \Rightarrow x \cdot y > 0\right]$
3. $(\forall x)\left[x \in \mathbb{F} \wedge x > 0 \Rightarrow (-x) < 0\right]$
4. $(\forall x)(\forall y)\left[x, y \in \mathbb{F} \wedge x < y \Rightarrow -y < -x\right]$
5. $(\forall x)(\forall y)(\forall z)\left[x, y, z \in \mathbb{F} \wedge x < y \wedge z < 0 \Rightarrow x \cdot z > y \cdot z\right]$
6. $(\forall x)\left[x \in \mathbb{F} \wedge x \neq 0 \Rightarrow x \cdot x = x^2 > 0\right]$
7. $(\forall x)(\forall y)\left[x, y \in \mathbb{F} \wedge 0 < x < y \Rightarrow 0 < y^{-1} < x^{-1}\right]$

In the Remark 1.2.8, we proved the second claim. We will prove two others. Proofs for all but two of the statements are given in our text.

Proof. (or #1) By the Trichotomy Law one and only one of $0 < 1$, $0 = 1$, or $1 < 0$ is true in the field. From the multiplicative identity property, $0 \neq 1$; thus, we have one and only one of $0 < 1$ or $1 < 0$. Suppose that $1 < 0$. From $OF1$, we have that $0 = 1 + (-1) < 0 + (-1) = -1$; i.e., $0 < -1$. Hence, $OF2$ implies that $(1) \cdot (-1) < (0) \cdot (-1)$ which, by Proposition 1.1.8(#3), is equivalent to $-1 < 0$. But, from the transitivity property, $0 < -1 \wedge -1 < 0 \Rightarrow 0 < 0$ which is a contradiction. ■

Excursion 1.2.10 *Fill in what is missing in order to complete the following proof of Proposition 1.2.9(#4).*

Proof. *Suppose that* $x, y \in \mathbb{F}$ *are such that* $x < y$. *In view of the additive inverse property,* $-x \in \mathbb{F}$ *and* $-y \in \mathbb{F}$ *satisfy*

$$-x + x = x + -x = 0 \qquad \text{and} \qquad \underset{(1)}{\underline{\hspace{8cm}}}.$$

From $\underset{(2)}{\underline{\hspace{4cm}}}$, $0 = x + -x < y + -x$; *i.e.,* $\underset{(3)}{\underline{\hspace{8cm}}}$ *and* $0 + -y < \left(\underset{(4)}{\underline{\hspace{4cm}}}\right) + -y$. *Repeated use of commutativity and associativity allows us to conclude that* $(y + -x) + -y = -x$. *Hence* $-y < -x$ *as claimed.* ■

Acceptable responses are: (1) $-y+y = y+-y = 0$, (2) OF1, (3) $0 < y+-x$, (4) $y+-x$.

Remark 1.2.11 *From Proposition 1.2.9(#1) we see that the two additional properties needed to get from an ordered set to an ordered field led to the requirement that* $(0,1)$ *be an element of the ordering (binary relation). From* $0 < 1$ *and* $(OF1)$, *we also have that* $1 < 1+1 = 2$, $2 < 2+1 = 3$; *etc. Using the convention* $\underbrace{1+1+1\cdots+1}_{n\,of\,them} = n$, *the general statement becomes* $0 < n < n+1$.

1.2.1 Special Subsets of an Ordered Field

There are three special subsets of any ordered field that are isolated for special consideration. We offer their formal definitions here for completeness and perspective.

Definition 1.2.12 *Let* $(\mathbb{F}, +, \cdot, 0, 1, \leq)$ *be an ordered field. A subset* S *of* $\mathbb{F}$ *is said to be* ***inductive*** *if and only if*

1. $1 \in S$ *and*
2. $(\forall x)\,(x \in S \Rightarrow x+1 \in S)$.

Definition 1.2.13 *For* $(\mathbb{F}, +, \cdot, 0, 1, \leq)$ *an ordered field, define*

$$\mathbb{N}_{\mathbb{F}} = \bigcap_{S \in \mathfrak{S}} S$$

where $\mathfrak{S} = \{S \subseteq \mathbb{F} : S \text{ is inductive}\}$. *We will call* $\mathbb{N}_{\mathbb{F}}$ *the set of* ***natural numbers*** *of* $\mathbb{F}$.

Note that, $T = \{x \in \mathbb{F} : x \geq 1\}$ is inductive because $1 \in T$ and closure of $\mathbb{F}$ under addition yields that $x+1 \in \mathbb{F}$ whenever $x \in \mathbb{F}$. Because $(\forall u)\,(u < 1 \Rightarrow u \notin T)$ and $T \in \mathfrak{S}$, we immediately have that any $n \in \mathbb{N}_{\mathbb{F}}$ satisfies $n \geq 1$.

Definition 1.2.14 *Let* $(\mathbb{F}, +, \cdot, 0, 1, \leq)$ *be an ordered field. The set of* ***integers of*** $\mathbb{F}$, *denoted* $\mathbb{Z}_{\mathbb{F}}$, *is*

$$\mathbb{Z}_{\mathbb{F}} = \{a \in \mathbb{F} : a \in \mathbb{N}_{\mathbb{F}} \vee \; -a \in \mathbb{N}_{\mathbb{F}} \vee a = 0\}.$$

It can be proved that both the natural numbers of a field and the integers of a field are closed under addition and multiplication. That is,

$$(\forall m)\,(\forall n)\,(n \in \mathbb{N}_{\mathbb{F}} \wedge m \in \mathbb{N}_{\mathbb{F}} \Rightarrow n+m \in \mathbb{N}_{\mathbb{F}} \wedge n \cdot m \in \mathbb{N}_{\mathbb{F}})$$

and

$$(\forall m)\,(\forall n)\,(n \in \mathbb{Z}_{\mathbb{F}} \wedge m \in \mathbb{Z}_{\mathbb{F}} \Rightarrow n+m \in \mathbb{Z}_{\mathbb{F}} \wedge n \cdot m \in \mathbb{Z}_{\mathbb{F}})\,.$$

This claim requires proof because the fact that addition and multiplication are binary operations on $\mathbb{F}$ only places $n+m$ and $n \cdot m$ in $\mathbb{F}$ because $\mathbb{N}_{\mathbb{F}} \subset \mathbb{F}$ and $\mathbb{Z}_{\mathbb{F}} \subset \mathbb{F}$.

Proofs of the closure of $\mathbb{N}_{\mathbb{F}} = \mathbb{N}$ under addition and multiplication that you might have seen in MAT108 made use of the Principle of Mathematical Induction. This is a useful tool for proving statements involving the natural numbers.

PRINCIPLE OF MATHEMATICAL INDUCTION (PMI). If S is an inductive set of natural numbers, then $S = \mathbb{N}$.

In MAT108, you should have had lots of practice using the Principle of Mathematical Induction to prove statements involving the natural numbers. Recall that to do this, you start the proof by defining a set S to be the set of natural numbers for which a given statement is true. Once we show that $1 \in S$ and $(\forall k)\,(k \in S \Rightarrow (k+1) \in S)$, we observe that S is an inductive set of natural numbers. Then we conclude, by the Principle of Mathematical Induction, that $S = \mathbb{N}$ which yields that the given statement is true for all $\mathbb{N}$.

Two other principles that are logically equivalent to the Principle of Mathematical Induction and still useful for some of the results that we will be proving in this course are the Well-Ordering Principle and the Principle of Complete Induction:

WELL-ORDERING PRINCIPLE (WOP). Any nonempty set S of natural numbers contains a smallest element.

PRINCIPLE OF COMPLETE INDUCTION (PCI). Suppose S is a nonempty set of natural numbers. If

$$((\forall m)\,(m \in \mathbb{N} \wedge \{k \in \mathbb{N} : k < m\} \subset S) \Rightarrow m \in S)$$

then $S = \mathbb{N}$.

Definition 1.2.15 *Let* $(\mathbb{F}, +, \cdot, 0, 1, \leq)$ *be an ordered field. Define*

$$\mathbb{Q}_{\mathbb{F}} = \left\{ r \in \mathbb{F} : (\exists m)\, (\exists n) \left(m, n \in \mathbb{Z}_{\mathbb{F}} \wedge n \neq 0 \wedge r = mn^{-1} \right) \right\}.$$

The set $\mathbb{Q}_{\mathbb{F}}$ *is called the set of* ***rational numbers of*** $\mathbb{F}$.

Properties #1 and #5 from Proposition 1.1.13 can be used to show the set of rationals of a field is also closed under both addition and multiplication.

The set of real numbers $\mathbb{R}$ is the ordered field with which you are most familiar. Theorem 1.19 in our text asserts that $\mathbb{R}$ is an ordered field; the proof is given in an appendix to the first chapter. The notation (and numerals) for the corresponding special subsets of $\mathbb{R}$ are:

$$\mathbb{N} = \mathbb{J} = \{1, 2, 3, 4, 5, ...\} \text{ the set of natural numbers}$$
$$\mathbb{Z} = \{m : (m \in \mathbb{N}) \vee (m = 0) \vee (-m \in \mathbb{N})\} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$$
$$\mathbb{Q} = \{p \cdot q^{-1} = \tfrac{p}{q} : p, q \in \mathbb{Z} \wedge q \neq 0\}.$$

Remark 1.2.16 *The set of natural numbers may also be referred to as the set of positive integers, while the set of nonnegative integers is* $\mathbb{J} \cup \{0\}$. *Another common term for* $\mathbb{J} \cup \{0\}$ *is the set of whole numbers which may be denoted by* $\mathbb{W}$. *In MAT108, the letter* $\mathbb{N}$ *was used to denote the set of natural numbers, while the author of our MAT127 text is using the letter* J. *To make it clearer that we are referring to special sets of numbers, we will use the "blackboard bold" form of the capital letter. Feel free to use either (the old)* $\mathbb{N}$ *or (the new)* $\mathbb{J}$ *for the natural numbers in the field of reals.*

While $\mathbb{N}$ and $\mathbb{Z}$ are not fields, both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields that have several distinguishing characteristics we will be discussing shortly. Since $\mathbb{Q} \subset \mathbb{R}$ and $\mathbb{R} - \mathbb{Q} \neq \emptyset$, it is natural to want a notation for the set of elements of $\mathbb{R}$ that are not rational. Towards that end, we let $\mathbb{I}rr \underset{def}{=} \mathbb{R} - \mathbb{Q}$ denote the set of irrationals. It was shown in MAT108 that $\sqrt{2}$ is irrational. Because $\sqrt{2} + \left(-\sqrt{2}\right) = 0 \notin \mathbb{I}rr$ and $\sqrt{2} \cdot \sqrt{2} = 2 \notin \mathbb{I}rr$, we see that $\mathbb{I}rr$ is not closed under either addition or multiplication.

1.2.2 Bounding Properties

Because both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields we note that "satisfaction of the set of ordered field axioms" is not enough to characterize the set of reals. This naturally prompts us to look for other properties that will distinguish the two algebraic systems. The distinction that we will illustrate in this section is that the set of rationals has "certain gaps." During this (motivational) part of the discussion, you might find it intuitively helpful to visualize the "old numberline" representation for the reals. Given two rationals r and s such that $r < s$, it can be shown that $m = (r + s) \cdot 2^{-1} \in \mathbb{Q}$ is such that $r < m < s$. Then $r_1 = (r + m) \cdot 2^{-1} \in \mathbb{Q}$ and $s_1 = (m + s) \cdot 2^{-1} \in \mathbb{Q}$ are such that $r < r_1 < m$ and $m < s_1 < s$. Continuing this process indefinitely and "marking the new rationals on an imagined numberline" might entice us into thinking that we can "fill in most of the points on the number line between r and s." A rigorous study of the situation will lead us to conclude that the thought is shockingly inaccurate. We certainly know that not all the reals can be found this way because, for example, $\sqrt{2}$ could never be written in the form of $(r + s) \cdot 2^{-1}$ for $r, s \in \mathbb{Q}$. The following excursion will motivate the property that we want to isolate in our formal discussion of bounded sets.

Excursion 1.2.17 *Let* $A = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 < 2\right\}$ *and* $B = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 > 2\right\}$. *Now we will expand a bit on the approach used in our text to show that* A *has no largest element and* B *has not smallest element. For* p *a positive rational, let*

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$

Then

$$q^2 - 2 = \frac{2\left(p^2 - 2\right)}{(p + 2)^2}.$$

(a) For $p \in A$, *justify that* $q > p$ *and* $q \in A$.

(b) For $p \in B$, *justify that* $q < p$ *and* $q \in B$.

Hopefully you took a few moments to find some elements of A and B in order to get a feel for the nature of the two sets. Finding a q that corresponds to a $p \in A$ and a $p \in B$ would pretty much tell you why the claims are true. For (a), you should have noted that $q > p$ because $\left(p^2 - 2\right)(p+2)^{-1} < 0$ whenever $p^2 < 2$; then $-\left(p^2 - 2\right)(p+2)^{-1} > 0$ implies that $q = p + \left(-\left(p^2 - 2\right)(p+2)^{-1}\right) > p + 0 = p$. That q is rational follows from the fact that the rationals are closed under multiplication and addition. Finally $q^2 - 2 = 2\left(p^2 - 2\right)(p+2)^{-2} < 0$ yields that $q \in A$ as claimed. For (b), the same reasons extend to the discussion needed here; the only change is that, for $p \in B$, $p^2 > 2$ implies that $\left(p^2 - 2\right)(p+2)^{-1} > 0$ from which it follows that $-\left(p^2 - 2\right)(p+2)^{-1} < 0$ and $q = p + \left(-\left(p^2 - 2\right)(p+2)^{-1}\right) < p + 0 = p$.

Now we formalize the terminology that describes the property that our example is intended to illustrate. Let $(S, \leq)$ be an ordered set; i.e., $<$ is an order on the set S. A subset A of S is said to be **bounded above** in S if

$$(\exists u)\,(u \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq u))\,.$$

Any element $u \in S$ satisfying this property is called an **upper bound** of A in S.

Definition 1.2.18 *Let $(S, \leq)$ be an ordered set. For $A \subset S$, u is a **least upper bound** or **supremum** of A in S if and only if*

1. $(u \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq u))$ *and*
2. $(\forall b)\,[(b \in S \wedge (\forall a)\,(a \in A \Rightarrow a \leq b)) \Rightarrow u \leq b]$.

Notation 1.2.19 *For $(S, \leq)$ an ordered set and $A \subset S$, the **least upper bound** of A is denoted by* $\mathrm{lub}\,(A)$ *or* $\sup\,(A)$.

Since a given set can be a subset of several ordered sets, it is often the case that we are simply asked to find the least upper bound of a given set without specifying the "parent ordered set." When asked to do this, simply find, if it exists, the u that satisfies

$$(\forall a)\,(a \in A \Rightarrow a \leq u) \qquad \textit{and} \qquad (\forall b)\,[(\forall a)\,(a \in A \Rightarrow a \leq b) \Rightarrow u \leq b]\,.$$

The next few examples illustrate how we can use basic "pre-advanced calculus" knowledge to find some least upper bounds of subsets of the reals.

Example 1.2.20 *Find the* lub $\left\{\frac{x}{1+x^2} : x \in \mathbb{R}\right\}$.

From Proposition 1.2.9(#5), we know that, for $x \in \mathbb{R}$, $(1-x)^2 \geq 0$*; this is equivalent to*

$$1 + x^2 \geq 2x$$

from which we conclude that $(\forall x)\left(x \in \mathbb{R} \Rightarrow \frac{x}{1+x^2} \leq \frac{1}{2}\right)$*. Thus,* $\frac{1}{2}$ *is an upper bound for* $\left\{\frac{x}{1+x^2} : x \in \mathbb{R}\right\}$*. Since* $\frac{1}{1+1^2} = \frac{1}{2}$*, it follows that*

$$\text{lub}\left\{\frac{x}{1+x^2} : x \in \mathbb{R}\right\} = \frac{1}{2}.$$

The way that this example was done and presented is an excellent illustration of the difference between scratch work (Phase II) and presentation of an argument (Phase III) in the mathematical process. From calculus (MAT21A or its equivalent) we can show that $f(x) = \frac{x}{1+x^2}$ has a relative minimum at $x = -1$ and a relative maximum at $x = 1$; we also know that $y = 0$ is a horizontal asymptote for the graph. Armed with the information that $\left(1, \frac{1}{2}\right)$ is a maximum for f, we know that all we need to do is use inequalities to show that $\frac{x}{1+x^2} \leq \frac{1}{2}$. In the scratch work phase, we can work backwards from this inequality to try to find something that we can claim from what we have done thus far; simple algebra gets use from $\frac{x}{1+x^2} \leq \frac{1}{2}$ to $1 - 2x + x^2 \geq 0$. Once we see that desire to claim $(1-x)^2 \geq 0$, we are home free because that property is given in one of our propositions about ordered fields.

Excursion 1.2.21 *Find the* lub(A) *for each of the following. Since your goal is simply to find the least upper bound, you can use any pre-advanced calculus information that is helpful.*

1. $A = \left\{ \dfrac{3 + (-1)^n}{2^{n+1}} : n \in \mathbb{J} \right\}$

2. $A = \{(\sin x)(\cos x) : x \in \mathbb{R}\}$

For (1), let $x_n = \dfrac{3 + (-1)^n}{2^{n+1}}$; then $x_{2j} = \dfrac{1}{2^{2j-1}}$ is a sequence that is strictly decreasing from $\dfrac{1}{2}$ to 0; while x_{2j-1} is also decreasing from $\dfrac{1}{2}$ to 0. Consequently the terms in A are never greater than $\dfrac{1}{2}$ with the value of $\dfrac{1}{2}$ being achieved when $n = 1$ and the terms get arbitrarily close to 0 as n approaches infinity. Hence, $\operatorname{lub}(A) = \dfrac{1}{2}$. For (2), it is helpful to recall that $\sin x \cos x = \dfrac{1}{2}\sin 2x$. The well known behavior of the sine function immediately yields that $\operatorname{lub}(A) = \dfrac{1}{2}$.

Example 1.2.22 *Find* lub (A) *where* $A = \left\{x \in \mathbb{R} : x^2 + x < 3\right\}$.

What we are looking for here is sup (A) *where* $A = f^{-1}((-\infty, 3))$ *for* $f(x) = x^2 + x$. *Because*

$$y = x^2 + x \Leftrightarrow y + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2,$$

f *is a parabola with vertex* $\left(-\dfrac{1}{2}, -\dfrac{1}{4}\right)$. *Hence,*

$$A = f^{-1}((-\infty, 3)) = \left\{x \in R : \frac{-1 - \sqrt{13}}{2} < x < \frac{-1 + \sqrt{13}}{2}\right\}$$

from which we conclude that $\sup(A) = \dfrac{-1 + \sqrt{13}}{2}$.

Note that the set $A = \left\{p \in \mathbb{Q} : p > 0 \wedge p^2 < 2\right\}$ is a subset of $\mathbb{Q}$ and a subset of $\mathbb{R}$. We have that $(\mathbb{Q}, \leq)$ and $(\mathbb{R}, \leq)$ are ordered sets where $<$ is defined by $r < s \Leftrightarrow (s + (-r))$ is positive. Now $\text{lub}\,(A) = \sqrt{2} \notin \mathbb{Q}$; hence, there is no least upper bound of A **in** $S = \mathbb{Q}$, but $A \subset S = \mathbb{R}$ has a least upper bound in $S = \mathbb{R}$. This tells us that the "parent set" is important, gives us a distinction between $\mathbb{Q}$ and $\mathbb{R}$ as ordered fields, and motivates us to name the important distinguishing property.

Definition 1.2.23 *An ordered set $(S, <)$ has the **least upper bound property** if and only if*

$$(\forall E)\left[\begin{array}{c}(E \subset S \wedge E \neq \emptyset \wedge (\exists \beta)\,(\beta \in S \wedge (\forall a)\,(a \in E \Rightarrow a \leq \beta))) \\ \Rightarrow ((\exists u)\,(u = \text{lub}\,(E) \wedge u \in S))\end{array}\right]$$

Remark 1.2.24 *As noted above, $(\mathbb{Q}, \leq)$ does not satisfy the "*lub *property", while $(\mathbb{R}, \leq)$ does satisfy this property.*

The proof of the following lemma is left an exercise.

Lemma 1.2.25 *Let $(X, \leq)$ be an ordered set and $A \subseteq X$. If A has a least upper bound in X, it is unique.*

We have analogous or companion definitions for subsets of an ordered set that are bounded below. Let $(S, \leq)$ be an ordered set; i.e., $<$ is an order on the set S. A subset A of S is said to be **bounded below** in S if

$$(\exists \upsilon)\,(\upsilon \in S \wedge (\forall a)\,(a \in A \Rightarrow \upsilon \leq a))\,.$$

Any element $u \in S$ satisfying this property is called a **lower bound** of A in S.

Definition 1.2.26 *Let $(S, \leq)$ be a linearly ordered set. A subset A of S is said to have a **greatest lower bound** or **infimum** in S if*

1. $(\exists g)\,(g \in S \wedge (\forall a)\,(a \in A \Rightarrow g \leq a))$, *and*
2. $(\forall c)\left[(c \in S \wedge (\forall a)\,(a \in A \Rightarrow c \leq a)) \Rightarrow c \leq g\right]$.

Example 1.2.27 *Find the* $\text{glb}\,(A)$ *where* $A = \left\{(-1)^n \left(\frac{1}{4} - \frac{2}{n}\right) : n \in \mathbb{N}\right\}$.

Let $x_n = (-1)^n \left(\frac{1}{4} - \frac{2}{n}\right)$*; then, for n odd,* $x_n = \frac{2}{n} - \frac{1}{4}$ *and, for n even,* $x_n = \frac{1}{4} - \frac{2}{n}$.

Suppose that $n \geq 4$. *By Proposition 1.2.9(#7), it follows that* $\frac{1}{n} \leq \frac{1}{4}$. *Then* $(OF2)$ *and* $(OF1)$ *yield that* $\frac{2}{n} \leq \frac{2}{4} = \frac{1}{2}$ *and* $\frac{2}{n} - \frac{1}{4} \leq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, *respectively. From* $\frac{2}{n} \leq \frac{1}{2}$ *and Proposition 1.2.9(#4), we have that* $-\frac{2}{n} \geq -\frac{1}{2}$. *Thus,* $\frac{1}{4} - \frac{2}{n} \geq \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$ *from* $(OF1)$. *Now, it follows from Proposition 1.2.9(#1) that* $n > 0$, *for any* $n \in \mathbb{N}$. *From Proposition 1.2.9(#7) and* $(OF1)$, $n > 0$ *and* $2 > 0$ *implies that* $\frac{2}{n} > 0$ *and* $\frac{2}{n} - \frac{1}{4} \geq -\frac{1}{4}$. *Similarly, from Proposition 1.2.9(#3) and* $(OF1)$, $\frac{2}{n} > 0$ *implies that* $-\frac{2}{n} < 0$ *and* $\frac{1}{4} - \frac{2}{n} < \frac{1}{4} + 0 = \frac{1}{4}$.

Combining our observations, we have that

$$(\forall n)\left[\left(n \in \mathbb{N} - \{1,2,3\} \wedge 2 \nmid n\right) \Rightarrow -\frac{1}{4} \leq x_n \leq \frac{1}{4}\right]$$

and

$$(\forall n)\left[\left(n \in \mathbb{N} - \{1,2,3\} \wedge 2 \mid n\right) \Rightarrow -\frac{1}{4} \leq x_n \leq \frac{1}{4}\right].$$

Finally, $x_1 = \frac{7}{4}$, $x_2 = -\frac{3}{4}$, *and* $x_3 = \frac{5}{12}$, *each of which is outside of* $\left[-\frac{1}{4}, \frac{1}{4}\right]$. *Comparing the values leads to the conclusion that* $\text{glb}(A) = -\frac{3}{4}$.

Excursion 1.2.28 *Find* glb (A) *for each of the following. Since your goal is simply to find the greatest lower bound, you can use any pre-advanced calculus information that is helpful.*

1. $A = \left\{\frac{3 + (-1)^n}{2^{n+1}} : n \in \mathbb{J}\right\}$

2. $A = \left\{ \frac{1}{2^n} + \frac{1}{3^m} : n, m \in \mathbb{N} \right\}$

Our earlier discussion in Excursion 1.2.21, the set given in (1) leads to the conclusion that $\text{glb}\,(A) = 0$. For (2), note that each of $\frac{1}{2^n}$ and $\frac{1}{3^m}$ are strictly decreasing to 0 as n and m are increasing, respectively. This leads us to conclude that $\text{glb}\,(A) = 0$; although it was not requested, we note that $\sup(A) = \frac{5}{6}$.

We close this section with a theorem that relates least upper bounds and greatest lower bounds.

Theorem 1.2.29 *Suppose $(S, <)$ is an ordered set with the least upper bound property and that B is a nonempty subset of S that is bounded below. Let*

$$L = \{g \in S : (\forall a)\,(a \in B \Rightarrow g \leq a)\}.$$

Then $\alpha = \sup(L)$ exists in S, and $\alpha = \inf(B)$.

Proof. Suppose that $(S, <)$ is an ordered set with the least upper bound property and that B is a nonempty subset of S that is bounded below. Then

$$L = \{g \in S : (\forall a)\,(a \in B \Rightarrow g \leq a)\}.$$

is not empty. Note that for each $b \in B$ we have that $g \leq b$ for all $g \in L$; i.e., each element of B is an upper bound for L. Since $L \subset S$ is bounded above and S satisfies the least upper bound property, the least upper bound of L exists and is in S. Let $\alpha = \sup(L)$.

Now we want to show that α is the greatest lower bound for B.

■

Definition 1.2.30 *An ordered set* $(S, <)$ *has the* ***greatest lower bound property*** *if and only if*

$$(\forall E)\left[\begin{array}{c}(E \subset S \wedge E \neq \emptyset \wedge (\exists \gamma)(\gamma \in S \wedge (\forall a)(a \in E \Rightarrow \gamma \leq a))) \\ \Rightarrow ((\exists w)(w = \mathrm{glb}(E) \wedge w \in S))\end{array}\right].$$

Remark 1.2.31 *Theorem 1.2.29 tells us that every ordered set that satisfies the least upper bound property also satisfies the greatest lower bound property.*

1.3 The Real Field

The Appendix for Chapter 1 of our text offers a construction of “the reals” from “the rationals”. In our earlier observation of special subsets of an ordered field, we offered formal definitions of the natural numbers of a field, the integers of a field, and the rationals of a field. Notice that the definitions were not tied to the objects (symbols) that we already accept as numbers. It is not the form of the objects in the ordered field that is important; it is the set of properties that must be satisfied. Once we accept the existence of an ordered field, all ordered fields are alike. While this identification of ordered fields and their corresponding special subsets can be made more formal, we will not seek that formalization.

It is interesting that our mathematics education actually builds up to the formulation of the real number field. Of course, the presentation is more hands-on and intuitive. At this point, we accept our knowledge of sums and products involving real numbers. I want to highlight parts of the building process simply to put the properties in perspective and to relate the least upper bound property to something tangible. None of this part of the discussion is rigorous. First, define the symbols 0 and 1, by $\{\} \underset{def}{=} 0$ and $\{\emptyset\} \underset{def}{=} 1$ and suppose that we have an ordered field $(R, +, \cdot, 0, 1, \leq)$. Furthermore, picture a representation of a straight horizontal line $(\longleftarrow\!\!\longrightarrow)$ on which we will place elements of this field in a way that attaches some geometric meaning to their location. The natural numbers of this field $\mathbb{N}_R$ is the "smallest" inductive subset; it is closed under addition and multiplication. It can be proved (Some of you saw the proofs in your MAT108 course.) that

$$(\forall x)\,(x \in \mathbb{N}_R \Rightarrow x \geq 1)$$

and

$$(\forall w)\,(w \in \mathbb{N}_R \Rightarrow \neg\,(\exists v)\,(v \in \mathbb{N}_R \wedge w < v < w + 1))\,.$$

This motivates our first set of markings on the representative line. Let's indicate the first mark as a "place for 1." Then the next natural number of the field is $1+1$, while the one after that is $(1 + 1) + 1$, followed by $[(1 + 1) + 1] + 1$, etc. This naturally leads us to choose a fixed length to represent 1 (or "1 unit") and place a mark for each successive natural number 1 away from and to the right of the previous natural number. It doesn't take too long to see that our collections of "added $1's$" is not a pretty or easy to read labelling system; this motivates our desire for neater representations. The symbols that we have come to accept are 1, 2, 3, 4, 5, 6, 7, 8, and 9. In the space provided draw a picture that indicates what we have thus far.

The fact that, in an ordered field, $0 < 1$ tells us to place 0 to the left of 1 on our representative line; then $0 + 1 = \{\ \} \cup \{\emptyset\} = \{\emptyset\} = 1$ justifies placing 0 "1 unit" away from the 1. Now the definition of the integers of a field $\mathbb{Z}_R$ adjoins the additive inverses of the natural numbers of a field; our current list of natural numbers leads to acceptance of $-1, -2, -3, -4, -5, -6, -7, -8$, and -9 as labels of the markings of the new special elements and their relationship to the natural numbers mandates their relative locations. Use the space provided to draw a picture that indicates what we have thus far.

Your picture should show several points with each neighboring pair having the same distance between them and "lots of space" with no labels or markings, but we still have the third special subset of the ordered field; namely, the rationals of the field $\mathbb{Q}_R$. We are about to prove an important result concerning the "density of the rationals" in an ordered field. But, for this intuitive discussion, our "grade school knowledge" of fractions will suffice. Picture (or use the last picture that you drew to illustrate) the following process: Mark the midpoint of the line segment from 0 to 1 and label it 2^{-1} or $\frac{1}{2}$; then mark the midpoint of each of the smaller line segments (the one from 0 to $\frac{1}{2}$ and the one from $\frac{1}{2}$ to 1) and label the two new points $\frac{1}{4}$ and $\frac{3}{4}$, respectively; repeat the process with the four smaller line segments to get $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}$ as the marked rationals between 0 and 1. It doesn't take too many iterations of this process to have it look like you have filled the interval. Of course, we know that we haven't because any rational in the from $p \cdot q^{-1}$ where $0 < p < q$ and $q \neq 2^n$ for any n has been omitted. What turned out to be a surprise, at the time of discovery, is that all the rationals r such the $0 \leq r \leq 1$ will not be "enough to fill the interval $[0, 1]$." At this point we have the set of elements of the field that are not in any of the special subsets, $R - \mathbb{Q}_R$, and the "set of vacancies" on our model line. We don't know that there is a one-to-one correspondence between them. That there is a correspondence follows from the what is proved in the Appendix to Chapter 1 of our text.

Henceforth, we use $(\mathbb{R}, +, \cdot, 0, 1, <)$ to denote the ordered field (of reals) that satisfies the least upper bound property and may make free use of the fact that for any $x \in \mathbb{R}$ we have that x is either rational or the least upper bound of a set of rationals. Note that the subfield $(\mathbb{Q}, +, \cdot, 0, 1, <)$ is an ordered field that does not satisfy the least upper bound property.

1.3.1 Density Properties of the Reals

In this section we prove some useful density properties for the reals.

Lemma 1.3.1 *If $S \subseteq \mathbb{R}$ has L as a least upper bound L, then*

$$(\forall \varepsilon)\left((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists s)\left(s \in S \wedge L - \varepsilon < s \leq L\right)\right).$$

Proof. Suppose S is a nonempty subset of $\mathbb{R}$ such that $L = \sup(S)$ and let $\varepsilon \in \mathbb{R}$ be such that $\varepsilon > 0$. By Proposition 1.2.9(#3) and $(OF1)$, $-\varepsilon < 0$ and $L - \varepsilon < L$. From the definition of least upper bound, each upper bound of S is greater than or equal to L. Hence, $L - \varepsilon$ is not an upper bound for S from which we conclude that $\neg(\forall s)(s \in S \Rightarrow s \leq L - \varepsilon)$ is satisfied; i.e.,

$$(\exists s)(s \in S \wedge L - \varepsilon < s).$$

Combining this with $L = \sup(S)$ yields that

$$(\exists s)(s \in S \wedge L - \varepsilon < s \leq L).$$

Since ε was arbitrary, $(\forall \varepsilon)\left((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists s)(s \in S \wedge L - \varepsilon < s \leq L)\right)$ as claimed. ∎

Theorem 1.3.2 (The Archimedean Principle for Real Numbers) *If α and β are positive real numbers, then there is some positive integer n such that $n\alpha > \beta$.*

Proof. The proof will be by contradiction. Suppose that there exist positive real numbers α and β such that $n\alpha \leq \beta$ for every natural number n. Since $\alpha > 0$, $\alpha < 2\alpha < 3\alpha < \cdots < n\alpha < \cdots$ is an increasing sequence of real numbers that is bounded above by β. Since $(\mathbb{R}, \leq)$ satisfies the least upper bound property $\{n\alpha : n \in \mathbb{N}\}$ has a least upper bound in $\mathbb{R}$, say L. Choose $\epsilon = \frac{1}{2}\alpha$ which is positive because $\alpha > 0$. Since $L = \sup\{n\alpha : n \in \mathbb{N}\}$, from Lemma 1.3.1, there exists $s \in \{n\alpha : n \in \mathbb{N}\}$ such that $L - \varepsilon < s \leq L$. If $s = N\alpha$, then for all natural numbers $m > N$, we also have that $L - \varepsilon < m\alpha \leq L$. Hence, for $m > N$, $0 \leq L - m\alpha < \varepsilon$. In particular,

$$0 \leq L - (N+1)\alpha < \varepsilon = \frac{1}{2}\alpha$$

and

$$0 \leq L - (N+2)\alpha < \varepsilon = \frac{1}{2}\alpha.$$

Thus, $L - \frac{1}{2}\alpha < (N+1)\alpha$ and $(N+2)\alpha < L < L + \frac{1}{2}\alpha$. But adding α to both sides of the first inequality, yields $L + \frac{1}{2}\alpha < (N+2)\alpha$ which contradicts $(N+2)\alpha < L + \frac{1}{2}\alpha$. Hence, contrary to our original assumption, there exists a natural number n such that $n\alpha > \beta$. ∎

Corollary 1.3.3 (Density of the Rational Numbers) *If α and β are real numbers with $\alpha < \beta$, then there is a rational number r such that $\alpha < r < \beta$.*

Proof. Since 1 and $\beta - \alpha$ are positive real numbers, by the Archimedean Principle, there exists a positive integer m such that $1 < m(\beta - \alpha)$, or equivalently

$$m\alpha + 1 \le m\beta.$$

Let n be the largest integer such that $n \le m\alpha$. It follows that

$$n + 1 \le m\alpha + 1 \le m\beta.$$

Since n is the largest integer such that $n \le m\alpha$, we know that $m\alpha < n + 1$. Consequently, $m\alpha < n + 1 < m\beta$, which is equivalent to having

$$\alpha < \frac{n+1}{m} < \beta.$$

Therefore, we have constructed a rational number that is between α and β. ■

Corollary 1.3.4 (Density of the Irrational Numbers) *If α and β are real numbers with $\alpha < \beta$, then there is an irrational number γ such that $\alpha < \gamma < \beta$.*

Proof. Suppose that α and β are real numbers with $\alpha < \beta$. By Corollary 1.3.3, there is a rational r that is between $\frac{\alpha}{\sqrt{2}}$ and $\frac{\beta}{\sqrt{2}}$. Since $\sqrt{2}$ is irrational, we conclude that $\gamma = r \cdot \sqrt{2}$ is an irrational that is between α and β. ■

1.3.2 Existence of nth Roots

The primary result in this connection that is offered by the author of our text is the following

Theorem 1.3.5 *For $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, we have that*

$$(\forall x)\,(\forall n)\left(x \in \mathbb{R}^+ \wedge n \in \mathbb{J} \Rightarrow (\exists! y)\left(y \in \mathbb{R} \wedge y^n = x\right)\right).$$

Before we start the proof, we note the following fact that will be used in the presentation.

Fact 1.3.6 $(\forall y)\,(\forall z)\,(\forall n)\left[(y, z \in \mathbb{R} \wedge n \in \mathbb{J} \wedge 0 < y < z) \Rightarrow y^n < z^n\right]$

To see this, for $y, z \in \mathbb{R}$ satisfying are $0 < y < z$, let

$$S = \left\{n \in \mathbb{J} : y^n < z^n\right\}.$$

Our set-up automatically places $1 \in S$. Suppose that $k \in S$; i.e., $k \in \mathbb{J}$ and $y^k < z^k$. Since $0 < y$, by $(OF2)$, $y^{k+1} = y \cdot y^k < y \cdot z^k$. From $0 < z$ and repeated use of Proposition 1.2.9(#2), we can justify that $0 < z^k$. Then $(OF2)$ with $0 < z^k$ and $y < z$ yields that $y \cdot z^k < z \cdot z^k = z^{k+1}$. As a consequence of the transitive law,

$$y^{k+1} < y \cdot z^k \wedge y \cdot z^k < z^{k+1} \Rightarrow y^{k+1} < z^{k+1};$$

that is, $k+1 \in S$. Since k was arbitrary, we conclude that $(\forall k)\,(k \in S \Rightarrow (k+1) \in S)$.

From $1 \in S \wedge (\forall k)\,(k \in S \Rightarrow (k+1) \in S)$, S is an inductive subset of the natural numbers. By the Principle of Mathematical Induction (PMI), $S = \mathbb{J}$. Since y and z were arbitrary, this completes the justification of the claim.

Fact 1.3.7 $(\forall w)\,(\forall n)\left[(w \in \mathbb{R} \wedge n \in \mathbb{J} - \{1\} \wedge 0 < w < 1) \Rightarrow w^n < w\right]$

Since $n \geq 2$, $n - 1 \geq 1$ and, by Fact 1.3.6, $w^{n-1} < 1^{n-1} = 1$. From $(OF2)$, $0 < w \wedge w^{n-1} < 1$ implies that $w^n = w^{n-1} \cdot w < 1 \cdot w = w$; i.e., $w^n < w$ as claimed.

Fact 1.3.8 $(\forall a)\,(\forall b)\,(\forall n)\,[(a, b \in \mathbb{R} \wedge n \in \mathbb{J} - \{1\} \wedge 0 < a < b) \Rightarrow (b^n - a^n) < (b-a)\,nb^{n-1}]$

From Fact 1.3.6, $n \geq 2 \wedge 0 < a < b \Rightarrow a^{n-1} < b^{n-1}$, while $(OF2)$ yields that $a \cdot b^j < b \cdot b^j = b^{j+1}$ for $j = 1, 2, ..., n-2$. It can be shown (by repeated application of Exercise 6(a)) that

$$b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1} < b^{n-1} + b^{n-1} + \cdots + b^{n-1} = nb^{n-1};$$

this, with $(OF2)$, implies that

$$b^n - a^n = (b-a)\left(b^{n-1} + b^{n-2}a + \cdots + ba^{n-2} + a^{n-1}\right) < (b-a)\,nb^{n-1}$$

as claimed.

Proof. (of the theorem.) Let $\mathbb{R}^+ = \{u \in \mathbb{R} : u > 0\}$. When $n = 1$, there is nothing to prove so we assume that $n \geq 2$. For fixed $x \in \mathbb{R}^+$ and $n \in \mathbb{J} - \{1\}$, set

$$E = \left\{t \in \mathbb{R}^+ : t^n < x\right\}.$$

Excursion 1.3.9 *Use* $w = \dfrac{x}{1+x}$ *to justify that* $E \neq \emptyset$.

Now let $u = 1 + x$ and suppose that $t > u > 0$. Fact 1.3.6 yields that $t^n > u^n$. From Proposition 1.2.9(#7), $u > 1 \Rightarrow 0 < \dfrac{1}{u} < 1$. It follows from Fact 1.3.7 and Proposition 1.2.9(#7) that $0 < \dfrac{1}{u^n} \leq \dfrac{1}{u}$ and $u^n \geq u$. By transitivity, $t^n > u^n \wedge u^n \geq u$ implies that $t^n > u$. Finally, since $u > x$ transitivity leads to the conclusion that $t^n > x$. Hence, $t \notin E$. Since t was arbitrary, $(\forall t)\,(t > u \Rightarrow t \notin E)$ which is equivalent to $(\forall t)\,(t \in E \Rightarrow t \leq u)$. Therefore, $E \subset \mathbb{R}$ is bounded above. From the least upper bound property, lub (E) exists. Let

$$y = \operatorname{lub}(E).$$

Since $E \subset \mathbb{R}^+$, we have that $y \geq 0$.

By the Trichotomy Law, one and only one of $y^n = x$, $y^n < x$, or $y^n > x$. In what follows we will that neither of the possibilities $y^n < x$, or $y^n > x$ can hold.

Case 1: If $y^n < x$, then $x - y^n > 0$. Since $y + 1 > 0$ and $n \geq 1$, $\dfrac{x - y^n}{n\,(y+1)^{n-1}} > 0$ and we can choose h such that $0 < h < 1$ and

$$h < \frac{x - y^n}{n\,(y+1)^{n-1}}.$$

Taking $a = y$ and $b = y + h$ in Fact 1.3.8 yields that

$$(y+h)^n - y^n < hn\,(y+h)^{n-1} < x - y^n.$$

Excursion 1.3.10 *Use this to obtain contradict that* $y = \sup(E)$.

Case 2: If $0 < x < y^n$, then $0 < y^n - x < ny^n$. Hence,

$$k = \frac{y^n - x}{ny^{n-1}}$$

is such that $0 < k < y$. For $t \geq (y - k)$, Fact 1.3.6 yields that $t^n \geq (y - k)^n$.

From Fact 1.3.8, with $b = y$ and $a = y - k$, we have that

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Excursion 1.3.11 *Use this to obtain another contradiction.*

From case 1 and case 2, we conclude that $y^n = x$. this concludes the proof that there exists a solution to the given equation.

The uniqueness of the solution follows from Fact 1.3.6. To see this, note that, if $y^n = x$ and w is such that $0 < w \neq y$, then $w < y$ implies that $w^n < y^n = x$, while $y < w$ implies that $x = y^n < w^n$. In either case, $w^n \neq x$. ■

*******For Excursion 1.3.9, you want to justify that the given w is in E. Because $0 < x < 1+x$, $0 < w = \dfrac{x}{1+x} < 1$. In view of fact 1.3.7, $w^n < w$ for $n \geq 2$ or $w^n \leq w$ for $n \geq 1$. But $x > 0 \wedge 1 + x > 1$ implies that $\dfrac{1}{1+x} < 1 \wedge \dfrac{x}{1+x} < x \cdot 1 = x$. From transitivity, $w^n < w \wedge w < x \Rightarrow w^n < x$; i.e., $w \in E$.

To obtain the desired contradiction for completion of Excursion 1.3.10, hopefully you notices that the given inequality implied that $(y + h)^n < x$ which would place $y + h$ in E; since $y + h > y$, this would contradict that $y = \sup(E)$ from which we conclude that $y^n < x$ is not true.

The work needed to complete Excursion 1.3.11 was a little more involved. In this case, the given inequality led to $-t^n < -x$ or $t^n > x$ which justifies that $t \notin E$; hence, $t > y - k$ implies that $t \notin E$ which is logically equivalent to $t \in E$ implies that $t < y - k$. This would make $y - k$ an upper bound for E which is a contradiction. Obtaining the contradiction yields that $x < y^n$ is also not true.***

Remark 1.3.12 *For x a positive real number and n a natural number, the number y that satisfies the equation $y^n = x$ is written as $\sqrt[n]{x}$ and is read as "the nth root of x."*

Repeated application of the associativity and commutativity of multiplication can be used to justify that, for positive real numbers α and β and n a natural number,

$$\alpha^n \beta^n = (\alpha\beta)^n .$$

From this identity and the theorem we have the following identity involving nth roots of positive real numbers.

Corollary 1.3.13 *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n} b^{1/n}.$$

Proof. For $\alpha = a^{1/n}$ and $\beta = b^{1/n}$, we have that $ab = \alpha^n \beta^n = (\alpha\beta)^n$. Hence $\alpha\beta$ is the unique solution to $y^n = ab$ from which we conclude that $(ab)^{1/n} = \alpha\beta$ as needed. ■

1.3.3 The Extended Real Number System

The extended real number system is $\mathbb{R} \cup \{-\infty, +\infty\}$ where $(\mathbb{R}, +, \cdot, 0, 1, <)$ is the ordered field that satisfies the least upper bound property as discussed above and the symbols $-\infty$ and $+\infty$ are defined to satisfy $-\infty < x < +\infty$ for all $x \in \mathbb{R}$. With this convention, any nonempty subset S of the extended real number system is bounded above by $+\infty$ and below by $-\infty$; if S has no finite upper bound, we write $\text{lub}(S) = +\infty$ and when S has no finite lower limit, we write $\text{glb}(S) = -\infty$.

The $+\infty$ and $-\infty$ are useful symbols; they are not numbers. In spite of their appearance, $-\infty$ is not an additive inverse for $+\infty$. This means that there is no meaning attached to any of the expressions $\infty - \infty$ or $\dfrac{\infty}{-\infty}$ or $\dfrac{\infty}{\infty}$; in fact, these expressions should never appear in things that you write. Because the symbols ∞ and $-\infty$ do not have additive (or multiplicative) inverses, $\mathbb{R} \cup \{-\infty, \infty\}$ is not a field. On the other hand, we do have some conventions concerning "interaction" of the special symbols with elements of the field $\mathbb{R}$; namely,

- If $x \in \mathbb{R}$, then $x + \infty = +\infty$, $x - \infty = -\infty$ and $\dfrac{x}{\infty} = \dfrac{x}{-\infty} = 0$.
- If $x > 0$, then $x \cdot (+\infty) = +\infty$ and $x \cdot (-\infty) = -\infty$.
- If $x < 0$, then $x \cdot (+\infty) = -\infty$ and $x \cdot (-\infty) = +\infty$.

Notice that nothing is said about the product of zero with either of the special symbols.

1.4 The Complex Field

For $\mathbb{C} = \mathbb{R} \times \mathbb{R}$, define addition $(+)$ and multiplication $(\cdot)$ by

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2),$$

respectively. That addition and multiplication are binary operations on $\mathbb{C}$ is a consequence of the closure of $\mathbb{R}$ under addition and multiplication. It follows immediately that

$$(x, y) + (0, 0) = (x, y) \qquad \textit{and} \qquad (x, y) \cdot (1, 0) = (x, y).$$

Hence, $(0, 0)$ and $(1, 0)$ satisfy the additive identity property and the multiplicative identity field property, respectively. Since the binary operations are defined as combinations of sums and products involving reals, direct substitution and appropriate manipulation leads to the conclusion that addition and multiplication over $\mathbb{C}$ are commutative and associative under addition and multiplication. (The actual manipulations are shown in our text on pages 12-13.)

To see that the additive inverse property is satisfied, note that $(x, y) \in \mathbb{C}$ implies that $x \in \mathbb{R} \wedge y \in \mathbb{R}$. The additive inverse property in the field $\mathbb{R}$ yields that $-x \in \mathbb{R}$

and $-y \in \mathbb{R}$. It follows that $(-x,-y) \in \mathbb{C}$ and $(x,y)+(-x,-y) = (0,0)$ and needed.

Suppose $(x,y) \in \mathbb{C}$ is such that $(x,y) \neq (0,0)$. Then $x \neq 0 \vee y \neq 0$ from which we conclude that $x^2+y^2 \neq 0$ and $(a,b) \underset{def}{=} \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$ is well defined. Now,

$$
\begin{aligned}
(x,y)\cdot(a,b) &= (x,y)\cdot\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \\
&= \left(x\cdot\frac{x}{x^2+y^2} - y\cdot\frac{-y}{x^2+y^2}, x\cdot\frac{-y}{x^2+y^2} + y\cdot\frac{x}{x^2+y^2}\right) \\
&= \left(\frac{x\cdot x+(-y)\cdot(-y)}{x^2+y^2}, \frac{x\cdot(-y)+y\cdot x}{x^2+y^2}\right) \\
&= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+yx}{x^2+y^2}\right) \\
&= (1,0).
\end{aligned}
$$

Hence, the multiplicative inverse property is satisfied for $(\mathbb{C},+,\cdot)$.

Checking that the distributive law is satisfied is a matter of manipulating the appropriate combinations over the reals. This is shown in our text on page 13.

Combining our observations justifies that $(\mathbb{C},+,\cdot,(0,0),(1,0))$ is a field. It is known as the complex field or the field of complex numbers.

Remark 1.4.1 *Identifying each element of* $\mathbb{C}$ *in the form* $(x,0)$ *with* $x \in \mathbb{R}$ *leads to the corresponding identification of the sums and products,* $x+a = (x,0)+(a,0) = (x+a,0)$ *and* $x\cdot a = (x,0)\cdot(a,0) = (x\cdot a,0)$. *Hence, the real field is a subfield of the complex field.*

The following definition will get us to an alternative formulation for the complex numbers that can make some of their properties easier to remember.

Definition 1.4.2 *The complex number* $(0,1)$ *is defined to be* i.

With this definition, it can be shown directly that

- $i^2 = (-1,0) = -1$ and
- if a and b are real numbers, then $(a,b) = a+bi$.

With these observations we can write

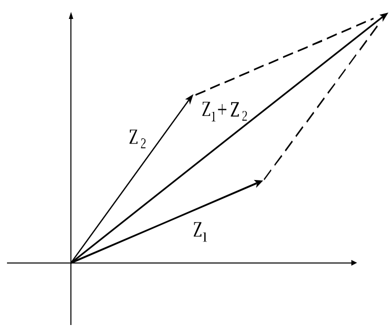
$$\mathbb{C} = \left\{a + bi : a, b \in \mathbb{R} \wedge i^2 = -1\right\}$$

with addition and multiplication being carried out using the distributive law, commutativity, and associativity.

We have two useful forms for complex numbers; the rectangular and trigonometric forms for the complex numbers are freely interchangeable and offer different geometric advantages.

From Rectangular Coordinates

Complex numbers can be represented geometrically as points in the plane. We plot them on a rectangular coordinate system that is called an Argand Graph. In $z = x + iy$, x is the real part of z, denoted by $\operatorname{Re} z$, and y is the imaginary part of z, denoted by $\operatorname{Im} z$. When we think of the complex number $x + iy$ as a vector $\overrightarrow{OP}$ joining the origin $O = (0, 0)$ to the point $P = (x, y)$, we grasp the natural geometric interpretation of addition $(+)$ in $\mathbb{C}$.



Definition 1.4.3 *The **modulus** of a complex number z is the magnitude of the vector representation and is denoted by* $|z|$*. If* $z = x + iy$*, then* $|z| = \sqrt{x^2 + y^2}$.

Definition 1.4.4 *The **argument** of a nonzero complex number* z*, denoted by* $\arg z$*, is a measurement of the angle that the vector representation makes with the positive real axis.*

Definition 1.4.5 *For* $z = x + iy$*, the **conjugate** of* z*, denoted by* $\bar{z}$*, is* $x - iy$.

Most of the properties that are listed in the following theorems can be shown fairly directly from the rectangular form.

Theorem 1.4.6 *For z and w complex numbers,*

1. $|z| \geq 0$ *with equality only if* $z = 0$,
2. $|\overline{z}| = |z|$,
3. $|zw| = |z|\,|w|$,
4. $|\operatorname{Re} z| \leq |z|$ *and* $|\operatorname{Im} z| \leq |z|$,
5. $|z + w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2$.

The proofs are left as exercises.

Theorem 1.4.7 (The Triangular Inequalities) *For complex numbers z_1 and z_2,*

$$|z_1 + z_2| \leq |z_1| + |z_2|\,, \text{ and } |z_1 - z_2| \geq ||z_1| - |z_2||\,.$$

Proof. To see the first one, note that

$$\begin{aligned} |z_1 + z_2|^2 &= |z_1|^2 + 2\operatorname{Re} z_1 z_2 + |z_2|^2 \\ &\leq |z_1|^2 + 2\,|z_1|\,|z_2| + |z_2|^2 = (|z_1| + |z_2|)^2 \end{aligned} .$$

The proof of the second triangular inequality is left as an exercise. ■

Theorem 1.4.8 *If z and w are complex numbers, then*

1. $\overline{z + w} = \bar{z} + \bar{w}$
2. $\overline{zw} = \bar{z}\bar{w}$
3. $\operatorname{Re} z = \dfrac{z + \bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z - \bar{z}}{2i}$,
4. $z\overline{z}$ *is a nonnegative real number.*

From Polar Coordinates

For nonzero $z = x + iy \in \mathbb{C}$, let $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\left(\dfrac{y}{x}\right) = \arg z$. Then the trigonometric form is

$$z = r\,(\cos\theta + i\,\sin\theta)\,.$$

In engineering, it is customary to use $\operatorname{cis}\theta$ for $\cos\theta + i\,\sin\theta$ in which case we write $z = r\,\operatorname{cis}\theta$.

NOTE: While (r, θ) uniquely determines a complex number, the converse is not true.

Excursion 1.4.9 *Use the polar form for complex numbers to develop a geometric interpretation of the product of two complex numbers.*

The following identity can be useful when working with complex numbers in polar form.

Proposition 1.4.10 (DeMoivre's Law) *For θ real and $n \in \mathbb{Z}$,*

$$[\operatorname{cis}\theta]^n = \operatorname{cis} n\theta.$$

Example 1.4.11 *Find all the complex numbers that when cubed give the value one.*

We are looking for all $\zeta \in \mathbb{C}$ such that $\zeta^3 = 1$. DeMoivre's Law offers us a nice tool for solving this equation. Let $\zeta = r \operatorname{cis}\theta$. Then $\zeta^3 = 1 \Leftrightarrow r^3 \operatorname{cis} 3\theta = 1$. Since $\left|r^3 \operatorname{cis} 3\theta\right| = r^3$, we immediately conclude that we must have $r = 1$. Hence, we need only solve the equation $\operatorname{cis} 3\theta = 1$. Due to the periodicity of the sine and cosine, we know that the last equation is equivalent to finding all θ such that $\operatorname{cis} 3\theta = \operatorname{cis}(2k\pi)$ for $k \in \mathbb{Z}$ which yields that $3\theta = 2k\pi$ for $k \in \mathbb{Z}$. But $\left\{\frac{2k\pi}{3} : k \in \mathbb{Z}\right\} = \left\{-\frac{2\pi}{3}, 0, \frac{2\pi}{3}\right\}$. Thus, we have three distinct complex numbers whose cubes are one; namely, $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$, $\operatorname{cis} 0 = 1$, and $\operatorname{cis}\left(\frac{2\pi}{3}\right)$. In rectangular form, the three complex numbers whose cubes are one are: $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$, 0, and $-\frac{1}{2} + \frac{\sqrt{3}}{2}$.

Theorem 1.4.12 (Schwarz's Inequality) *If $a_1, ..., a_n$ and $b_1, ..., b_n$ are complex numbers, then*

$$\left|\sum_{j=1}^{n} a_j \overline{b}_j\right|^2 \le \left(\sum_{j=1}^{n} \left|a_j\right|^2\right)\left(\sum_{j=1}^{n} \left|b_j\right|^2\right).$$

Proof. First the statement is certainly true if $b_k = 0$ for all k, $1 \leq k \leq n$. Thus we assume that not all the b_k are zero. Now, for any $\lambda \in \mathbb{C}$, note that

$$\sum_{j=1}^{n} \left| a_j - \lambda \overline{b_j} \right|^2 \geq 0.$$

Excursion 1.4.13 *Make use of this inequality and the choice of*

$$\lambda = \left(\sum_{j=1}^{n} a_j b_j \right) \left(\sum_{j=1}^{n} \left| b_j \right|^2 \right)^{-1}$$

to complete the proof.

■

Remark 1.4.14 *A special case of Schwarz's Lemma contains information relating the modulus of two vectors with the absolute value of their dot product. For example, if* $\overrightarrow{v_1} = (a_1, a_2)$ *and* $\overrightarrow{v_2} = (b_1, b_2)$ *are vectors in* $\mathbb{R} \times \mathbb{R}$*, then Schwarz's Lemma merely reasserts that* $\left| \overrightarrow{v_1} \bullet \overrightarrow{v_2} \right| = |a_1 b_1 + a_2 b_2| \leq \left| \overrightarrow{v_1} \right| \left| \overrightarrow{v_2} \right|$.

1.4.1 Thinking Complex

Complex variables provide a very convenient way of describing shapes and curves. It is important to gain a facility at representing sets in terms of expressions involving complex numbers because we will use them for mappings and for applications to various phenomena happening within "shapes." Towards this end, let's do some work on describing sets of complex numbers given by equations involving complex variables.

One way to obtain a description is to translate the expressions to equations involving two real variables by substituting $z = x + iy$.

Example 1.4.15 *Find all complex numbers z that satisfy*

$$2|z| = 2\operatorname{Im} z - 1.$$

Let $z = x + iy$. *Then*

$$\begin{aligned} 2|z| = 2\operatorname{Im} z - 1 &\Leftrightarrow \quad 2\sqrt{x^2 + y^2} = 2y - 1 \\ &\Leftrightarrow \left(4\left(x^2 + y^2\right) = 4y^2 - 4y + 1\right) \wedge \left(y \geq \frac{1}{2}\right) \\ &\Leftrightarrow \quad 4x^2 = -4y + 1 \wedge y \geq \frac{1}{2} \\ &\Leftrightarrow \quad x^2 = -\left(y - \frac{1}{4}\right) \wedge y \geq \frac{1}{2}. \end{aligned}$$

The last equation implies that $y \leq \frac{1}{4}$. *Since* $y \leq \frac{1}{4} \wedge y \geq \frac{1}{2}$ *is never satisfied, we conclude that the set of solutions for the given equation is empty.*

Excursion 1.4.16 *Find all* $z \in \mathbb{C}$ *such that* $|z| - z = 1 + 2i$.

*******Your work should have given the $\frac{3}{2} - 2i$ as the only solution.*******

Another way, which can be quite a time saver, is to reason by TRANSLATING TO THE GEOMETRIC DESCRIPTION. In order to do this, there are some geometric descriptions that are useful for us to recall:

$\{z : \|z - z_0\| = r\}$	is the locus of all points z equidistant from the fixed point, z_0, with the distance being $r > 0$. (a circle)
$\{z : \|z - z_1\| = \|z - z_2\|\}$	is the locus of all points z equidistant from two fixed points, z_1 and z_2. (the perpendicular bisector of the line segment joining z_1 and z_2.)
$\{z : \|z - z_1\| + \|z - z_2\| = \rho\}$ for a constant $\rho > \|z_1 - z_2\|$	is the locus of all points for which the sum of the distances from 2 fixed points, z_1 and z_2, is a constant greater than $\|z_1 - z_2\|$. (an ellipse)

Excursion 1.4.17 *For each of the following, without substituting $x + iy$ for z, sketch the set of points z that satisfy the given equations. Provide labels, names, and/or important points for each object.*

1. $\left| \dfrac{z - 2i}{z + 3 + 2i} \right| = 1$

2. $|z - 4i| + |z + 7i| = 12$

3. $|4z + 3 - i| \le 3$

*******The equations described a straight line, an ellipse, and a disk, respectively. In set notation, you should have obtained $\left\{x + iy \in \mathbb{C} : y = -\frac{3}{4}x - \frac{9}{8}\right\}$,
$\left\{x + iy \in \mathbb{C} : \frac{x^2}{\left(\frac{23}{4}\right)} + \frac{\left(y + \frac{3}{2}\right)^2}{6^2} = 1\right\}$, and
$\left\{x + iy \in \mathbb{C} : \left(x + \frac{3}{4}\right)^2 + \left(y - \frac{1}{4}\right)^2 \le \left(\frac{3}{4}\right)^2\right\}$.***

Remark 1.4.18 *In general, if k is a positive real number and $a, b \in \mathbb{C}$, then*

$$\left\{z \in \mathbb{C} : \left|\frac{z-a}{z-b}\right| = k, k \neq 1\right\}$$

describes a circle.

Excursion 1.4.19 *Use the space below to justify this remark.*

***Simplifying $\left|\frac{z-a}{z-b}\right| = k$ leads to

$$\left(1-k^2\right)|z|^2 - 2\,\mathrm{Re}\left(a\bar{z}\right) + 2k^2\,\mathrm{Re}\left(b\bar{z}\right) + \left(|a|^2 - k^2\,|b|^2\right)$$

from which the remark follows.***

1.5 Problem Set A

1. For $\mathbb{F} = \{p, q, r\}$, let the binary operations of addition, $\oplus$, and multiplication, $\otimes$, be defined by the following tables.

$\oplus$	r	q	p
r	r	q	p
q	q	p	r
p	p	r	q

$\otimes$	r	q	p
r	r	r	r
q	r	q	p
p	r	p	q

   (a) Is there an additive identity for the algebraic structure $(\mathbb{F}, \oplus, \otimes)$? Briefly justify your position.

   (b) Is the multiplicative inverse property satisfied? If yes, specify a multiplicative inverse for each element of $\mathbb{F}$ that has one.

(c) Assuming the notation from our field properties, find $(r \oplus q) \otimes \left(p \oplus p^{-1}\right)$.

(d) Is $\{(p, p), (p, r), (q, q), (p, q), (r, r)\}$ a field ordering on $\mathbb{F}$? Briefly justify your claim.

2. For a field $(\mathbb{F}, +, \cdot, e, f)$, prove each of the following parts of Proposition 1.1.6.

   (a) The multiplicative identity of a field is unique.

   (b) The multiplicative inverse of any element in $\mathbb{F} - \{e\}$ is unique.

3. For a field $(\mathbb{F}, +, \cdot, e, f)$, prove each of the following parts of Proposition 1.1.8.

   (a) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow -(a+b) = (-a) + (-b))$

   (b) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow a \cdot (-b) = -(a \cdot b))$

   (c) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot b = -(a \cdot b))$

   (d) $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (-a) \cdot (-b) = a \cdot b)$

   (e) $(\forall a)(\forall b)\left(a, b \in \mathbb{F} - \{e\} \Rightarrow (a \cdot b)^{-1} = \left(a^{-1}\right)\left(b^{-1}\right)\right)$

4. For a field $(\mathbb{F}, +, \cdot, 0, 1)$, prove Proposition 1.1.10(#1):
   $(\forall a)(\forall b)(a, b \in \mathbb{F} \Rightarrow (\exists! x)(x \in \mathbb{F} \wedge a + x = b))$

5. For a field $(\mathbb{F}, +, \cdot, 0, 1)$, show that, for $a, b, c \in \mathbb{F}$,

$$a - (b + c) = (a - b) - c \qquad \textit{and} \qquad a - (b - c) = (a - b) + c.$$

   Give reasons for each step of your demonstration.

6. For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$, prove that

   (a) $(\forall a)(\forall b)(\forall c)(\forall d)[(a, b, c, d \in \mathbb{F} \wedge a < b \wedge c < d) \Rightarrow a + c < b + d]$

   (b) $(\forall a)(\forall b)(\forall c)(\forall d)[(a, b, c, d \in \mathbb{F} \wedge 0 < a < b \wedge 0 < c < d) \Rightarrow ac < bd]$

7. For an ordered field $(\mathbb{F}, +, \cdot, 0, 1, <)$, prove each of the following

(a) $(\forall a)\,(\forall b)\,(\forall c)\left[(a, b, c \in \mathbb{F} \wedge c \neq 0) \Rightarrow \left(a \cdot c^{-1}\right) + \left(b \cdot c^{-1}\right) = (a + b) \cdot c^{-1}\right]$

(b) $(\forall a)\,(\forall b)\,(\forall c)\,(\forall d)\,[(a, c \in \mathbb{F} \wedge b, d \in \mathbb{F} - \{0\}) \Rightarrow b \cdot d \neq 0 \wedge \left(a \cdot b^{-1}\right) + \left(c \cdot d^{-1}\right) = (a \cdot d + b \cdot c) \cdot (b \cdot d)^{-1}]$

8. Find the least upper bound and the greatest lower bound for each of the following.

   (a) $\left\{\dfrac{n + (-1)^n}{n} : n \in \mathbb{N}\right\}$

   (b) $\left\{(-1)^n \left(\pi + \dfrac{1}{n}\right) : n \in \mathbb{N}\right\}$

   (c) $\left\{\dfrac{1}{m} + \dfrac{1}{n} : m, n \in \mathbb{J}\right\}$

   (d) $\left\{\dfrac{1}{1 + x^2} : x \in \mathbb{R}\right\}$

   (e) $\left\{\dfrac{1}{3^n} + \dfrac{1}{5^{n-1}} : n \in \mathbb{J}\right\}$

   (f) $\left\{x + \dfrac{1}{x} : x \in \mathbb{R} - \{0\}\right\}$

   (g) $\left\{x + \dfrac{1}{x} : \dfrac{1}{2} < x < 2\right\}$

9. Let $(X, \leq)$ be an ordered set and $A \subseteq X$. Prove that, if A has a least upper bound in X, it is unique.

10. Suppose that $S \subseteq \mathbb{R}$ is such that $\inf(S) = M$. Prove that

$$(\forall \varepsilon)\,((\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists g)\,(g \in S \wedge M \leq g < M + \varepsilon)).$$

11. For $f(x) = \dfrac{2}{x} + \dfrac{1}{x^2}$, find

    (a) $\sup f^{-1}((-\infty, 3))$

    (b) $\inf f^{-1}((3, \infty))$

12. Suppose that $P \subset Q \subset \mathbb{R}$ and $P \neq \emptyset$. If P and Q are bounded above, show that $\sup(P) \leq \sup(Q)$.

13. Let $A = \left\{x \in \mathbb{R} : (x+2)(x-3)^{-1} < -2\right\}$. Find the $\sup(A)$ and the $\inf(A)$.

14. Use the Principle of Mathematical Induction to prove that, for $a \geq 0$ and n a natural number, $(1+a)^n \geq 1 + na$.

15. Find all the values of

    (a) $(-2, 3)(4, -1)$.

    (b) $(1+2i)[3(2+i) - 2(3+6i)]$.

    (c) $(1+i)^3$.

    (d) $(1+i)^4$.

    (e) $(1+i)^n - (1-i)^n$.

16. Show that the following expressions are both equal to one.

    (a) $\left[\dfrac{-1+i\sqrt{3}}{2}\right]^3$ (b) $\left[\dfrac{-1-i\sqrt{3}}{2}\right]^3$

17. For any integers k and n, show that $i^n = i^{n+4k}$. How many distinct values can be assumed by i^n?

18. Use the Principle of Mathematical Induction to prove DeMoivre's Law.

19. If $z_1 = 3 - 4i$ and $z_2 = -2 + 3i$, obtain graphically and analytically

    (a) $2z_1 + 4z_2$.

    (b) $3z_1 - 2\overline{z_2}$.

    (c) $z_1 - \overline{z_2} - 4$.

    (d) $|z_1 + z_2|$.

    (e) $|z_1 - z_2|$.

    (f) $|2\overline{z_1} + 3\overline{z_2} - 1|$.

20. Prove that there is no ordering on the complex field that will make it an ordered field.

21. Carefully justify the following parts of Theorem 1.4.6. For z and w complex numbers,

    (a) $|z| \geq 0$ with equality only if $z = 0$,

    (b) $|\overline{z}| = |z|$,

    (c) $|zw| = |z|\,|w|$,

(d) $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$,

(e) $|z+w|^2 = |z|^2 + 2\operatorname{Re} z\overline{w} + |w|^2$.

22. Prove the "other" triangular inequality: For complex numbers z_1 and z_2, $|z_1 - z_2| \geq ||z_1| - |z_2||$.

23. Carefully justify the following parts of Theorem 1.4.8. If z and w are complex numbers, then

    (a) $\overline{z+w} = \bar{z} + \bar{w}$

    (b) $\overline{zw} = \bar{z}\bar{w}$

    (c) $\operatorname{Re} z = \dfrac{z+\bar{z}}{2}$, $\operatorname{Im} z = \dfrac{z-\bar{z}}{2i}$,

    (d) $z\bar{z}$ is a nonnegative real number.

24. Find the set of all $z \in \mathbb{C}$ that satisfy:

    (a) $1 < |z| \leq 3$.

    (b) $\left|\dfrac{z-3}{z+2}\right| = 1$.

    (c) $\operatorname{Re} z^2 > 0$.

    (d) $|z-1| + |z+1| = 2$.

    (e) $\operatorname{Im} z^2 > 0$.

    (f) $\left|\dfrac{z+2}{z-1}\right| = 2$.

    (g) $|z-2| + |z+2| = 5$.

    (h) $|z| = 1 + \operatorname{Re}(z)$.

25. When does $az + b\bar{z} + c = 0$ represent a line?

26. Prove that the vector z_1 is parallel to the vector z_2 if and only if $\operatorname{Im}(z_1\bar{z}_2) = 0$.