Chapter 2

From Finite to Uncountable Sets

A considerable amount of the material offered in this chapter is a review of terminology and results that were covered in MAT108. Our brief visit allows us to go beyond some of what we saw and to build a deeper understanding of some of the material for which a revisit would be beneficial.

2.1 Some Review of Functions

We have just seen how the concept of function gives precise meaning for binary operations that form part of the needed structure for a field. The other "big" use of function that was seen in MAT108 was with defining "set size" or cardinality. For precise meaning of what constitutes set size, we need functions with two additional properties.

Definition 2.1.1 *Let A and B be nonempty sets and $f : A \longrightarrow B$. Then*

1. *f is **one-to-one**, written $f : A \overset{1-1}{\longrightarrow} B$, if and only if*

$$(\forall x)\,(\forall y)\,(\forall z)\,((x, z) \in f \wedge (y, z) \in f \Rightarrow x = y)\,,$$

2. *f is **onto**, written $f : A \twoheadrightarrow B$, if and only if*

$$(\forall y)\,(y \in B \Rightarrow (\exists x)\,(x \in A \wedge (x, y) \in f))\,,$$

3. *f is a **one-to-one correspondence**, written $f : A \overset{1-1}{\twoheadrightarrow} B$, if and only if f is one-to-one and onto.*

Remark 2.1.2 *In terms of our other definitions, $f : A \longrightarrow B$ is onto if and only if*

$$\operatorname{rng}(f) \underset{def}{=} \{y \in B : (\exists x)(x \in A \wedge (x, y) \in f)\} = B$$

which is equivalent to $f[A] = B$.

In the next example, the first part is shown for completeness and to remind the reader about how that part of the argument that something is a function can be proved. As a matter of general practice, as long as we are looking at basic functions that result in simple algebraic combinations of variables, you can assume that was is given in that form in a function on either its implied domain or on a domain that is specified.

Example 2.1.3 *For $f = \left\{\left(x, \dfrac{x}{1-|x|}\right) \in \mathbb{R} \times \mathbb{R} : -1 < x < 1\right\}$, prove that*

$$f : (-1, 1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}.$$

(a) *By definition, $f \subseteq \mathbb{R} \times \mathbb{R}$; i.e., f is a relation from $(-1, 1)$ to $\mathbb{R}$.*

Now suppose that $x \in (-1, 1)$. Then $|x| < 1$ from which it follows that $1 - |x| \neq 0$. Hence, $(1 - |x|)^{-1} \in \mathbb{R} - \{0\}$ and $y \underset{def}{=} x \cdot (1 - |x|)^{-1} \in \mathbb{R}$ because multiplication is a binary operation on $\mathbb{R}$. Since x was arbitrary, we have shown that

$$(\forall x)(x \in (-1, 1) \Rightarrow (\exists y)(y \in \mathbb{R} \wedge (x, y) \in f))\text{; i.e.,}$$

$\operatorname{dom}(f) = (-1, 1)$.

Suppose that $(x, y) \in f \wedge (x, v) \in f$. Then $u = x \cdot (1 - |x|)^{-1} = v$ because multiplication is single-valued on $\mathbb{R} \times \mathbb{R}$. Since x, u, and v were arbitrary,

$$(\forall x)(\forall u)(\forall v)((x, u) \in f \wedge (x, v) \in f \Rightarrow u = v);$$

i.e., f is single-valued.

Because f is a single-valued relation from $(-1, 1)$ to $\mathbb{R}$ whose domain is $(-1, 1)$, we conclude that $f : (-1, 1) \to \mathbb{R}$.

(b) *Suppose that* $f(x_1) = f(x_2)$*; i.e.,* $x_1, x_2 \in (-1, 1)$ *and* $\dfrac{x_1}{1 - |x_1|} = \dfrac{x_2}{1 - |x_2|}$. *Since* $f(x_1) = f(x_2)$ *we must have that* $f(x_1) < 0 \wedge f(x_2) < 0$ *or* $f(x_1) \geq 0 \wedge f(x_2) \geq 0$ *which implies that* $-1 < x_1 < 0 \wedge -1 < x_2 < 0$ *or* $1 > x_1 \geq 0 \wedge 1 > x_2 \geq 0$. *Now* $x_1, x_2 \in (-1, 0)$ *yields that* $f(x_1) = \dfrac{x_1}{1 + x_1} = \dfrac{x_2}{1 + x_2} = f(x_2)$, *while* $x_1, x_2 \in [0, 1)$ *leads to* $f(x_1) = \dfrac{x_1}{1 - x_1} = \dfrac{x_2}{1 - x_2} = f(x_2)$. *In either case, a simple calculation gives that* $x_1 = x_2$. *Since* x_1 *and* x_2 *were arbitrary,* $(\forall x_1)(\forall x_2)(f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$. *Therefore,* f *is one-to-one.*

(c) *Finally, fill in what is missing to finish showing that* f *is onto. Let* $w \in \mathbb{R}$. *Then either* $w < 0$ *or* $w \geq 0$. *For* $w < 0$, *let* $x = \dfrac{w}{1 - w}$. *Then* $(1 - w) > 0$ *and, because* $-1 < 0$, *we have that* $-1 + w < w$ *or* $-(1 - w) < w$. *Hence,* $-1 < \dfrac{w}{1 - w}$ *and we conclude the* $x \in$ $\underset{(1)}{\underline{\hspace{3cm}}}$. *It follows that* $|x| = \underset{(2)}{\underline{\hspace{3cm}}}$ *and*

$$f(x) = \frac{x}{1 - |x|} = \underset{(3)}{\underline{\hspace{8cm}}}.$$

For $w \geq 0$, *let* $x = \dfrac{w}{1 + w}$. *Because* $1 > 0$ *and* $w \geq 0$ *implies that* $\underset{(4)}{\underline{\hspace{3cm}}} > w > 0$ *which is equivalent to having* $1 > x = \dfrac{w}{1 + w} > 0$. *Hence,* $|x| = \underset{(5)}{\underline{\hspace{3cm}}}$ *and*

$$f(x) = \underset{(6)}{\underline{\hspace{8cm}}}.$$

Since $w \in \mathbb{R}$ *was arbitrary, we conclude that* f *maps* $(-1, 1)$ *onto* $\mathbb{R}$.

***Acceptable responses are: (1) $(-1, 0)$, (2) $\dfrac{-w}{1 - w}$,
(3) $\left(\dfrac{w}{1 - w}\right)\left(1 + \dfrac{w}{1 - w}\right)^{-1} = w$, (4) $1 + w$, (5) $\dfrac{w}{1 + w}$,
(6) $\left(\dfrac{w}{1 + w}\right)\left(1 - \dfrac{w}{1 + w}\right)^{-1} = w$.***

Given a relation from a set A to a set B, we saw two relations that could be used to describe or characterize properties of functions.

Definition 2.1.4 *Given sets A, B, and C, let $R \in \mathcal{P}(A \times B)$ and $S \in P(B \times C)$ where $\mathcal{P}(X)$ denotes the power set of X.*

1. *the **inverse** of R, denoted by R^{-1}, is $\{(y, x) : (x, y) \in R\}$;*
2. *the **composition** of R and S, denoted by $S \circ R$, is*

$$\{(x, z) \in A \times C : (\exists y)((x, y) \in R \wedge (y, z) \in S)\}.$$

Example 2.1.5 *For $R = \left\{(x, y) \in \mathbb{N} \times \mathbb{Z} : x^2 + y^2 \leq 4\right\}$ and $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 1\}$, $R^{-1} = \{(0, 1), (-1, 1), (1, 1), (0, 2)\}$, $S^{-1} = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \dfrac{x - 1}{2}\right\}$, and $S \circ R = \{(1, 1), (1, 3), (1, -1), (2, 1)\}$.*

Note that the inverse of a relation from a set A to a set B is always a relation from B to A; this is because a relation is an arbitrary subset of a Cartesian product that neither restricts nor requires any extent to which elements of A or B must be used. On the other hand, while the inverse of a function must be a relation, it need not be a function; even if the inverse is a function, it need not be a function with domain B. The following theorem, from MAT108, gave us necessary and sufficient conditions under which the inverse of a function is a function.

Theorem 2.1.6 *Let $f : A \to B$. Then f^{-1} is a function if and only if f is one–to–one. If f^{-1} is a function, then f^{-1} is a function from B into A if and only if f is a function from A onto B.*

We also saw many results that related inverses, compositions and the identity function. These should have included all or a large subset of the following.

Theorem 2.1.7 *Let $f : A \to B$ and $g : B \to C$. Then $g \circ f$ is a function from A into C.*

Theorem 2.1.8 *Suppose that A, B, C, and D are sets, $R \in \mathcal{P}(A \times B)$, $S \in \mathcal{P}(B \times C)$, and $T \in \mathcal{P}(C \times D)$. Then*

$$T \circ (S \circ R) = (T \circ S) \circ R.$$

and

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

Theorem 2.1.9 *Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then*

1. $R \circ R^{-1} \in \mathcal{P}(B \times B)$ *and, whenever R is single-valued,* $R \circ R^{-1} \subseteq I_B$
2. $R^{-1} \circ R \in \mathcal{P}(A \times A)$ *and, whenever R is one-to-one,* $R^{-1} \circ R \subseteq I_A$
3. $\left(R^{-1}\right)^{-1} = R$
4. $I_B \circ R = R$ *and* $R \circ I_A = R$.

Theorem 2.1.10 *For $f : A \to B$ and $g : B \to C$,*

1. *If f and g are one–to–one, then $g \circ f$ is one–to–one.*
2. *If f is onto B and g is onto C, then $g \circ f$ is onto C.*
3. *If $g \circ f$ is one–to–one, then f is one–to–one.*
4. *If $g \circ f$ is onto C then g is onto C.*

Theorem 2.1.11 *Suppose that A, B, C, and D are sets in the universe $\mathcal{U}$.*

1. *If h is a function having* $\operatorname{dom} h = A$, *g is a function such that* $\operatorname{dom} g = B$, *and $A \cap B = \emptyset$, then $h \cup g$ is a function such that* $\operatorname{dom}(h \cup g) = A \cup B$.
2. *If $h : A \twoheadrightarrow C$, $g : B \twoheadrightarrow D$ and $A \cap B = \emptyset$, then $h \cup g : A \cup B \twoheadrightarrow C \cup D$.*
3. *If $h : A \overset{1-1}{\to} C$, $g : B \overset{1-1}{\to} D$, $A \cap B = \emptyset$, and $C \cap D = \emptyset$, then*
$$h \cup g : A \cup B \overset{1-1}{\to} C \cup D.$$

Remark 2.1.12 *Theorem 2.1.11 can be used to give a slightly different proof of the result that was shown in Example 2.1.3. Notice that the relation f that was given in Example 2.1.3 can be realized as $f_1 \cup f_2$ where*

$$f_1 = \left\{ \left(x, \frac{x}{1+x} \right) \in \mathbb{R} \times \mathbb{R} : -1 < x < 0 \right\}$$

and

$$f_2 = \left\{ \left(x, \frac{x}{1-x} \right) \in \mathbb{R} \times \mathbb{R} : 0 \le x < 1 \right\};$$

for this set-up, we would show that $f_1 : (-1, 0) \overset{1-1}{\twoheadrightarrow} (-\infty, 0)$ and $f_2 : [0, 1) \overset{1-1}{\twoheadrightarrow} [0, \infty)$ and claim $f_1 \cup f_2 : (-1, 1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}$ from Theorem 2.1.11, parts (#2) and (#3).

2.2 A Review of Cardinal Equivalence

Definition 2.2.1 *Two sets A and B are said to be* ***cardinally equivalent****, denoted by $A \sim B$, if and only if $(\exists f)\left(f : A \overset{1-1}{\twoheadrightarrow} B\right)$. If $A \sim B$ (read "A is equivalent to B"), then A and B are said to have the same cardinality.*

Example 2.2.2 *Let $A = \{0\}$ and $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Make use of the relation*

$$\{(x, f(x)) : x \in [0, 1]\}$$

where

$$f(x) = \begin{cases} \frac{1}{2} & , \text{if } x \in A \\ \frac{x}{1+2x} & , \text{if } x \in B \\ x & , \text{if } x \in [0,1] - (A \cup B) \end{cases}$$

to prove that the closed interval $[0, 1]$ is cardinally equivalent to the open interval $(0, 1)$.

Proof. *Let $F = \{(x, f(x)) : x \in [0, 1]\}$ where f is defined above. Then $F = \{(x, g_1) : x \in A \cup B\} \cup \{(x, g_2) : x \in ([0, 1] - (A \cup B))\}$ where*

$$g_1(x) = \begin{cases} \frac{1}{2} & , \text{if } x \in A \\ \frac{x}{1+2x} & , \text{if } x \in B \end{cases} \quad \text{and} \quad g_2 = f \restriction_{[0,1]-(A\cup B)}.$$

Suppose that $x \in A \cup B$. Then either $x = 0$ or there exists $n \in \mathbb{N}$ such that $x = \frac{1}{n}$. It follows that $g_1(x) = g_1(0) = \frac{1}{2}$ or $g_1(x) = g_1\left(\frac{1}{n}\right) = \frac{\frac{1}{n}}{1+2\frac{1}{n}} = \frac{1}{n+2} \in (0, 1)$. Since x was arbitrary, we have that

$$(\forall x)\left(x \in A \cup B \Rightarrow (\exists y)\left(y \in (0, 1) \wedge g_1(x) = y\right)\right).$$

Thus, $\operatorname{dom}(g_1) = A \cup B$*. Furthermore, since* $(1 + 2 \cdot x) \neq 0$ *for* $x \in B$

$$\frac{x}{1+2x} = x \cdot (1 + 2 \cdot x)^{-1}$$

is defined and single-valued because $\cdot$ *and* $+$ *are the binary operations on the field* $\mathbb{R}$*. Hence,* $g_1 : A \cup B \to (0, 1)$.

Since

$$g_1 [A \cup B] = \left\{ \frac{1}{n+2} : n \in \mathbb{N} \right\} \underset{def}{=} C,$$

we have that $g_1 : A \cup B \twoheadrightarrow C$*. Now suppose that* $x_1, x_2 \in A \cup B$ *are such that* $g_1(x_1) = g_1(x_2)$*. Then either* $g_1(x_1) = g_1(x_2) = \frac{1}{2}$ *or* $g_1(x_1), g_1(x_2) \in \left\{ \frac{1}{n+2} : n \in \mathbb{N} \right\}$*. In the first case, we have that* $x_1 = x_2 = 0$*. In the second case, we have that* $g_1(x_1) = g_1(x_2) \Rightarrow$

$$\frac{x_1}{1+2x_1} = \frac{x_2}{1+2x_2} \Leftrightarrow x_1 + 2x_1x_2 = x_2 + 2x_2x_1 \Leftrightarrow x_1 = x_2.$$

Since x_1 *and* x_2 *were arbitrary,*

$$(\forall x_1, x_2) (x_1, x_2 \in A \cup B \wedge g_1(x_1) = g_1(x_2) \Rightarrow x_1 = x_2) \text{; i.e.,}$$

g_1 *is one-to-one. Therefore,*

$$g_1 : A \cup B \overset{1-1}{\twoheadrightarrow} C.$$

Note that $[0, 1] - (A \cup B) = (0, 1) - C$*. Thus,* g_2*, as the identity function on* $(0, 1) - C$*, is one-to-one and onto. That is,*

$$g_2 : ((0, 1) - C) \overset{1-1}{\twoheadrightarrow} ((0, 1) - C).$$

From Theorem 2.1.11 (#2) *and* (#3), $g_1 : A \cup B \overset{1-1}{\twoheadrightarrow} C$, $g_2 : ((0, 1) - C) \overset{1-1}{\twoheadrightarrow} ((0, 1) - C)$, $([0, 1] - (A \cup B)) \cap (A \cup B) = \emptyset$ *and* $((0, 1) - C) \cap C = \emptyset$ *implies that*

$$g_1 \cup g_2 : (A \cup B) \cup ((0, 1) - C) \overset{1-1}{\twoheadrightarrow} C \cup ((0, 1) - C). \qquad (*)$$

Substituting $((0,1) - C) = [0,1] - (A \cup B)$ *in addition to noting that*

$$(A \cup B) \cup ([0,1] - (A \cup B)) = [0,1]$$

and

$$C \cup ((0,1) - C) = (0,1)\,,$$

we conclude from (*) that

$$F = g_1 \cup g_2 : [0,1] \overset{1-1}{\twoheadrightarrow} (0,1)\,.$$

Therefore, $|[0,1]| = |(0,1)|$. ■

For the purpose of describing and showing that sets are "finite", we make use of the following collection of "master sets." For each $k \in \mathbb{J}$, let

$$\mathbb{J}_k = \{j \in \mathbb{J} : 1 \leq j \leq k\}.$$

For $k \in \mathbb{J}$, the set $\mathbb{J}_k$ is defined to have cardinality k. The following definition offers a classification that distinguishes set sizes of interest.

Definition 2.2.3 *Let S be a set in the universe $\mathcal{U}$. Then*

1. *S is **finite** $\Leftrightarrow ((S = \emptyset) \vee (\exists k)\,(k \in \mathbb{J} \wedge S \sim \mathbb{J}_k)$.*
2. *S is **infinite** $\Leftrightarrow$ S is not finite.*
3. *S is **countably infinite** or **denumerable** $\Leftrightarrow S \sim \mathbb{J}$.*
4. *S is **at most countable** $\Leftrightarrow$ ((S is finite) $\vee$ (S is denumerable)).*
5. *S is **uncountable** $\Leftrightarrow$ S is neither finite nor countably infinite.*

Recall that if $S = \emptyset$, then it is said to have cardinal number 0, written $|S| = 0$. If $S \sim \mathbb{J}_k$, then S is said to have cardinal number k; i.e., $|S| = k$.

Remark 2.2.4 *Notice that the term countable has been omitted from the list given in Definition 2.2.3; this was done to stress that the definition of countable given by the author of our textbook is different from the definition that was used in all the MAT108 sections. The term "at most countable" corresponds to what was defined as countable in MAT108. In these Companion Notes, we will avoid confusion by not using the term countable; when reading your text, keep in mind that Rudin uses the term countable for denumerable or countably infinite.*

We know an infinite set is one that is not finite. Now it would be nice to have some meaningful infinite sets. The first one we think of is $\mathbb{N}$ or $\mathbb{J}$. While this claim may seem obvious, it needs proving. This leads to the following

Proposition 2.2.5 *The set $\mathbb{J}$ is infinite.*

Space for comments.

Proof. Since $\{\emptyset\} \underset{def}{=} 1 \in \mathbb{J}$, $\mathbb{J}$ is not empty. To prove that $\neg(\exists k)(k \in \mathbb{J} \wedge \mathbb{J}_k \sim \mathbb{J})$ is suffices to show that $(\forall k)(\forall f)\left(\left(k \in \mathbb{J} \wedge f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}\right) \Rightarrow f[\mathbb{J}_k] \neq \mathbb{J}\right)$. Suppose that $k \in \mathbb{J}$ and f is such that $f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}$. Let $n = f(1) + f(2) + \cdots + f(k) + 1$. For each j, $1 \leq j \leq k$, we have that $f(j) > 0$. Hence, n is a natural number that is greater than each $f(j)$. Thus, $n \neq f(j)$ for any $j \in \mathbb{J}_k$. But then $n \notin \operatorname{rng}(f)$ from which we conclude that f is not onto $\mathbb{J}$; i.e., $f[\mathbb{J}_k] \neq \mathbb{J}$. Since k and f were arbitrary, we have that $(\forall k)(\forall f)\left(\left(k \in \mathbb{J} \wedge f : \mathbb{J}_k \xrightarrow{1-1} \mathbb{J}\right) \Rightarrow f[\mathbb{J}_k] \neq \mathbb{J}\right)$ which is equivalent to the claim that $(\forall k)(k \in \mathbb{J} \Rightarrow \mathbb{J}_k \nsim \mathbb{J})$. Because

$$((\mathbb{J} \neq \emptyset) \wedge \neg(\exists k)(k \in \mathbb{J} \wedge \mathbb{J} \sim \mathbb{J}_k),$$

it follows that $\mathbb{J}$ is not finite as claimed. ■

Remark 2.2.6 *From the Pigeonhole Principle (various forms of which were visited in MAT108), we know that, for any set X,*

$$X \text{ finite } \Rightarrow (\forall Y)(Y \subset X \wedge Y \neq X \Rightarrow Y \nsim X).$$

The contrapositive tautology yields that

$$\neg(\forall Y)(Y \subset X \wedge Y \neq X \Rightarrow Y \nsim X) \Rightarrow \neg(X \text{ is finite})$$

which is equivalent to

$$(\exists Y)(Y \subset X \wedge Y \neq X \wedge Y \sim X) \Rightarrow X \text{ is infinite.} \qquad (\triangle)$$

In fact, $(\triangle)$ *could have been used as a alternative definition of infinite set. To see how* $(\triangle)$ *can be used to prove that a set is infinite, note that*

$$\mathbb{J}_e = \{n \in \mathbb{J} : 2|n\}$$

is such that $\mathbb{J}_e \subsetneq \mathbb{J}$ *and* $\mathbb{J}_e \sim \mathbb{J}$ *where that latter follows because* $f(x) = 2x : \mathbb{J} \overset{1-1}{\twoheadrightarrow} \mathbb{J}_e$*; consequently,* $\mathbb{J}$ *is infinite.*

Recall that the cardinal number assigned to $\mathbb{J}$ is $\aleph_0$ which is read as "aleph naught." Also shown in MAT108 was that the set $\mathcal{P}(\mathbb{J})$ cannot be (cardinally) equivalent to $\mathbb{J}$; this was a special case of

Theorem 2.2.7 (Cantor's Theorem) *For any set* S, $|S| < |\mathcal{P}(S)|$.

Remark 2.2.8 *It can be shown, and in some sections of MAT108 it was shown, that* $\mathcal{P}(\mathbb{J}) \sim \mathbb{R}$*. Since* $|\mathbb{J}| < |\mathbb{R}|$*, the cardinality of* $\mathbb{R}$ *represents a different "level of infinite." The symbol given for the cardinality of* $\mathbb{R}$ *is* **c**, *an abbreviation for continuum.*

Excursion 2.2.9 *As a memory refresher concerning proofs of cardinal equivalence, complete each of the following.*

1. *Prove* $(2,4) \sim (-5,20)$.

2. *Use the function* $f(n) = \begin{cases} \dfrac{n}{2} & , \quad n \in \mathbb{J} \wedge 2 \mid n \\ -\dfrac{n-1}{2} & , \quad n \in \mathbb{J} \wedge 2 \nmid n \end{cases}$

to prove that $\mathbb{Z}$ is denumerable.

*******For (1), one of the functions that would have worked is $f(x) = \frac{25}{2}x - 30$; justifying that $f : (2,4) \xrightarrow[]{1-1} \twoheadrightarrow (-5,20)$ involves only simple algebraic manipulations. Showing that the function given in (2) in one-to-one and onto involves applying elementary algebra to the several cases that need to be considered for members of the domain and range.*******

We close this section with a proposition that illustrates the general approach that can be used for drawing conclusions concerning the cardinality of the union of two sets having known cardinalities

Proposition 2.2.10 *The union a denumerable set and a finite set is denumerable; i.e.,*

$$(\forall A)\,(\forall B)\,(A \text{ denumerable} \ \wedge\ B \text{ finite} \ \Rightarrow (A \cup B) \ \text{ is denumerable}).$$

Proof. Let A and B be sets such that A is denumerable and B is finite. First we will prove that $A \cup B$ is denumerable when $A \cap B = \emptyset$. Since B is finite, we have that either $B = \emptyset$ or there exists a natural number k and a function f such that $f : B \overset{1-1}{\twoheadrightarrow} \{j \in \mathbb{N} : j \leq k\}$.

If $B = \emptyset$, then $A \cup B = A$ is denumerable. If $B \neq \emptyset$, then let f be such that $f : B \overset{1-1}{\twoheadrightarrow} \mathbb{N}_k$ where $\mathbb{N}_k \underset{def}{=} \{j \in \mathbb{N}_k : j \leq k\}$. Since A is denumerable, there exists a function g such that $g : A \overset{1-1}{\twoheadrightarrow} \mathbb{N}$. Now let $h = \{(n, n+k) : n \in \mathbb{N}\}$. Because addition is a binary operation on $\mathbb{N}$ and $\mathbb{N}$ is closed under addition, for each $n \in \mathbb{N}$, $n + k$ is a uniquely determined natural number. Hence, we have that $h : \mathbb{N} \to \mathbb{N}$. Since $n \in \mathbb{N}$ implies that $n \geq 1$, from OF1, $n+k \geq 1+k$; consequently, $\{j \in \mathbb{N} : j \geq 1 + k\} = \mathbb{N} - \mathbb{N}_k$ is a codomain for h. Thus, $h : \mathbb{N} \to \mathbb{N} - \mathbb{N}_k$.

We will now show that h is one-to-one and onto $\mathbb{N}-\mathbb{N}_k$.

(i) Suppose that $h(n_1)=h(n_2)$; i.e., $n_1+k=n_2+k$. Since $\mathbb{N}$ is the set of natural numbers for the field of real numbers, there exists an additive inverse $(-k)\in\mathbb{R}$ such that $k+(-k)=(-k)+k=0$. From associativity and substitution, we have that

$$\begin{aligned} n_1 = n_1+(k+(-k)) &= (n_1+k)+(-k) \\ &= (n_2+k)+(-k) \\ &= n_2+(k+(-k)) \\ &= n_2. \end{aligned}$$

Since n_1 and n_2 were arbitrary, $(\forall n_1)(\forall n_2)(h(n_1)=h(n_2)\Rightarrow n_1=n_2)$; i.e., h is one-to-one.

(ii) Let $w\in\mathbb{N}-\mathbb{N}_k$. Then $w\in\mathbb{N}$ and $w\geq 1+k$. By OF1, associativity of addition, and the additive inverse property,

$$w+(-k)\geq(1+k)+(-k)=1+(k+(-k))=1.$$

Hence, $x \underset{def}{=} w+(-k)\in\mathbb{N}=\mathrm{dom}(h)$. Furthermore,

$$h(x)=x+k=(w+(-k))+k=w+((-k)+k)=w.$$

Since w was arbitrary, we have shown that

$$(\forall w)(w\in\mathbb{N}-\mathbb{N}_k\Rightarrow(\exists x)(x\in\mathbb{N}\wedge(x,w)\in h));$$

that is, h is onto.

From (i) and (ii), we conclude that $h:\mathbb{N}\overset{1-1}{\twoheadrightarrow}\mathbb{N}-\mathbb{N}_k$. From Theorem 2.1.10, parts (1) and (2), $g:A\overset{1-1}{\twoheadrightarrow}\mathbb{N}$ and $h:\mathbb{N}\overset{1-1}{\twoheadrightarrow}\mathbb{N}-\mathbb{N}_k$ implies that

$$h\circ g:A\overset{1-1}{\twoheadrightarrow}\mathbb{N}-\mathbb{N}_k.$$

Now we consider the new function $F=f\cup(h\circ g)$ from $B\cup A$ into $\mathbb{N}$ which can also be written as

$$F(x)=\begin{cases} f(x) & \text{for } x\in B \\ (h\circ g)(x) & \text{for } x\in A \end{cases}.$$

Since $A\cap B=\emptyset$, $\mathbb{N}\cap(\mathbb{N}-\mathbb{N}_k)=\emptyset$, $f:B\overset{1-1}{\twoheadrightarrow}\mathbb{N}_k$ and $h\circ g:A\overset{1-1}{\twoheadrightarrow}\mathbb{N}-\mathbb{N}_k$, by Theorems 2.1.11, part (1) and (2), $F:B\cup A\overset{1-1}{\twoheadrightarrow}\mathbb{N}\cup(\mathbb{N}-\mathbb{N}_k)=\mathbb{N}$. Therefore, $B\cup A$ or $A\cup B$ is cardinally equivalent to $\mathbb{N}$; i.e., $A\cup B$ is denumerable.

If $A \cap B \neq \emptyset$, then we consider the sets $A - B$ and B. In this case, $(A - B) \cap B = \emptyset$ and $(A - B) \cup B = A \cup B$. Now the set B is finite and the set $A - B$ is denumerable. The latter follows from what we showed above because our proof for the function h was for k arbitrary, which yields that

$$(\forall k)\,(k \in \mathbb{N} \Rightarrow |\mathbb{N} - \mathbb{N}_k| = \aleph_0)\,.$$

From the argument above, we again conclude that $A \cup B = (A - B) \cup B$ is denumerable.

Since A and B were arbitrary,

$$(\forall A)\,(\forall B)\,(A \text{ denumerable } \wedge\ B \text{ finite } \Rightarrow (A \cup B) \text{ is denumerable})\,.$$

■

2.2.1 Denumerable Sets and Sequences

An important observation that we will use to prove some results concerning at most countable sets and families of such sets is the fact that a denumerable set can be "arranged in an (infinite) sequence." First we will clarify what is meant by arranging a set as a sequence.

Definition 2.2.11 *Let A be a nonempty set. A **sequence** of elements of A is a function $f : \mathbb{J} \longrightarrow A$. Any $f : \mathbb{J}_k \longrightarrow A$ for a $k \in \mathbb{J}$ is a **finite sequence** of elements of A.*

For $f : \mathbb{J} \longrightarrow A$, letting $a_n = f(n)$ leads to the following common notations for the sequence: $\{a_n\}_{n=1}^{\infty}$, $\{a_n\}_{n\in\mathbb{J}}$, $\{a_n\}$, or $a_1, a_2, a_3, ..., a_n, ...$ It is important to notice the distinction between $\{a_n\}_{n=1}^{\infty}$ and $\{a_n : n \in \mathbb{J}\}$; the former is a sequence where the listed terms need not be distinct, while the latter is a set. For example, if $f : \mathbb{J} \longrightarrow \{1, 2, 3\}$ is the constant function $f(n) = 1$, then

$$\{a_n\}_{n=1}^{\infty} = 1, 1, 1, ...$$

while $\{a_n : n \in \mathbb{J}\} = \{1\}$.

Now, if A is a denumerable or countably infinite set then there exists a function g such that $g : \mathbb{J} \overset{1-1}{\twoheadrightarrow} A$. In this case, letting $g(n) = x_n$ leads to a sequence $\{x_n\}_{n\in\mathbb{J}}$ of elements of A that exhausts A; i.e., every element of A appears someplace in the sequence. This phenomenon explains our meaning to saying that the "elements of A can be arranged in an infinite sequence." The proof of the following theorem illustrates an application of this phenomenon.

Theorem 2.2.12 *Every infinite subset of a countably infinite set is countably infinite.*

Proof. Let A be a denumerable set and E be an infinite subset of A. Because A is denumerable, it can be arranged in an infinite sequence, say $\{a_n\}_{n=1}^{\infty}$. Let

$$S_1 = \{m \in \mathbb{J} : a_m \in E\}.$$

Because S_1 is a nonempty set of natural numbers, by the Well-Ordering Principle, S_1 has a least element. Let n_1 denote the least element of S_1 and set

$$S_2 = \{m \in \mathbb{J} : a_m \in E\} - \{n_1\}.$$

Since E is infinite, S_2 is a nonempty set of natural numbers. By the Well-Ordering Principle, S_2 has a least element, say n_2. In general, for S_1, S_2, ..., S_{k-1} and $n_1, n_2, ..., n_{k-1}$, we choose

$$n_k = \min S_k \qquad where \qquad S_k = \{m \in \mathbb{J} : a_m \in E\} - \{n_1, n_2, ..., n_{k-1}\}$$

Use the space provided to convince yourself that this choice "arranges E into an infinite sequence $\left\{a_{n_k}\right\}_{k=1}^{\infty}$."

■

2.3 Review of Indexed Families of Sets

Recall that if $\mathcal{F}$ is an indexed family of subsets of a set S and Δ denotes the indexing set, then the **union of the sets in** $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$, denoted by $\bigcup_{\alpha\in\Delta} A_\alpha$, is

$$\{p \in S : (\exists\beta)\left(\beta \in \Delta \wedge p \in A_\beta\right)\};$$

and the **intersection of the sets in** $\mathcal{F}$, denoted by $\bigcap_{\alpha\in\Delta} A_\alpha$, is

$$\{p \in S : (\forall\beta)\left(\beta \in \Delta \Rightarrow p \in A_\beta\right)\}.$$

Remark 2.3.1 *If $\mathcal{F}$ is a countably infinite or denumerable family of sets (subsets of a set S), then the indexing set is $\mathbb{J}$ or $\mathbb{N}$; in this case, the union and intersection over $\mathcal{F}$ are commonly written as $\bigcup_{j=1}^{\infty} A_j$ and $\bigcap_{j=1}^{\infty} A_j$, respectively. If $\mathcal{F}$ is a nonempty finite family of sets, then $\mathbb{J}_k$, for some $k \in \mathbb{J}$, can be used as an indexing set; in this case, the union and intersection over $\mathcal{F}$ are written as $\bigcup_{j=1}^{k} A_j$ and $\bigcap_{j=1}^{k} A_j$, respectively.*

It is important to keep in mind that, in an indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$, different subscript assignments does not ensure that the sets represented are different. An example that you saw in MAT108 was with equivalence classes. For the relation $\equiv_3$that was defined over $\mathbb{Z}$ by $x \equiv_3 y \Leftrightarrow 3|\,(x-y)$, for any $\alpha \in \mathbb{Z}$, let $A_\alpha = [\alpha]_{\equiv_3}$; then $A_{-4} = A_2 = A_5$, though the subscripts are different. The set of equivalence classes from an equivalence relation do, however, form a pairwise disjoint family.

An indexed family $\mathcal{F} = \{A_\alpha : \alpha \in \Delta\}$ is **pairwise disjoint** if and only if

$$(\forall\alpha)\,(\forall\beta)\left(\alpha,\beta \in \Delta \wedge A_\alpha \cap A_\beta \neq \emptyset \Rightarrow A_\alpha = A_\beta\right);$$

it is **disjoint** if and only if $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$. Note that being disjoint is a weaker condition that being pairwise disjoint.

Example 2.3.2 *For each $j \in \mathbb{Z}$, where $\mathbb{Z}$ denotes the set of integers, let*

$$A_j = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} : |x_1 - j| \leq 1 \wedge |x_2| \leq 1\},$$

Find $\bigcup_{j\in\mathbb{Z}} A_j$ and $\bigcap_{j\in\mathbb{Z}} A_j$.

Each A_j consists of a "2 by 2 square" that is symmetric about the x-axis. For each $j \in \mathbb{Z}$, A_j and A_{j+1} overlap in the section where $j \le x_1 \le j+1$, while A_j and A_{j+3} have nothing in common. Consequently, $\bigcup_{j\in\mathbb{Z}} A_j = \left\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \le 1\right\}$ and $\bigcap_{j\in\mathbb{Z}} A_j = \emptyset$.

Excursion 2.3.3 *For $n \in \mathbb{N}$, let $C_n = \left[-3 + \frac{1}{2n}, \frac{5n + (-1)^n}{n}\right)$ and $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$. Find $\bigcup_{j\in\mathbb{N}} C_j$ and $\bigcap_{j\in\mathbb{N}} C_j$.*

For this one, hopefully you looked at C_n for a few n. For example, $C_1 = \left[-\frac{5}{2}, 4\right)$, $C_2 = \left[-\frac{11}{4}, 5\frac{1}{2}\right)$, and $C_3 = \left[-\frac{17}{6}, 4\frac{2}{3}\right)$. Upon noting that the left endpoints of the intervals are decreasing to -3 while the right endpoints are oscillating above and below 5 and closing in on 5, we conclude that $\bigcup_{j\in\mathbb{N}} C_j = \left(-3, 5\frac{1}{2}\right)$ and $\bigcap_{j\in\mathbb{N}} C_j = \left[-\frac{5}{2}, 4\right)$.

Excursion 2.3.4 *For $j \in \mathbb{J}$, let $A_j = \left\{x \in \mathbb{R} : x \ge \sqrt{j}\right\}$. Justify the claim that $\mathcal{A} = \left\{A_j : j \in \mathbb{J}\right\}$ is disjoint but not pairwise disjoint.*

Hopefully, your discussion led to your noticing that $A_k \cap A_m = A_{\max\{k,m\}}$. On the other hand, to justify that $\bigcap_{j \in \mathbb{N}} A_j = \emptyset$, you needed to note that given any fixed positive real number w there exists $p \in \mathbb{J}$ such that $w \notin A_p$; taking $p = \lfloor w^2 + 1 \rfloor$, where $\lfloor \bullet \rfloor$ denotes the greatest integer function, works.

2.4 Cardinality of Unions Over Families

We saw the following result, or a slight variation of it, in MAT108.

Lemma 2.4.1 *If A and B are disjoint finite sets, then $A \cup B$ is finite and*

$$|A \cup B| = |A| + |B| \,.$$

Excursion 2.4.2 *Fill in what is missing to complete the following proof of the Lemma.*

Space for Scratch Work

Proof. Suppose that A and B are finite sets such that $A \cap B = \emptyset$. If $A = \emptyset$ or $B = \emptyset$, then $A \cup B = \underset{(1)}{\underline{\qquad\qquad}}$ or $A \cup B = \underset{(2)}{\underline{\qquad\qquad}}$, respectively. In either case $A \cup B$ is $\underset{(3)}{\underline{\qquad\qquad\qquad\qquad}}$, and $|\emptyset| = 0$ yields that $|A| + |B| = |A \cup B|$. If $A \neq \emptyset$ and $B \neq \emptyset$, then there exists $k, n \in \mathbb{N}$ such that $|A| = |\{i \in \mathbb{N} : i \leq k\}|$ and $|B| = |\{i \in \mathbb{N} : i \leq n\}|$. Hence there exist functions f and g such that $f : A \overset{1-1}{\twoheadrightarrow} \underset{(4)}{\underline{\qquad\qquad\qquad}}$ and $g : \underset{(5)}{\underline{\qquad\qquad\qquad\qquad\qquad}}$. Now let $H = \{k+1, k+2, \cdots, k+n\}$. Then the function $h(x) = k + x$ is such that $h : \{i \in \mathbb{N} : i \leq n\} \overset{1-1}{\twoheadrightarrow} H$. Since the composition of one–to–one onto functions is a one–to–one correspondence, $F = h \circ g : \underset{(6)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}}$.

From Theorem 2.1.11, $A \cap B = \emptyset$,
$\{i \in \mathbb{N} : i \leq k\} \cap H = \emptyset$, $f : A \overset{1-1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq k\}$, and
$F : B \overset{1-1}{\twoheadrightarrow} H$ implies that
$f \cup F :$ $\underset{(7)}{\underline{\hspace{6cm}}}$. Since
$\{i \in \mathbb{N} : i \leq k\} \cup H =$ $\underset{(8)}{\underline{\hspace{5cm}}}$, we
conclude that $A \cup B$ is $\underset{(9)}{\underline{\hspace{2cm}}}$ and
$|A \cup B| = \underset{(10)}{\underline{\hspace{2cm}}} = \underset{(11)}{\underline{\hspace{2cm}}}$. ■

*******Acceptable responses: (1) B, (2) A, (3) finite, (4) $\{i \in \mathbb{N} : i \leq k\}$ or A_k, (5) $B \overset{1-1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq n\}$, (6) $B \overset{1-1}{\twoheadrightarrow} H$, (7) $A \cup B \overset{1-1}{\twoheadrightarrow} \{i \in \mathbb{N} : i \leq k\} \cup H$, (8) $\{i \in \mathbb{N} : i \leq k + n\}$ or A_{k+n}, (9) finite, (10) $k + n$, and (11) $|A| + |B|$.*******

Lemma 2.4.1 and the Principle of Mathematical Induction can be used to prove

Theorem 2.4.3 *The union of a finite family of finite sets is finite.*

Proof. The proof is left as an exercise. ■

Now we want to extend the result of the theorem to a comparable result concerning denumerable sets. The proof should be reminiscent of the proof that $|\mathbb{Q}| = \aleph_0$.

Theorem 2.4.4 *The union of a denumerable family of denumerable sets is denumerable.*

Proof. For each $n \in \mathbb{J}$, let E_n be a denumerable set. Each E_n can be arranged as an infinite sequence, say $\{x_{nj}\}_{j=1}^{\infty}$. Then

$$\bigcup_{k \in \mathbb{J}} E_k = \{x_{nj} : n \in \mathbb{J} \wedge j \in \mathbb{J}\}.$$

Because E_1 is denumerable and $E_1 \subset \bigcup_{j \in \mathbb{J}} E_j$, we know that $\bigcup_{j \in \mathbb{J}} E_j$ is an infinite set. We can use the sequential arrangement to establish an infinite array; let the sequence corresponding to E_n form the nth row.

$$\begin{array}{cccccccc}
x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} & \dots & \dots \\
x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} & \dots & \dots \\
x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} & \dots & \dots \\
x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & x_{46} & \dots & \dots \\
x_{51} & x_{52} & x_{53} & x_{54} & x_{55} & x_{56} & \dots & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}$$

.The terms in the infinite array can be rearranged in an expanding triangular array, such as

$$\begin{array}{cccccccc}
x_{11} & & & & & & & \\
x_{21} & x_{12} & & & & & & \\
x_{31} & x_{22} & x_{13} & & & & & \\
x_{41} & x_{32} & x_{23} & x_{14} & & & & \\
x_{51} & x_{42} & x_{33} & x_{24} & x_{15} & & & \\
x_{61} & x_{52} & x_{43} & x_{34} & x_{25} & x_{16} & & \\
x_{71} & x_{62} & x_{53} & x_{44} & x_{35} & x_{26} & x_{17} & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots
\end{array} .$$

This leads us to the following infinite sequence:

$$x_{11}, x_{21}, x_{21}, x_{31}, x_{22}, x_{13}, \dots$$

Because we have not specified that each E_n is distinct, the infinite sequence may list elements from $\bigcup_{k\in\mathbb{J}} E_k$ more than once; in this case, $\bigcup_{k\in\mathbb{J}} E_k$ would correspond to an infinite subsequence of the given arrangement. Consequently, $\bigcup_{k\in\mathbb{J}} E_k$ is denumerable, as needed. ■

Corollary 2.4.5 *If A is at most countable, and, for each $\alpha \in A$, B_α is at most countable, then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is at most countable.

The last theorem in this section determines the cardinality of sets of $n - tuples$ that are formed from a given countably infinite set.

Theorem 2.4.6 *For A a denumerable set and $n \in \mathbb{J}$, let $T_n = \underbrace{A \times A \times \cdots \times A}_{n\,of\,them} = A^n$; i.e.,*

$$T_n = \left\{ (a_1, a_2, ..., a_n) : (\forall j) \left(j \in \mathbb{J} \wedge 1 \leq j \leq n \Rightarrow a_j \in A \right) \right\}.$$

Then T_n is denumerable.

Proof. Let $S = \{n \in \mathbb{J} : T_n \sim \mathbb{J}\}$. Since $T_1 = A$ and A is denumerable, $1 \in S$. Suppose that $k \in S$; i.e., $k \in \mathbb{J}$ and T_k is denumerable. Now $T_{k+1} = T_k \times A$ where it is understood that $((x_1, x_2, \ldots, x_k), a) = (x_1, x_2, \ldots, x_k, a)$. For each $b \in T_k$, $\{(b, a) : a \in A\} \sim A$. Hence,

$$(\forall b) (b \in T_k \Rightarrow \{(b, a) : a \in A\} \sim \mathbb{J}).$$

Because T_k is denumerable and

$$T_{k+1} = \bigcup_{b \in T_k} \{(b, a) : a \in A\}$$

it follows from Theorem 2.4.4 that T_{k+1} is denumerable; i.e., $(k + 1) \in S$. Since k was arbitrary, we conclude that $(\forall k) (k \in S \Rightarrow (k + 1) \in S)$.

By the Principle of Mathematical Induction,

$$1 \in S \wedge (\forall k) (k \in S \Rightarrow (k + 1) \in S)$$

implies that $S = \mathbb{J}$. ∎

Corollary 2.4.7 *The set of all rational numbers is denumerable.*

Proof. This follows immediately upon noting that

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z} \wedge q \in \mathbb{J} \wedge \gcd(p, q) = 1 \right\} \sim \{(p, q) \in \mathbb{Z} \times \mathbb{J} : \gcd(p, q) = 1\}$$

and $\mathbb{Z} \times \mathbb{J}$ is an infinite subset of $\mathbb{Z} \times \mathbb{Z}$ which is denumerable by the theorem. ∎

2.5 The Uncountable Reals

In Example 2.1.3, it was shown that $f(x) = \dfrac{x}{1+|x|} : (-1,1) \overset{1-1}{\twoheadrightarrow} \mathbb{R}$. Hence, the interval $(-1,1)$ is cardinally equivalent to $\mathbb{R}$. The map $g(x) = \frac{1}{2}(x+1)$ can be used to show that $(-1,1) \sim (0,1)$. We noted earlier that $|\mathbb{J}| < |\mathbb{R}|$. For completeness, we restate the theorem and quickly review the proof.

Theorem 2.5.1 *The open interval* $(0,1)$ *is uncountable. Consequently,* $\mathbb{R}$ *is uncountable.*

Proof. Since $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \subseteq (0,1)$ and $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \sim \mathbb{J}$, we know that $(0,1)$ is not finite.

Suppose that

$$f : \mathbb{J} \overset{1-1}{\to} (0,1).$$

Then we can write

$$\begin{array}{lcl} f(1) & = & 0.a_{11}a_{12}a_{13}a_{14}\cdots\cdots\cdots\cdot\cdot \\ f(2) & = & 0.a_{21}a_{22}a_{23}a_{24}\cdots\cdots\cdots\cdot\cdot \\ f(3) & = & 0.a_{31}a_{32}a_{33}a_{34}\cdots\cdots\cdots\cdot \\ \vdots & & \\ \vdots & & \\ f(n) & = & 0.a_{n1}a_{n2}a_{n3}a_{n4}\cdots\cdots\cdots\cdot\cdot \\ \vdots & & \\ \vdots & & \end{array}$$

where $a_{km} \in \{0,1,2,3,4,5,6,7,8,9\}$. Because f is one-to-one, we know that, if .20000... is in the listing, then .199999... is not.

Finally, let $m = 0.b_1b_2b_3b_4\cdots$, where $b_j = \begin{cases} \langle \quad \rangle, & \text{if} \quad a_{jj} \neq \langle \quad \rangle \\ [\quad], & \text{if} \quad a_{jj} = \langle \quad \rangle \end{cases}$ (The substitutions for $\langle \bullet \rangle$.and $[\bullet]$ are yours to choose.). Now justify that there is no $q \in \mathbb{J}$

such that $f(q) = m$.

Hence, $(\exists m)(\forall k)(k \in J \Rightarrow f(k) \neq m)$; i.e., f is not onto.

Since f was arbitrary, we have shown that

$$(\forall f)\left(f : \mathbb{J} \to (0,1) \wedge f \text{ one-to-one} \Rightarrow f \text{ is not onto}\right).$$

Because $[(P \wedge Q) \Rightarrow \neg M]$ is logically equivalent to $[P \Rightarrow \neg(Q \wedge M)]$ and $\neg[P \Rightarrow Q]$ is equivalent to $[P \wedge \neg Q]$ for any propositions P, Q and M, we conclude that

$$\begin{aligned} &\left[(\forall f)\left(f : \mathbb{J} \to (0,1) \Rightarrow \neg(f \text{ one-to-one} \wedge f \text{ is onto})\right)\right] \\ \Leftrightarrow \quad &(\forall f)\neg\left(f : \mathbb{J} \to (0,1) \wedge f \text{ one-to-one} \wedge f \text{ is onto}\right); \end{aligned}$$

i.e., $\neg(\exists f)\left(f : \mathbb{J} \overset{1-1}{\twoheadrightarrow} (0,1)\right)$. Hence, the open interval $(0,1)$ is an infinite set that is not denumerable. ■

Corollary 2.5.2 *The set of sequences whose terms are the digits* 0 *and* 1 *is an uncountable set.*

2.6 Problem Set B

1. For each of the following relations, find R^{-1}.

   (a) $R = \{(1,3), (1,5), (5,7), (10,12)\}$

   (b) $R = \left\{(x,y) \in \mathbb{R} \times \mathbb{R} : y = x^2\right\}$

   (c) $R = \{(a,b) \in A \times B : a|b\}$ where $A = \mathbb{J}$ and $B = \{j \in \mathbb{Z} : |j| \leq 6\}$

2. Prove that each of the following is one-to-one on its domain.

   (a) $f(x) = \dfrac{2x+5}{3x-2}$

(b) $f(x) = x^3$

3. Prove that $f(x) = x^2 - 6x + 5$ maps $\mathbb{R}$ onto $[-4, \infty)$.

4. Prove each of the following parts of theorems that were stated in this chapter.

   (a) Suppose that A, B, C, and D are sets, $R \in \mathcal{P}(A \times B)$, $S \in \mathcal{P}(B \times C)$, and $T \in \mathcal{P}(C \times D)$. Then $T \circ (S \circ R) = (T \circ S) \circ R$

   (b) Suppose that A, B, and C are sets, $R \in \mathcal{P}(A \times B)$ and $S \in \mathcal{P}(B \times C)$. Then
   $$(S \circ R)^{-1} = R^{-1} \circ S^{-1}.$$

   (c) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$R \circ R^{-1} \in \mathcal{P}(B \times B) \textit{ and, whenever } R \text{ is single-valued, } R \circ R^{-1} \subseteq I_B.$$

   (d) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$R^{-1} \circ R \in \mathcal{P}(A \times A) \textit{ and, wnenever } R \text{ is one-to-one, } R^{-1} \circ R \subseteq I_A.$$

   (e) Suppose that A and B are sets and that $R \in \mathcal{P}(A \times B)$. Then
   $$\left(R^{-1}\right)^{-1} = R, I_B \circ R = Rand\ R \circ I_A = R.$$

5. For $f : A \to B$ and $g : B \to C$, prove each of the following.

   (a) If f and g are one–to–one, then $g \circ f$ is one–to–one.

   (b) If f is onto B and g is onto C, then $g \circ f$ is onto C.

   (c) If $g \circ f$ is one–to–one, then f is one–to–one.

   (d) If $g \circ f$ is onto C then g is onto C.

6. For A, B, C, and D sets in the universe $\mathcal{U}$, prove each of the following.

   (a) If h is a function having $\operatorname{dom} h = A$, g is a function such that $\operatorname{dom} g = B$, and $A \cap B = \emptyset$, then $h \cup g$ is a function such that $\operatorname{dom}(h \cup g) = A \cup B$.

   (b) If $h : A \twoheadrightarrow C$, $g : B \twoheadrightarrow D$ and $A \cap B = \emptyset$, then $h \cup g : A \cup B \twoheadrightarrow C \cup D$.

(c) If $h : A \overset{1-1}{\to} C$, $g : B \overset{1-1}{\to} D$, $A \cap B = \emptyset$, and $C \cap D = \emptyset$, then

$$h \cup g : A \cup B \overset{1-1}{\to} C \cup D.$$

7. Prove each of the following cardinal equivalences.

   (a) $[-6, 10] \sim [1, 4]$
   (b) $(-\infty, 3) \sim (1, \infty)$
   (c) $(-\infty, 1) \sim (1, 2)$
   (d) $\mathbb{W} = \mathbb{J} \cup \{0\} \sim \mathbb{Z}$

8. Prove that the set of natural numbers that are primes is infinite.

9. Let A be a nonempty finite set and B be a denumerable set. Prove that $A \times B$ is denumerable.

10. Find the union and intersection of each of the following families of sets.

   (a) $\mathcal{A} = \{\{1, 3, 5\}, \{2, 3, 4, 5, 6\}, \{0, 3, 7, 9\}\}$
   (b) $\mathcal{A} = \{A_n : n \in \mathbb{J}\}$ where $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n}\right)$
   (c) $\mathcal{B} = \{B_n : n \in \mathbb{J}\}$ where $B_n = \left(-\frac{1}{n}, n\right)$
   (d) $\mathcal{C} = \{C_n : n \in \mathbb{J}\}$ where $C_n = \left\{x \in \mathbb{R} : 4 - \frac{3}{n} < x < 6 + \frac{2}{3n}\right\}$

11. Prove that the finite union of finite sets is finite.

12. For $\mathbb{W} = \mathbb{J} \cup \{0\}$, let $F : \mathbb{W} \times \mathbb{W} \longrightarrow \mathbb{W}$ be defined by

$$F(i, j) = j + \frac{k(k+1)}{2}$$

   where $k = i + j$. Prove that F is a one-to-one correspondence.

13. Prove that $\mathbb{Q} \times \mathbb{Q}$ is denumerable.