Chapter 3

METRIC SPACES and SOME BASIC TOPOLOGY

Thus far, our focus has been on studying, reviewing, and/or developing an understanding and ability to make use of properties of $\mathbb{R} = \mathbb{R}^1$. The next goal is to generalize our work to $\mathbb{R}^n$ and, eventually, to study functions on $\mathbb{R}^n$.

3.1 Euclidean n-space

The set $\mathbb{R}^n$ is an extension of the concept of the Cartesian product of two sets that was studied in MAT108. For completeness, we include the following

Definition 3.1.1 *Let S and T be sets. The **Cartesian product** of S and T, denoted by $S \times T$, is*

$$\{(p, q) : p \in S \wedge q \in T\}.$$

The Cartesian product of any finite number of sets $S_1, S_2, \ldots, S_N$, denoted by $S_1 \times S_2 \times \cdots \times S_N$, is

$$\left\{(p_1, p_2, \ldots, p_N) : (\forall j)\left((j \in \mathbb{J} \wedge 1 \leq j \leq N) \Rightarrow p_j \in S_j\right)\right\}.$$

*The object $(p_1, p_2, \ldots, p_N)$ is called an N-**tuple**.*

Our primary interest is going to be the case where each set is the set of real numbers.

Definition 3.1.2 ***Real*** ***n*-space**, *denoted* $\mathbb{R}^n$, *is the set all ordered n-tuples of real numbers; i.e.,*

$$\mathbb{R}^n = \{(x_1, x_2, ..., x_n) : x_1, x_2, ..., x_n \in \mathbb{R}\}.$$

Thus, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ of them}}$, *the Cartesian product of* $\mathbb{R}$ *with itself n times.*

Remark 3.1.3 *From MAT108, recall the definition of an* ***ordered pair****:*

$$(a, b) \underset{def}{=} \{\{a\}, \{a, b\}\}.$$

This definition leads to the more familiar statement that $(a, b) = (c, d)$ *if and only if* $a = b$ *and* $c = d$. *It also follows from the definition that, for sets* A, B *and* C, $(A \times B) \times C$ *is, in general, not equal to* $A \times (B \times C)$; *i.e., the Cartesian product is not associative. Hence, some conventions are introduced in order to give meaning to the extension of the binary operation to more that two sets. If we define ordered triples in terms of ordered pairs by setting* $(a, b, c) = ((a, b), c)$; *this would allow us to claim that* $(a, b, c) = (x, y, z)$ *if and only if* $a = x$, $b = y$, *and* $c = z$. *With this in mind, we interpret the Cartesian product of sets that are themselves Cartesian products as "big" Cartesian products with each entry in the tuple inheriting restrictions from the original sets. The point is to have helpful descriptions of objects that are described in terms of n-tuple.*

Addition and scalar multiplication on n-tuple is defined by

$$(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

and

$$\alpha\,(x_1, x_2, ..., x_n) = (\alpha x_1, \alpha x_2, ..., \alpha x_n)\,, \text{ for } \alpha \in \mathbb{R}, \text{ respectively.}$$

The geometric meaning of addition and scalar multiplication over $\mathbb{R}^2$ and $\mathbb{R}^3$ as well as other properties of these vector spaces was the subject of extensive study in vector calculus courses (MAT21D on this campus). For each n, $n \geq 2$, it can be shown that $\mathbb{R}^n$ is a real vector space.

Definition 3.1.4 *A* ***real vector space*** $\mathbb{V}$ *is a set of elements called* ***vectors****, with given operations of* ***vector addition*** $+ : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$ *and* ***scalar multiplication*** $\cdot : \mathbb{R} \times \mathbb{V} \longrightarrow \mathbb{V}$ *that satisfy each of the following:*

1. $(\forall \mathbf{v})\,(\forall \mathbf{w})\,(\mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v})$ *commutativity*

2. $(\forall \mathbf{u})\,(\forall \mathbf{v})\,(\forall \mathbf{w})\,(\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w})$ *associativity*

3. $(\exists \mathbf{0})\,(\mathbf{0} \in \mathbb{V} \wedge (\forall \mathbf{v})\,(\mathbf{v} \in \mathbb{V} \Rightarrow \mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}))$ *zero vector*

4. $(\forall \mathbf{v})\,(\mathbf{v} \in \mathbb{V} \Rightarrow (\exists\,(-\mathbf{v}))\,((-\mathbf{v}) \in \mathbb{V} \wedge \mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}))$ *negatives*

5. $(\forall \lambda)\,(\forall \mathbf{v})\,(\forall \mathbf{w})\,(\lambda \in \mathbb{R} \wedge \mathbf{v}, \mathbf{w} \in \mathbb{V} \Rightarrow \lambda \cdot (\mathbf{v} + \mathbf{w}) = \lambda \cdot \mathbf{v} + \lambda \cdot \mathbf{w})$ *distributivity*

6. $(\forall \lambda)\,(\forall \gamma)\,(\forall \mathbf{w})\,(\lambda, \gamma \in \mathbb{R} \wedge \mathbf{w} \in \mathbb{V} \Rightarrow \lambda\,(\gamma \cdot \mathbf{w}) = (\lambda\gamma) \cdot \mathbf{w})$ *associativity*

7. $(\forall \lambda)\,(\forall \gamma)\,(\forall \mathbf{w})\,(\lambda, \gamma \in \mathbb{R} \wedge \mathbf{w} \in \mathbb{V} \Rightarrow (\lambda + \gamma) \cdot \mathbf{w} = \lambda \cdot \mathbf{w} + \gamma \cdot \mathbf{w})$ *distributivity*

8. $(\forall \mathbf{v})\,(\mathbf{v} \in \mathbb{V} \Rightarrow 1 \cdot \mathbf{v} = \mathbf{v} \cdot 1 = \mathbf{v})$ *multiplicative identity*

Given two vectors, $\mathbf{x} = (x_1, x_2, ..., x_n)$ and $\mathbf{y} = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$, the **inner product** (also known as the scalar product) is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j;$$

and the **Euclidean norm** (or magnitude) of $\mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is given by

$$|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{j=1}^{n} \left(x_j\right)^2}.$$

The vector space $\mathbb{R}^n$ together with the inner product and Euclidean norm is called **Euclidean** n-space. The following two theorems pull together the basic properties that are satisfied by the Euclidean norm.

Theorem 3.1.5 *Suppose that* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$. *Then*

(a) $|\mathbf{x}| \geq 0$;

(b) $|\mathbf{x}| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$;

(c) $|\alpha \mathbf{x}| = |\alpha|\,|\mathbf{x}|$; *and*

(d) $|\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}|$.

Excursion 3.1.6 *Use Schwarz's Inequality to justify part (d). For* $\mathbf{x} = (x_1, x_2, ..., x_n)$ *and* $\mathbf{y} = (y_1, y_2, ..., y_n)$ *in* $\mathbb{R}^n$,

$|\mathbf{x}\cdot\mathbf{y}|^2 =$

Remark 3.1.7 *It often helps to take our observations back to the setting that is "once removed" from* $\mathbb{R}^1$. *For the case* $\mathbb{R}^2$, *the statement given in part (d) of the theorem relates to the dot product of two vectors: For* $\xi = \overrightarrow{(x_1, x_2)}$ *and* $\eta = \overrightarrow{(y_1, y_2)}$, *we have that*

$$\xi\cdot\eta = x_1y_1 + x_2y_2$$

which, in vector calculus, was shown to be equivalent to $|\xi||\eta|cos\theta$ *where* θ *is the angle between the vectors* ξ *and* η.

Theorem 3.1.8 (The Triangular Inequalities) *Suppose that* $\mathbf{x} = (x_1, x_2, ..., x_N)$, $\mathbf{y} = (y_1, y_2, ..., y_N)$ *and* $\mathbf{z} = (z_1, z_2, ..., z_N)$ *are elements of* $\mathbb{R}^N$. *Then*

(a) $|\mathbf{x}+\mathbf{y}| \le |\mathbf{x}| + |\mathbf{y}|$; *i.e.,*

$$\left(\sum_{j=1}^{N}(x_j+y_j)^2\right)^{1/2} \le \left(\sum_{j=1}^{N}x_j^2\right)^{1/2} + \left(\sum_{j=1}^{N}y_j^2\right)^{1/2}$$

where $(\cdots)^{1/2}$ *denotes the positive square root and equality holds if and only if either all the* x_j *are zero or there is a nonnegative real number* λ *such that* $y_j = \lambda x_j$ *for each* j, $1 \le j \le N$; *and*

(b) $|\mathbf{x}-\mathbf{z}| \le |\mathbf{x}-\mathbf{y}| + |\mathbf{y}-\mathbf{z}|$; *i.e.,*

$$\left(\sum_{j=1}^{N}(x_j-z_j)^2\right)^{1/2} \le \left(\sum_{j=1}^{N}(x_j-y_j)^2\right)^{1/2} + \left(\sum_{j=1}^{N}(y_j-z_j)^2\right)^{1/2}$$

where $(\cdots)^{1/2}$ *denotes the positive square root and equality holds if and only if there is a real number* r*, with* $0 \le r \le 1$*, such that* $y_j = rx_j + (1-r)z_j$ *for each* j, $1 \le j \le N$.

Remark 3.1.9 *Again, it is useful to view the triangular inequalities on "familiar ground." Let* $\xi = \overrightarrow{(x_1, x_2)}$ *and* $\eta = \overrightarrow{(y_1, y_2)}$*. Then the inequalities given in Theorem 3.1.8 correspond to the statements that were given for the complex numbers; i.e., statements concerning the lengths of the vectors that form the triangles that are associated with finding* $\xi + \eta$ *and* $\xi - \eta$*.*

Observe that, for $C = \{(x, y) : x^2 + y^2 = 1\}$ and $I = \{x : a \le x \le b\}$ where $a < b$, the Cartesian product of the circle C with I, $C \times I$, is the right circular cylinder,

$$U = \{(x, y, z) : x^2 + y^2 = 1 \wedge a \le z \le b\},$$

and the Cartesian product of I with C, $I \times C$, is the right circular cylinder,

$$V = \{(x, y, z) : a \le x \le b, y^2 + z^2 = 1\}.$$

If graphed on the same $\mathbb{R}^3$-coordinate system, U and V are different objects due to different orientation; on the other hand, U and V have the same height and radius which yield the same volume, surface area; etc. Consequently, distinguishing U from V depends on perspective and reason for study. In the next section, we lay the foundation for properties that place U and V in the same category.

3.2 Metric Spaces

In the study of $\mathbb{R}^1$ and functions on $\mathbb{R}^1$ the length of intervals and intervals to describe set properties are useful tools. Our starting point for describing properties for sets in $\mathbb{R}^n$ is with a formulation of a generalization of distance. It should come as no surprise that the generalization leads us to multiple interpretations.

Definition 3.2.1 *Let* S *be a set and suppose that* $d : S \times S \longrightarrow \mathbb{R}^1$*. Then* d *is said to be a **metric (distance function)** on* S *if and only if it satisfies the following three properties:*

(i) $(\forall x)\,(\forall y)\left[(x, y) \in S \times S \Rightarrow d(x, y) \ge 0 \wedge (d(x, y) = 0 \Leftrightarrow x = y)\right]$,

(ii) $(\forall x)(\forall y)\left[(x,y)\in S\times S \Rightarrow d(y,x)=d(x,y)\right]$ *(symmetry), and*

(iii) $(\forall x)(\forall y)(\forall z)\left[x,y,z\in S \Rightarrow d(x,z)\le d(x,y)+d(y,z)\right]$ *(triangle inequality).*

Definition 3.2.2 *A **metric space** consists of a pair (S,d)–a set, S, and a metric, d, on S.*

Remark 3.2.3 *There are three commonly used (studied) metrics for the set $\mathbb{R}^N$. For $\mathbf{x}=(x_1,x_2,...,x_N)$ and $\mathbf{y}=(y_1,y_2,...,y_N)$, we have:*

- $(\mathbb{R}^N,d)$ *where* $d(\mathbf{x},\mathbf{y})=\sqrt{\sum_{j=1}^{N}\left(x_j-y_j\right)^2}$*, the Euclidean metric,*
- $(\mathbb{R}^N,D)$ *where* $D(\mathbf{x},\mathbf{y})=\sum_{j=1}^{N}|x_j-y_j|$*, and*
- $(\mathbb{R}^N,d_\infty)$ *where* $d_\infty(\mathbf{x},\mathbf{y})=\max_{1\le j\le N}\left|x_j-y_j\right|$.

Proving that d, D, and d_∞ are metrics is left as an exercise.

Excursion 3.2.4 *Graph each of the following on Cartesian coordinate systems*

1. $A=\{x\in\mathbb{R}^2 : d(\mathbf{0},\mathbf{x})\le 1\}$

2. $B=\{x\in\mathbb{R}^2 : D(\mathbf{0},\mathbf{x})\le 1\}$

3. $C = \{x \in \mathbb{R}^2 : d_\infty(\mathbf{0}, \mathbf{x}) \leq 1\}$

For (1), you should have gotten the closed circle with center at origin and radius one; for (2), your work should have led you to a "diamond" having vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$; the closed shape for (3) is the square with vertices $(1, -1)$, $(1, 1)$, $(-1, 1)$, and $(-1, -1)$.

Though we haven't defined continuous and integrable functions yet as a part of this course, we offer the following observation to make the point that metric spaces can be over different objects. Let $\mathcal{C}$ be the set of all functions that are continuous real valued functions on the interval $I = (x : 0 < x \leq 1\}$. Then there are two natural metrics to consider on the set $\mathcal{C}$; namely, for f and g in $\mathcal{C}$ we have

(1) $(\mathcal{C}, d)$ where $d(f, g) = \max\limits_{0 \leq x \leq 1} |f(x) - g(x)|$, and

(2) $(\mathcal{C}, \overline{d})$ where $\overline{d}(f, g) = \int_0^1 |f(x) - g(x)| dx$.

Because metrics on the same set can be distinctly different, we would like to distinguish those that are related to each other in terms of being able to "travel between" information given by them. With this in mind, we introduce the notion of equivalent metrics.

Definition 3.2.5 *Given a set S and two metric spaces (S, d_1) and (S, d_2), d_1 and d_2 are said to be* ***equivalent metrics*** *if and only if there are positive constants c and C such that $cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y)$ for all x, y in S.*

Excursion 3.2.6 *As the result of one of the Exercises in Problem Set C, you will know that the metrics d and d_∞ on $\mathbb{R}^2$ satisfy $d_\infty(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{2} \cdot d_\infty(\mathbf{x}, \mathbf{y})$.*

1. Let $A = \{x \in \mathbb{R}^2 : d(0, x) \leq 1\}$. Draw a figure showing the boundary of A and then show the largest circumscribed square that is symmetric about *the origin and the square, symmetric about the origin, that circumscribes the boundary of A.*

2. *Let* $C = \{x \in \mathbb{R}^2 : d_\infty(0, x) \leq 1\}$. *Draw a figure showing the boundary of* C *and then show the largest circumscribed circle that is centered at the origin and the circle, centered at the origin, that circumscribes the boundary of* C.

*******For (1), your outer square should have corresponded to $\left\{\mathbf{x} = (x_1 x_2) \in \mathbb{R}^2 : d_\infty(\mathbf{0}, \mathbf{x}) = \sqrt{2}\right\}$; the outer circle that you showed for part of (2) should have corresponded to $\left\{\mathbf{x} = (x_1 x_2) \in \mathbb{R}^2 : d(\mathbf{0}, \mathbf{x}) = \sqrt{2}\right\}$.*******

Excursion 3.2.7 *Let* $E = \{(cos\theta, sin\theta) : 0 \leq \theta < 2\pi\}$ *and define* $d^*(p_1, p_2) = |\theta_1 - \theta_2|$ *where* $p_1 = (cos\theta_1, sin\theta_1)$ *and* $p_2 = (cos\theta_2, sin\theta_2)$. *Show that* (E, d^*) *is* *a metric space.*

The author of our textbook refers to an open interval $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ as a **segment** which allows the term **interval** to be reserved for a closed interval $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; half-open intervals are then in the form of $[a, b)$ or $(a, b]$.

Definition 3.2.8 *Given real numbers $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that $a_j < b_j$ for $j = 1, 2, ..., n$,*

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j)\left(1 \leq j \leq n \Rightarrow a_j \leq x_j \leq b_j\right)\right\}$$

*is called an n-**cell**.*

Remark 3.2.9 *With this terminology, a 1-cell is an interval and a 2-cell is a rectangle.*

Definition 3.2.10 *If $\mathbf{x} \in \mathbb{R}^n$ and r is a positive real number, then the **open ball** with center $\mathbf{x}$ and radius r is given by*

$$B(\mathbf{x}, r) = \left\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\right\};$$

and the ***closed ball*** *with center x and radius r is given by*

$$\overline{B(\mathbf{x}, r)} = \left\{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| \le r\right\}.$$

Definition 3.2.11 *A subset E of $\mathbb{R}^n$ is* ***convex*** *if and only if*

$$(\forall \mathbf{x})(\forall \mathbf{y})(\forall \lambda)\left[\mathbf{x}, \mathbf{y} \in E \wedge 0 < \lambda < 1 \Rightarrow \lambda \mathbf{x} + (1-\lambda)\mathbf{y} \in E\right]$$

Example 3.2.12 *For $\mathbf{x} \in \mathbb{R}^n$ and r a positive real number, suppose that $\mathbf{y}$ and $\mathbf{z}$ are in $B(\mathbf{x}, r)$. If λ real is such that $0 < \lambda < 1$, then*

$$\begin{aligned} |\lambda \mathbf{y} + (1-\lambda)\mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1-\lambda)(\mathbf{z} - \mathbf{x})| \\ &\le \lambda|\mathbf{y} - \mathbf{x}| + (1-\lambda)|\mathbf{z} - \mathbf{x}| \\ &< \lambda r + (1-\lambda) r = r \quad . \end{aligned}$$

Hence, $\lambda \mathbf{y} + (1-\lambda)\mathbf{z} \in B(\mathbf{x}, r)$. Since y and z were arbitrary,

$$(\forall \mathbf{y})(\forall \mathbf{z})(\forall \lambda)\left[\mathbf{y}, \mathbf{z} \in B(\mathbf{x}, r) \wedge 0 < \lambda < 1 \Rightarrow \lambda \mathbf{y} + (1-\lambda)\mathbf{z} \in B(\mathbf{x}, r)\right];$$

that is, $B(\mathbf{x}, r)$ is a convex subset of $\mathbb{R}^n$.

3.3 Point Set Topology on Metric Spaces

Once we have a distance function on a set, we can talk about the proximity of points. The idea of a segment (interval) in $\mathbb{R}^1$ is replaced by the concept of a neighborhood (closed neighborhood). We have the following

Definition 3.3.1 *Let p_0 be an element of a metric space S whose metric is denoted by d and r be any positive real number. The* ***neighborhood of the point*** *p_0 with radius r is denoted by $N(p_0, r)$ or $N_r(p_0)$ and is given by*

$$N_r(p_0) = \{p \in S : d(p, p_0) < r\}.$$

The closed neighborhood with center p_0 and radius r is denoted by $\overline{N_r(p_0)}$ and is given by

$$\overline{N_r(p_0)} = \{p \in S : d(p, p_0) \le r\}.$$

Remark 3.3.2 *The sets A, B and C defined in Excursion 3.2.4 are examples of closed neighborhoods in* $\mathbb{R}^2$ *that are centered at* $(0, 0)$ *with unit radius.*

What does the unit neighborhood look like for $(\mathbb{R}^2, \hat{d})$ where

$$\hat{d}(x, y) = \begin{cases} 0, & \text{if} \quad x = y \\ \\ 1, & \text{if} \quad x \neq y \end{cases} \quad \text{is known as the discrete metric?}$$

We want to use the concept of neighborhood to describe the nature of points that are included in or excluded from sets in relationship to other points that are in the metric space.

Definition 3.3.3 *Let* A *be a set in a metric space* (S, d).

1. *Suppose that* p_0 *is an element of* A. *We say the* p_0 *is an* ***isolated point*** *of* A *if and only if*

$$(\exists N_r(p_0)) \left[N_r(p_0) \cap A = \{p_0\} \right]$$

2. *A point* p_0 *is a* ***limit point*** *of the set* A *if and only if*

$$(\forall N_r(p_0)) (\exists p) \left[p \neq p_0 \wedge p \in A \cap N_r(p_0) \right].$$

   *(**N.B.** A limit point need not be in the set for which it is a limit point.)*

3. *The set* A *is said to be* ***closed*** *if and only if* A *contains all of its limit points.*

4. *A point* p *is an* ***interior point*** *of* A *if and only if*

$$\left(\exists N_{r_p}(p)\right) \left[N_{r_p}(p) \subset A \right]$$

5. *The set* A *is* ***open*** *if and only if*

$$(\forall p)\left(p \in A \Rightarrow \left(\exists N_{r_p}(p)\right)\left[N_{r_p}(p) \subset A\right]\right);$$

i.e., every point in A *is an interior point of* A*.*

Example 3.3.4 *For each of the following subsets of* $\mathbb{R}^2$ *use the space that is provided to justify the claims that are made for the given set.*

(a) $\{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \in \mathbb{J} \wedge |x_1 + x_2| < 5\}$ *is closed because is contains all none of its limit points.*

(b) $\{(x_1, x_2) \in \mathbb{R}^2 : 4 < x_1^2 \wedge x_2 \in \mathbb{J}\}$ *is neither open not closed.*

(c) $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > |x_1|\}$ *is open.*

Our next result relates neighborhoods to the "open" and "closed" adjectives.

Theorem 3.3.5 *(a) Every neighborhood is an open set.*
(b) Every closed neighborhood is a closed set.

Use this space to draw some helpful pictures related to proving the results.

Proof. (a) Let $N_r(p_0)$ be a neighborhood. Suppose that $q \in N_r(p_0)$ and set $r_1 = d(p_0, q)$. Let $\rho = \frac{r - r_1}{4}$. If $x \in N_\rho(q)$, then $d(x, q) < \frac{r - r_1}{4}$ and the triangular inequality yields that

$$d(p_0, x) \leq d(p_0, q) + d(q, x) < r_1 + \frac{r - r_1}{4} = \frac{3r_1 + r}{4} < r.$$

Hence, $x \in N_r(p_0)$. Since x was arbitrary, we conclude that

$$(\forall x)\left(x \in N_\rho(q) \Rightarrow x \in N_r(p_0)\right);$$

i.e., $N_\rho(q) \subset N_r(p_0)$.Therefore, q is an interior point of $N_r(p_0)$. Because q was arbitrary, we have that each element of $N_r(p_0)$ is an interior point. Thus, $N_r(p_0)$ is open, as claimed.

Excursion 3.3.6 *Fill in what is missing in order to complete the following proof of (b)*

Let $\overline{N_r(p_0)}$ be a closed neighborhood and suppose that q is a limit point of $\overline{N_r(p_0)}$. Then, for each $r_n = \dfrac{1}{n}$, $n \in \mathbb{J}$, there exists $p_n \neq q$ such that $p_n \in \overline{N_r(p_0)}$ and $d(q, p_n) < \dfrac{1}{n}$. Because $p_n \in \overline{N_r(p_0)}$, $d(p_0, p_n) \leq r$ for each $n \in \mathbb{J}$. Hence, by the triangular inequality

$$d(q, p_0) \leq d(q, p_n) + \underset{(1)}{\underline{\hspace{3cm}}} \leq \underset{(2)}{\underline{\hspace{3cm}}}.$$

Since q and p_0 are fixed and $\dfrac{1}{n}$ goes to 0 as n goes to infinity, it follows that $d(q, p_0) \leq r$; that is, $q \in \underset{(3)}{\underline{\hspace{3cm}}}$. Finally, q and arbitrary limit point of $\overline{N_r(p_0)}$ leads to the conclusion that $\overline{N_r(p_0)}$ contains $\underset{(4)}{\underline{\hspace{6cm}}}$. Therefore, $\overline{N_r(p_0)}$ is closed.

■

Acceptable responses are: (1) $d(p_n, p_0)$, (2) $\dfrac{1}{n} + r$, (3) $\overline{N_r(p_0)}$, (4) all of its limit points.

The definition of limit point leads us directly to the conclusion that only infinite subsets of metric spaces will have limit points.

Theorem 3.3.7 *Suppose that (X, d) is a metric space and $A \subset X$. If p is a limit point of A, then every neighborhood of p contains infinitely many points of A.*

Space for scratch work.

Proof. For a metric space (X, d) and $A \subset X$, suppose that $p \in X$ is such that there exists a neighborhood of p, $N(p)$, with the property that $N(p) \cap A$

is a finite set. If $N(p) \cap A = \emptyset$ or , $N(p) \cap A = \{p\}$, then p is not a limit point. Otherwise, $N(p) \cap A$ being finite implies that it can be realized as a finite sequence, say $q_1, q_2, , q_3, ..., q_n$ for some fixed $n \in \mathbb{J}$. For each j, $1 \le j \le n$, let $r_j = d(x, q_j)$. Set $\rho = \min_{\substack{1 \le j \le n \\ q_j \neq p}} d(x, q_j)$. If $p \in \{q_1, q_2, , q_3, ..., q_n\}$, then $N_\rho(p) \cap A = \{p\}$; otherwise $N_\rho(p) \cap A = \emptyset$. In either case, we conclude that p is not a limit point of A.

We have shown that if $p \in X$ has a neighborhood, $N(p)$, with the property that $N(p) \cap A$ is a finite set, then p is not a limit point of $A \subset X$. From the contrapositive tautology it follows immediately that if p is a limit point of $A \subset X$, then every neighborhood of p contains infinitely many points of A. ■

Corollary 3.3.8 *Any finite subset of a metric space has no limit point.*

From the Corollary, we note that every finite subset of a metric space is closed because it contains all none of its limit points.

3.3.1 Complements and Families of Subsets of Metric Spaces

Given a family of subsets of a metric space, it is natural to wonder about whether or not the properties of being open or closed are passed on to the union or intersection. We have already seen that these properties are not necessarily transmitted when we look as families of subsets of $\mathbb{R}$.

Example 3.3.9 *Let* $\mathcal{A} = \{A_n : n \in \mathbb{J}\}$ *where* $A_n = \left[\frac{-3n+2}{n}, \frac{2n^2-n}{n^2}\right]$. *Note that* $A_1 = [-1, 1]$, $A_2 = \left[-2, \frac{3}{2}\right]$, *and* $A_3 = \left[-3 + \frac{2}{3}, 2 - \frac{1}{3}\right]$. *More careful inspection reveals that* $\frac{-3n+2}{n} = -3 + \frac{2}{n}$ *is strictly decreasing to* -3 *and* $n \to \infty$, $\frac{2n^2-n}{n^2} = 2 - \frac{1}{n}$ *is strictly increasing to* 2 *as* $n \to \infty$, *and* $A_1 = [-1, 1] \subset A_n$ *for each* $n \in \mathbb{J}$. *It follows that* $\bigcup_{n \in \mathbb{J}} A_n = (-3, 2)$ *and* $\bigcap_{n \in \mathbb{J}} A_n = A_1 = [-1, 1]$.

The example tells us that we may need some special conditions in order to claim preservation of being open or closed when taking unions and/or intersections over families of sets.

The other set operation that is commonly studied is complement or relative complement. We know that the complement of a segment in $\mathbb{R}^1$ is closed. This motivates us to consider complements of subsets of metric spaces in general. Recall the following

Definition 3.3.10 *Suppose that A and B are subsets of a set S. Then the **set difference** (or **relative complement**) $A - B$, read "A not B", is given by*

$$A - B = \{p \in S : p \in A \wedge p \notin B\};$$

*the **complement** of A, denoted by A^c, is $S - A$.*

Excursion 3.3.11 *Let $A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\}$ and*

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1| \leq 1 \wedge |x_2 - 1| \leq 1\}.$$

On separate copies of Cartesian coordinate systems, show the sets $A - B$ and $A^c = \mathbb{R}^2 - A$.

The following identities, which were proved in MAT108, are helpful when we are looking at complements of unions and intersections. Namely, we have

Theorem 3.3.12 (deMorgan's Laws) *Suppose that S is any space and $\mathcal{F}$ is a family of subsets of S. Then*

$$\left[\bigcup_{A\in\mathcal{F}} A\right]^c = \bigcap_{A\in\mathcal{F}} A^c$$

and

$$\left[\bigcap_{A\in\mathcal{F}} (A)\right]^c = \bigcup_{A\in\mathcal{F}} A^c.$$

The following theorem pulls together basic statement concerning how unions, intersections and complements effect the properties of being open or closed. Because their proofs are straightforward applications of the definitions, most are left as exercises.

Theorem 3.3.13 *Let S be a metric space.*

1. *The union of any family $\mathcal{F}$ of open subsets of S is open.*
2. *If $A_1, A_2, \ldots, A_m$ is a finite family of open subsets of S, then the intersection $\bigcap_{j=1}^{m} A_j$ is open.*
3. *For any subset A of S, A is closed if and only if A^c is open.*
4. *The intersection of any family $\mathcal{F}$ of closed subsets of S is closed.*
5. *If $A_1, A_2, \ldots, A_m$ is a finite family of closed subsets of S, then the union $\bigcup_{j=1}^{m} A_j$ is closed.*
6. *The space S is both open and closed.*
7. *The null set is both open and closed.*

Proof. (of #2) Suppose that $A_1, A_2, \ldots, A_m$ is a finite family of open subsets of S, and $x \in \bigcap_{j=1}^{m} A_j$. From $x \in \bigcap_{j=1}^{m} A_j$, it follows that $x \in A_j$ for each j, $1 \leq j \leq m$. Since each A_j is open, for each j, $1 \leq j \leq m$, there exists $r_j > 0$ such that $N_{r_j}(x) \subset A_j$. Let $\rho = \min_{1 \leq j \leq m} r_j$. Because $N_\rho(x) \subset A_j$ for each j, $1 \leq j \leq m$, we conclude that $N_\rho(x) \subset \bigcap_{j=1}^{m} A_j$. Hence, x is an interior point of $\bigcap_{j=1}^{m} A_j$. Finally, since x was arbitrary, we can claim that each element of $\bigcap_{j=1}^{m} A_j$ is an interior point. Therefore, $\bigcap_{j=1}^{m} A_j$ is open.

(or #3) Suppose that $A \subset S$ is closed and $x \in A^c$. Then $x \notin A$ and, because A contains all of its limit points, x is not a limit point of A. Hence, $x \notin A \wedge \neg(\forall N_r(x))\left[A \cap (N_r(x) - \{x\}) \neq \emptyset\right]$ is true. It follows that $x \notin A$ and there exists a $\rho > 0$ such that $A \cap \left(N_\rho(x) - \{x\}\right) = \emptyset$. Thus, $A \cap N_\rho(x) = \emptyset$ and we conclude that $N_\rho(x) \subset A^c$; i.e., x is an interior point of A^c. Since x was arbitrary, we have that each element of A^c is an interior point. Therefore, A^c is open.

To prove the converse, suppose that $A \subset S$ is such that A^c is open. If p is a limit point of A, then $(\forall N_r(p))\left[A \cap (N_r(p) - \{p\}) \neq \emptyset\right]$. But, for any $\rho > 0$, $A \cap \left(N_\rho(p) - \{p\}\right) \neq \emptyset$ implies that $\left(N_\rho(p) - \{p\}\right)$ is not contained in A^c. Hence,

p is not an interior point of A^c and we conclude that $p \notin A^c$. Therefore, $p \in A$. Since p was arbitrary, we have that A contains all of its limit points which yields that A is closed. ■

Remark 3.3.14 *Take the time to look back at the proof of (#2) to make sure that you where that fact that the intersection was over a finite family of open subsets of S was critical to the proof.*

Given a subset of a metric space that is neither open nor closed we'd like to have a way of describing the process of "extracting an open subset" or "building up to a closed subset." The following terminology will allow us to classify elements of a metric space S in terms of their relationship to a subset $A \subset S$.

Definition 3.3.15 *Let A be a subset of a metric space S. Then*

1. *A point $p \in S$ is an **exterior point** of A if and only if*

$$(\exists N_r(p))\left[N_r(p) \subset A^c\right],$$

   where $A^c = S - A$.

2. *The **interior of** A, denoted by $\operatorname{Int}(A)$ or $A^{(0)}$, is the set of all interior points of A.*

3. *The **exterior of** A, denoted by $\operatorname{Ext}(A)$, is the set of all exterior points of A.*

4. *The **derived set** of A, denoted by A', is the set of all limit points of A.*

5. *The **closure of** A, denoted by $\overline{A}$, is the union of A and its derived set; i.e., $\overline{A} = A \cup A'$.*

6. *The **boundary of** A, denoted by ∂A, is the difference between the closure of A and the interior of A; i.e., $\partial A = \overline{A} - A^{(0)}$.*

Remark 3.3.16 *Note that, if A is a subset of a metric space S, then* $\operatorname{Ext}(A) = \operatorname{Int}(A^c)$ *and*

$$x \in \partial A \Leftrightarrow (\forall N_r(x))\left[N_r(x) \cap A \neq \emptyset \wedge N_r(x) \cap A^c \neq \emptyset\right].$$

The proof of these statements are left as exercises.

Excursion 3.3.17 *For $A \cup B$ where*

$$A = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

and

$$B = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1 - 1| \leq 1 \wedge |x_2 - 1| \leq 1\}.$$

1. *Sketch a graph of $A \cup B$.*

2. *On separate representations for $\mathbb{R}^2$, show each of the following*

$$\text{Int}\,(A \cup B)\,,\ \text{Ext}\,(A \cup B)\,,\ (A \cup B)'\,,\ \textit{and}\ \overline{(A \cup B)}.$$

***Hopefully, your graph of $A \cup B$ consisted of the union of the open disc that is centered at the origin and has radius one with the closed square having vertices

$(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$; the disc and square overlap in the first quadrant and the set is not open and not closed. Your sketch of $\text{Int}\,(A \cup B)$ should have shown the disc and square without the boundaries (;i.e., with the outline boundaries as not solid curves), while your sketch of $\text{Ext}\,(A \cup B)$ should have shown everything that is outside the combined disc and square–also with the outlining boundary as not solid curves. Finally, because $A \cup B$ has no isolated points, $(A \cup B)'$ and $\overline{(A \cup B)}$ are shown as the same sets–looking like $\text{Int}\,(A \cup B)$ with the outlining boundary now shown as solid curves.***

The following theorem relates the properties of being open or closed to the concepts described in Definition 3.3.15.

Theorem 3.3.18 *Let A be any subset of a metric space S.*

(a) The derived set of A, A', is a closed set.

(b) The closure of A, $\overline{A}$, is a closed set.

(c) Then $A = \overline{A}$ if and only if A is closed.

(d) The boundary of A, ∂A, is a closed set.

(e) The interior of A, $\text{Int}\,(A)$, is an open set.

(f) If $A \subset B$ and B is closed, then $\overline{A} \subset B$.

(g) If $B \subset A$ and B is open, $B \subset \text{Int}(A)$.

(h) Any point (element) of S is a closed set.

The proof of part *(a)* is problem #6 in WRp43, while (e) and (g) are parts of problem #9 in WRp43.

Excursion 3.3.19 *Fill in what is missing to complete the following proofs of parts (b), (c), and (f).*

Part (b): In view of Theorem 3.3.13(#3), it suffices to show that $\underset{(1)}{\underline{\hspace{3cm}}}$.

Suppose that $x \in S$ is such that $x \in \left(\overline{A}\right)^c$. Because $\overline{A} = A \cup A'$, it follows that $x \notin A$ and $\underset{(2)}{\underline{\hspace{3cm}}}$. *From the latter, there exists a neighborhood of x, $N(x)$, such* *that* $\left(\underset{(3)}{\underline{\hspace{3cm}}}\right) \cap A = \emptyset$*; while the former yields that* $\left(\underset{(4)}{\underline{\hspace{3cm}}}\right) \cap A = \emptyset$*. Hence,* $N(x) \subset A^c$*. Suppose that* $y \in N(x)$*. Since* $\underset{(5)}{\underline{\hspace{3cm}}}$*, there exists a neighborhood* $N^*(y)$ *such that* $N^*(y) \subset N(x)$*. From the transitivity of subset,* $\underset{(6)}{\underline{\hspace{3cm}}}$ *from which we conclude that* y *is not a limit point of* A*; i.e.,* $y \in (A')^c$*. Because* y *was arbitrary,*

$$(\forall y)\left[y \in N(x) \Rightarrow \underset{(7)}{\underline{\hspace{3cm}}}\right];$$

i.e., $\underset{(8)}{\underline{\hspace{6cm}}}$*. Combining our containments yields that* $N(x) \subset A^c$ *and* $\underset{(8)}{\underline{\hspace{6cm}}}$*. Hence,*

$$N(x) \subset A^c \cap (A')^c = \left[\underset{(9)}{\underline{\hspace{3cm}}}\right]^c.$$

Since x *was arbitrary, we have shown that*

$$\underset{(10)}{\underline{\hspace{9cm}}}.$$

Therefore, $\left(\overline{A}\right)^c$ *is open.*

Part (c): From part (b), if $A = \overline{A}$*, then* $\underset{(11)}{\underline{\hspace{3cm}}}$*.*

Conversely, if $\underset{(12)}{\underline{\hspace{3cm}}}$*, then* $A' \subset A$*. Hence,* $A \cup A' = \underset{(13)}{\underline{\hspace{0.7cm}}}$*; that is,* $\overline{A} = A$*.*

Part (f): Suppose that $A \subset B$*,* B *is closed, and* $x \in \overline{A}$*. Then* $x \in A$ *or* $\underset{(14)}{\underline{\hspace{3cm}}}$*. If* $x \in A$*, then* $x \in B$*; if* $x \in A'$*, then for every neighborhood of* x*,* $N(x)$*, there exists* $w \in A$ *such that* $w \neq x$ *and* $\underset{(15)}{\underline{\hspace{3cm}}}$*. But then*

$w \in B$ and $(N(x) - \{x\}) \cap B \neq \emptyset$. Since $N(x)$ was arbitrary, we conclude that $\underset{(16)}{\underline{\hspace{3cm}}}$. Because B is closed, $\underset{(17)}{\underline{\hspace{3cm}}}$. Combining the conclusions and noting that $x \in \overline{A}$ was arbitrary, we have that

$$(\forall x)\left[\underset{(18)}{\underline{\hspace{6cm}}}\right].$$

Thus, $\overline{A} \subset B$.

*******Acceptable responses are (1) the complement of A closure is open, (2) $x \notin A'$, (3) $N(x) - \{x\}$, (4) $N(x)$, (5) $N(x)$ is open, (6) $N^*(y) \subset A^c$, (7) $y \in (A')^c$, (8) $N(x) \subset (A')^c$; (9) $A \cup A'$, and (10) $(\forall x)\left(x \in \left(\overline{A}\right)^c \Rightarrow (\exists N_r(x))\left(N_r(x) \subset \left(\overline{A}\right)^c\right)\right)$; (11) A is closed, (12) A is closed, (13) A; (14) x is a limit point of A (or $x \in A'$); (15) $w \in N(x)$; (16) x is a limit point of B (or $x \in B'$); (17) $x \in B$, (18) $x \in \overline{A} \Rightarrow x \in B$.*******

Definition 3.3.20 *For a metric space (X, d) and $E \subset X$, the set E is **dense** in X if and only if*

$$(\forall x)\left(x \in X \Rightarrow x \in E \vee x \in E'\right).$$

Remark 3.3.21 *Note that for a metric space (X, d), $E \subset X$ implies that $\overline{E} \subset X$ because the space X is closed. On the other hand, if E is dense in X, then $X \subset E \cup E' = \overline{E}$. Consequently, we see that E is dense in a metric space X if and only if $\overline{E} = X$.*

Example 3.3.22 *We have that the sets of rationals and irrationals are dense in Euclidean 1-space. This was shown in the two Corollaries the Archimedean Principle for Real Numbers that were appropriately named "Density of the Rational Numbers" and "Density of the Irrational Numbers."*

Definition 3.3.23 *For a metric space (X, d) and $E \subset X$, the set E is **bounded** if and only if*

$$(\exists M)(\exists q)\left[M \in \mathbb{R}^+ \wedge q \in X \wedge (E \subset N_M(q))\right].$$

Excursion 3.3.24 *Justify that each of the following sets is bounded in Euclidean space.*

1. $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : -1 \le x_1 < 2 \wedge |x_2 - 3| < 1\right\}$

2. $B = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \ge 0 \wedge x_2 \ge 0 \wedge x_3 \ge 0 \wedge 2x_1 + x_2 + 4x_3 = 2\right\}$

Remark 3.3.25 *Note that, for* $(\mathbb{R}^2, \hat{d})$*, where*

$$\hat{d}(x, y) = \begin{cases} 0, & \text{if } \quad x = y \\ \\ 1, & \text{if } \quad x \neq y \end{cases},$$

the space $\mathbb{R}^2$ *is bounded. This example stresses that classification of a set as bounded is tied to the metric involved and may allow for a set to be bounded*

The definitions of least upper bound and greatest lower bound directly lead to the observation that they are limit points for bounded sets of real numbers.

Theorem 3.3.26 *Let* E *be a nonempty set of real numbers that is bounded,* $\alpha = \sup(E)$*, and* $\beta = \inf(E)$*. Then* $\alpha \in \overline{E}$ *and* $\beta \in \overline{E}$*.*

Space for illustration.

Proof. It suffices to show the result for least upper bounds. Let E be a nonempty set of real numbers that is bounded above and $\alpha = \sup(E)$. If $\alpha \in E$, then $\alpha \in \overline{E} =$

$E \cup E'$. For $\alpha \notin E$, suppose that h is a positive real number. Because $\alpha - h < \alpha$ and $\alpha = \sup(E)$, there exists $x \in E$ such that $\alpha - h < x < \alpha$. Since h was arbitrary,

$$(\forall h)\,(h > 0 \Rightarrow (\exists x)\,(\alpha - h < x < \alpha))\,;$$

i.e., α is a limit point for E. Therefore, $\alpha \in \overline{E}$ as needed. ■

Remark 3.3.27 *In view of the theorem we note that any closed nonempty set of real numbers that is bounded above contains its least upper bound and any closed nonempty set of real numbers that is bounded below contains its greatest lower bound.*

3.3.2 Open Relative to Subsets of Metric Spaces

Given a metric space (X, d), for any subset Y of X, $d \upharpoonright_Y$ is a metric on Y. For example, given the Euclidean metric d_e on $\mathbb{R}^2$ we have that $d_e \upharpoonright_{\mathbb{R}\times\{0\}}$ corresponds to the (absolute value) Euclidean metric, $d = |x - y|$, on the reals. It is natural to ask about how properties studied in the (parent) metric space transfer to the subset.

Definition 3.3.28 *Given a metric space (X, d) and $Y \subset X$. A subset E of Y is* ***open relative to*** *Y if and only if*

$$(\forall p)\left[p \in E \Rightarrow (\exists r)\left(r > 0 \wedge (\forall q)\left[q \in Y \wedge d\,(p,q) < r \Rightarrow q \in E\right]\right)\right]$$

which is equivalent to

$$(\forall p)\left[p \in E \Rightarrow (\exists r)\,(r > 0 \wedge Y \cap N_r\,(p) \subset E)\right].$$

Example 3.3.29 *For Euclidean 2-space, $\left(\mathbb{R}^2, d\right)$, consider the subsets*

$$Y = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 3\right\} \text{ and } Z = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \wedge 2 \leq x_2 < 5\right\}.$$

(a) The set $X_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : 3 \leq x_1 < 5 \wedge 1 < x_2 < 4\right\} \cup \{(3,1)\,, (3,4)\}$ is not open relative to Y, while $X_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : 3 \leq x_1 < 5 \wedge 1 < x_2 < 4\right\}$ is open relative to Y.

(b) The half open interval $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \wedge 2 \leq x_2 < 3\right\}$ is open relative to Z.

From the example we see that a subset of a metric space can be open relative to another subset though it is not open in the whole metric space. On the other hand, the following theorem gives us a characterization of open relative to subsets of a metric space in terms of sets that are open in the metric space.

Theorem 3.3.30 *Suppose that* (X, d) *is a metric space and* $Y \subset X$. *A subset* E *of* Y *is open relative to* Y *if and only if there exists an open subset* G *of* X *such that* $E = Y \cap G$.

Space for scratch work.

Proof. Suppose that (X, d) is a metric space, $Y \subset X$, and $E \subset Y$.

If E is open relative to Y, then corresponding to each $p \in E$ there exists a neighborhood of p, $N_{r_p}(p)$, such that $Y \cap N_{r_p}(p) \subset E$. Let $\mathcal{A} = \left\{N_{r_p}(p) : p \in E\right\}$. By Theorems 3.3.5(a) and 3.3.13(#1), $G \underset{def}{=} \cup \mathcal{A}$ is an open subset of X. Since $p \in N_{r_p}(p)$ for each $p \in E$, we have that $E \subset G$ which, with $E \subset Y$, implies that $E \subset G \cap Y$. On the other hand, the neighborhoods $N_{r_p}(p)$ were chosen such that $Y \cap N_{r_p}(p) \subset E$; hence,

$$\bigcup_{p \in E} \left(Y \cap N_{r_p}(p)\right) = Y \cap \left(\bigcup_{p \in E} N_{r_p}(p)\right) = Y \cap G \subset E.$$

Therefore, $E = Y \cap G$, as needed.

Now, suppose that G is an open subset of X such that $E = Y \cap G$ and $p \in E$. Then $p \in G$ and G open in X yields the existence of a neighborhood of p, $N(p)$, such that $N(p) \subset G$. It follows that $N(p) \cap Y \subset G \cap Y = E$. Since p was arbitrary, we have that

$$(\forall p) \left[p \in E \Rightarrow (\exists N(p)) \left[N(p) \cap Y \subset E\right]\right];$$

i.e., E is open relative to Y. ■

3.3.3 Compact Sets

In metric spaces, many of the properties that we study are described in terms of neighborhoods. The next set characteristic will allow us to extract finite collections of neighborhoods which can lead to bounds that are useful in proving other results about subsets of metric spaces or functions on metric spaces.

Definition 3.3.31 *Given a metric space (X, d) and $A \subset X$, the family $\{G_\alpha : \alpha \in \Delta\}$ of subsets of X is an* ***open cover*** *for A if and only if G_α is open for each $\alpha \in \Delta$ and $A \subset \bigcup_{\alpha \in \Delta} G_\alpha$.*

Definition 3.3.32 *A subset K of a metric space (X, d) is* ***compact*** *if and only if every open cover of K has a finite subcover; i.e., given any open cover $\{G_\alpha : \alpha \in \Delta\}$ of K, there exists an $n \in \mathbb{J}$ such that $\left\{G_{\alpha_k} : k \in \mathbb{J} \wedge 1 \leq k \leq n\right\}$ is a cover for K.*

We have just seen that a subset of a metric space can be open relative to another subset without being open in the whole metric space. Our first result on compact sets is tells us that the situation is different when we look at compactness relative to subsets.

Theorem 3.3.33 *For a metric space (X, d), suppose that $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

Excursion 3.3.34 *Fill in what is missing to complete the following proof of Theorem 3.3.33.*

Space for scratch work.

Proof. *Let (X, d) be a metric space and $K \subset Y \subset X$.*

Suppose that K is compact relative to X and $\{U_\alpha : \alpha \in \Delta\}$ is a family of sets such that, for each α, U_α is open relative to Y such that

$$K \subset \bigcup_{\alpha \in \Delta} U_\alpha.$$

By Theorem 3.3.30, corresponding to each $\alpha \in \Delta$, there exists a set G_α such that G_α is open relative to X and $\underset{(1)}{\underline{\hspace{3cm}}}$.

Since $K \subset Y$ and

$$K \subset \bigcup_{\alpha \in \Delta} U_\alpha = \bigcup_{\alpha \in \Delta} \Bigg(\underset{(1)}{\underline{\hspace{3cm}}} \Bigg) = Y \cap \bigcup_{\alpha \in \Delta} G_\alpha, \textit{ if}$$

follows that

$$K \subset \bigcup_{\alpha \in \Delta} G_\alpha.$$

Because K is compact relative to X, there exists a finite number of elements of Δ, $\alpha_1, \alpha_2, ..., \alpha_n$, such that

$$\underset{(2)}{\underline{\hspace{6cm}}}.$$

Now $K \subset Y$ and $K \subset \bigcup_{j=1}^{n} G_{\alpha_j}$ yields that

$$K \subset Y \cap \bigcup_{j=1}^{n} G_{\alpha_j} = \underset{(3)}{\underline{\hspace{3cm}}} = \underset{(4)}{\underline{\hspace{3cm}}}.$$

Since $\{U_\alpha : \alpha \in \Delta\}$ was arbitrary, we have shown that every open relative to Y cover of K has a finite subcover. Therefore,

$\underset{(5)}{\underline{\hspace{6cm}}}$.

Conversely, suppose that K is compact relative to Y and that $\{W_\alpha : \alpha \in \Delta\}$ is a family of sets such that, for each α, W_α is open relative to X and

$$K \subset \bigcup_{\alpha \in \Delta} W_\alpha.$$

For each $\alpha \in \Delta$, *let* $U_\alpha = Y \cap W_\alpha$. *Now* $K \subset Y$ *and* $K \subset \bigcup_{\alpha \in \Delta} W_\alpha$ *implies that*

$$\underset{(6)}{\underline{\hspace{6cm}}}.$$

Consequently, $\{U_\alpha : \alpha \in \Delta\}$ *is an open relative to* Y *cover for* K. *Now* K *compact relative to* Y *yields that there exists a finite number of elements of* Δ, $\alpha_1, \alpha_2, ..., \alpha_n$, *such that* $\underset{(7)}{\underline{\hspace{3cm}}}$. *Since*

$$\bigcup_{j=1}^{n} U_{\alpha_j} = \bigcup_{j=1}^{n} \left(Y \cap W_{\alpha_j}\right) = Y \cap \bigcup_{j=1}^{n} W_{\alpha_j}$$

and $K \subset Y$, *it follows that* $\underset{(8)}{\underline{\hspace{4.5cm}}}$.
Since $\{W_\alpha : \alpha \in \Delta\}$ *was arbitrary, we conclude that every family of sets that form an open relative to* X *cover of* K *has a finite subcover. Therefore,*

$\underset{(9)}{\underline{\hspace{6cm}}}$.

■

Acceptable fill-ins: (1) $U_\alpha = Y \cap G_\alpha$, (2) $K \subset G_{\alpha_1} \cup G_{\alpha_2} \cup \cdots \cup G_{\alpha_n}$ (or $K \subset \bigcup_{j=1}^{n} G_{\alpha_j}$), (3) $\bigcup_{j=1}^{n} \left(Y \cap G_{\alpha_j}\right)$, (4) $\bigcup_{j=1}^{n} U_{\alpha_j}$, (5) K is compact relative to Y, (6) $K \subset Y \cap \bigcup_{\alpha \in \Delta} W_\alpha = \bigcup_{\alpha \in \Delta} (Y \cap W_\alpha) = \bigcup_{\alpha \in \Delta} U_\alpha$, (7) $K \subset \bigcup_{j=1}^{n} U_{\alpha_j}$, (8) $K \subset \bigcup_{j=1}^{n} W_{\alpha_j}$, (9) K is compact in X.

Our next set of results show relationships between the property of being compact and the property of being closed.

Theorem 3.3.35 *If A is a compact subset of a metric space (S, d), then A is closed.*

Excursion 3.3.36 *Fill-in the steps of the proof as described*

Proof. *Suppose that* A *is a compact subset of a metric space* (S, d) *and* $p \in S$ *is such that* $p \notin A$. *For* $q \in A$, *let* $r_q = \frac{1}{4} d(p, q)$. *The* $\left\{ N_{r_q}(q) : q \in A \right\}$ *is an open cover for* A. *Since* A *is compact, there exists a finite number of* q, *say* $q_1, q_2, ..., q_n$, *such that*

$$A \subset N_{r_{q_1}}(q_1) \cup N_{r_{q_2}}(q_2) \cup \cdots \cup N_{r_{q_n}}(q_n) \underset{def}{=} W.$$

(a) Justify that the set $V = N_{r_{q_1}}(p) \cap N_{r_{q_2}}(p) \cap \cdots \cap N_{r_{q_n}}(p)$ *is a neighborhood of* p *such that* $V \cap W = \emptyset$.

(b) Justify that A^c *is open.*

(c) Justify that the result claimed in the theorem is true.

■

*******For (a), hopefully you noted that taking $r = \min_{1 \leq j \leq n} r_{q_j}$ yields that $N_{r_{q_1}}(p) \cap N_{r_{q_2}}(p) \cap \cdots \cap N_{r_{q_n}}(p) = N_r(p)$. To complete (b), you needed to observe that $N_r(p) \subset A^c$ made p an interior point of A^c; since p was an arbitrary point satisfying $p \notin A$, it followed that A^c is open. Finally, part (c) followed from Theorem 3.3.13(#3) which asserts that the complement of an open set is closed; thus, $(A^c)^c = A$ is closed.*******

Theorem 3.3.37 *In any metric space, closed subsets of a compact sets are compact.*

Space for scratch work.

Excursion 3.3.38 *Fill in the two blanks in order to complete the following proof of the theorem.*

Proof. For a metric space (X, d), suppose that $F \subset K \subset X$ are such that F is closed (relative to X) and K is compact. Let $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ be an open cover for F. Then the family $\Omega = \{V : V \in \mathcal{G} \vee V = F^c\}$ is an open cover for K. It follows from K being compact that there exists a finite number of elements of Ω, say $V_1, V_2, ..., V_n$, such that

______________________.

Because $F \subset K$, we also have that

______________________.

If for some $j \in \mathbb{J}$, $1 \le j \le n$, $F^c = V_j$, the family $\{V_k : 1 \le k \le n \wedge k \neq j\}$ would still be a finite open cover for F. Since $\mathcal{G}$ was an arbitrary open cover for F, we conclude that every open cover of F has a finite subcover. Therefore, F is compact. ■

Corollary 3.3.39 *If F and K are subsets of a metric space such that F is closed and K is compact, then $F \cap K$ is compact.*

Proof. As a compact subset of a metric space, from Theorem 3.3.35, K is closed. Then, it follows directly from Theorems 3.3.13(#5) and 3.3.37 that $F \cap K$ is compact as a closed subset of the compact set K. ■

Remark 3.3.40 *Notice that Theorem 3.3.35 and Theorem 3.3.37 are not converses of each other. The set $\left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \ge 2 \wedge x_2 = 0\right\}$ is an example of a closed set in Euclidean 2-space that is not compact.*

Definition 3.3.41 *Let* $\{S_n\}_{n=1}^{\infty}$ *be a sequence of subsets of a metric space* X. *Then* $\{S_n\}_{n=1}^{\infty}$ *is a* ***nested sequence of sets*** *if and only if* $(\forall n)\ (n \in \mathbb{J} \Rightarrow S_{n+1} \subset S_n)$.

Definition 3.3.42 *A family* $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ *of sets in the universe* $\mathcal{U}$ *has the* ***finite intersection property*** *if and only if the intersection over any finite subfamily of* $\mathcal{A}$ *is nonempty; i.e.,*

$$(\forall \Omega)\left[\Omega \subset \Delta \wedge \Omega \text{ finite} \Rightarrow \bigcap_{\beta \in \Omega} A_\beta \neq \emptyset\right].$$

The following theorem gives a sufficient condition for a family of nonempty compact sets to be disjoint. The condition is not being offered as something for you to apply to specific situations; it leads us to a useful observation concerning nested sequences of nonempty compact sets.

Theorem 3.3.43 *If* $\{K_\alpha : \alpha \in \Delta\}$ *is a family of nonempty compact subsets of a metric space* X *that satisfies the finite intersection property, then* $\bigcap_{\alpha \in \Delta} K_\alpha \neq \emptyset$.

Space for notes.

Proof. Suppose that $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$ and choose $K_\delta \in \{K_\alpha : \alpha \in \Delta\}$. Since $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$,

$$(\forall x)\left[x \in K_\delta \Rightarrow x \notin \bigcap_{\alpha \in \Delta} K_\alpha\right].$$

Let

$$\mathcal{G} = \{K_\alpha : \alpha \in \Delta \wedge K_\alpha \neq K_\delta\}.$$

Because each K_α is compact, by Theorems 3.3.35 and 3.3.13(#3), K_α is closed and K_α^c is open. For any $w \in K_\delta$, we have that $w \notin \bigcap_{\alpha \in \Delta} K_\alpha$. Hence, there exists a

$\beta \in \Delta$ such that $w \notin K_\beta$ from which we conclude that $w \in K_\beta^c$ and $K_\beta \neq K_\delta$. Since w was arbitrary, we have that

$$(\forall w)\left[w \in K_\delta \Rightarrow (\exists \beta)\left(\beta \in \Delta \wedge K_\beta \neq K_\delta \wedge w \in K_\beta^c\right)\right].$$

Thus, $K_\delta \subset \bigcup_{G \in \mathcal{G}} G$ which establishes $\mathcal{G}$ as an open cover for K_δ. Because K_δ is compact there exists a finite number of elements of $\mathcal{G}$, $K_{\alpha_1}^c, K_{\alpha_2}^c, \ldots, K_{\alpha_n}^c$, such that

$$K_\delta \subset \bigcup_{j=1}^{n} K_{\alpha_j}^c = \left(\bigcap_{j=1}^{n} K_{\alpha_j}\right)^c$$

from DeMorgan's Laws from which it follows that

$$K_\delta \cap \left(\bigcap_{j=1}^{n} K_{\alpha_j}\right) = \emptyset.$$

Therefore, there exists a finite subfamily of $\{K_\alpha\}$ that is disjoint.

We have shown that if $\bigcap_{\alpha \in \Delta} K_\alpha = \emptyset$, then there exists a finite subfamily of $\{K_\alpha : \alpha \in \Delta\}$ that has empty intersection. From the Contrapositive Tautology, if $\{K_\alpha : \alpha \in \Delta\}$ is a family of nonempty compact subsets of a metric space such that the intersection of any finite subfamily is nonempty, then $\bigcap_{\alpha \in \Delta} K_\alpha \neq \emptyset$. ■

Corollary 3.3.44 *If $\{K_n\}_{n=1}^\infty$ is a nested sequence of nonempty compact sets, then $\bigcap_{n \in \mathbb{J}} K_n \neq \emptyset$.*

Proof. For Δ any finite subset of $\mathbb{J}$, let $m = \max\{j : j \in \Delta\}$. Because $\{K_n\}_{n=1}^\infty$ is a nested sequence on nonempty sets, $K_m \subset \bigcap_{j \in \Delta} K_j$ and $\bigcap_{j \in \Delta} K_j \neq \emptyset$. Since Δ was arbitrary, we conclude that $\{K_n : n \in \mathbb{J}\}$ satisfies the finite intersection property. Hence, by Theorem 3.3.43, $\bigcap_{n \in \mathbb{J}} K_n \neq \emptyset$. ■

Corollary 3.3.45 *If $\{S_n\}_{n=1}^\infty$ is a nested sequence of nonempty closed subsets of a compact sets in a metric space, then $\bigcap_{n \in \mathbb{J}} S_n \neq \emptyset$.*

Theorem 3.3.46 *In a metric space, any infinite subset of a compact set has a limit point in the compact set.*

Space for notes and/or scratch work.

Proof. Let K be a compact subset of a metric space and E is a nonempty subset of K. Suppose that no element of K is a limit point for E. Then for each x in K there exists a neighborhood of x, say $N(x)$, such that $(N(x) - \{x\}) \cap E = \emptyset$. Hence, $N(x)$ contains at most one point from E; namely x. The family $\{N(x) : x \in K\}$ forms an open cover for K. Since K is compact, there exists a finite number of elements in $\{N(x) : x \in K\}$, say $N(x_1), N(x_2), ..., N(x_n)$, such that $K \subset N(x_1) \cup N(x_2) \cup \cdots \cup N(x_n)$. Because $E \subset K$, we also have that $E \subset N(x_1) \cup N(x_2) \cup \cdots \cup N(x_n)$. From the way that the neighborhoods were chosen, it follows that $E \subset \{x_1, x_2, ..., x_n\}$. Hence, E is finite.

We have shown that for any compact subset K of metric space, every subset of K that has not limit points in K is finite. Consequently, any infinite subset of K must have at least one limit point that is in K. ■

3.3.4 Compactness in Euclidean n-space

Thus far our results related to compact subsets of metric spaces described implications of that property. It would be nice to have some characterizations for compactness. In order to achieve that goal, we need to restrict our consideration to specific metric spaces. In this section, we consider only real n-space with the Euclidean metric. Our first goal is to show that every n-cell is compact in $\mathbb{R}^n$. Leading up to this we will show that every nested sequence of nonempty n-cells is not disjoint.

Theorem 3.3.47 (Nested Intervals Theorem) *If $\{I_n\}_{n=1}^{\infty}$ is a nested sequence of intervals in $\mathbb{R}^1$, then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.*

Proof. For the nested sequence of intervals $\{I_n\}_{n=1}^{\infty}$, let $I_n = [a_n, b_n]$ and $A = \{a_n : n \in \mathbb{J}\}$. Because $\{I_n\}_{n=1}^{\infty}$ is nested, $[a_n, b_n] \subset [a_1, b_1]$ for each $n \in \mathbb{J}$. It follows that $(\forall n)\,(n \in \mathbb{J} \Rightarrow a_n \leq b_1)$. Hence, A is a nonempty set of real numbers that is bounded above. By the Least Upper Bound Property, $x \underset{def}{=} \sup A$ exists and is real. From the definition of least upper bound, $a_n \leq x$ for each $n \in \mathbb{J}$. For any positive integers k and m, we have that

$$a_k \leq a_{k+m} \leq b_{k+m} \leq b_k$$

from which it follows that $x \leq b_n$ for all $n \in \mathbb{J}$. Since $a_n \leq x \leq b_n$ for each $n \in J$, we conclude that $x \in \bigcap_{n=1}^{\infty} I_n$. Hence, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$. ■

Remark 3.3.48 *Note that, for* $B = \{b_n : n \in J\}$ *appropriate adjustments in the proof that was given for the Nested Intervals Theorem would allow us to conclude that* $\inf B \in \bigcap_{n=1}^{\infty} I_n$. *Hence, if lengths of the nested integrals go to* 0 *as n goes to* ∞, *then* $\sup A = \inf B$ *and we conclude that* $\bigcap_{n=1}^{\infty} I_n$ *consists of one real number.*

The Nested Intervals Theorem generalizes to nested n-cells. The key is to have the set-up that makes use of the n intervals $\left[x_j, y_j\right]$, $1 \leq j \leq n$, that can be associated with $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$.

Theorem 3.3.49 (Nested n-Cells Theorem) *Let n be a positive integer. If* $\{I_k\}_{k=1}^{\infty}$ *is a nested sequence of n-cells, then* $\bigcap_{k=1}^{\infty} I_k \neq \emptyset$.

Proof. For the nested sequence of intervals $\{I_k\}_{k=1}^{\infty}$, let

$$I_k = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : a_{k,j} \leq x_j \leq b_{k,j} \text{ for } j = 1, 2, ..., n\right\}.$$

For each j, $1 \leq j \leq n$, let $I_{k,j} = \left[a_{k,j}, b_{k,j}\right]$. Then each $\left\{I_{k,j}\right\}_{k=1}^{\infty}$ satisfies the conditions of the Nested Intervals Theorem. Hence, for each j, $1 \leq j \leq n$, there exists $w_j \in R$ such that $w_j \in \bigcap_{k=1}^{\infty} I_{k,j}$. Consequently, $(w_1, w_2, ..., w_n) \in \bigcap_{k=1}^{\infty} I_k$ as needed. ■

Theorem 3.3.50 *Every n-cell is compact.*

Proof. For real constants $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ such that $a_j < b_j$ for each $j = 1, 2, ..., n$, let

$$I_0 = I = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J}) \left(1 \le j \le n \Rightarrow a_j \le x_j \le b_j\right)\right\}$$

and

$$\delta = \sqrt{\sum_{j=1}^{n} \left(b_j - a_j\right)^2}.$$

Then $(\forall \mathbf{x}) (\forall \mathbf{y}) \left[\mathbf{x}, \mathbf{y} \in I_0 \Rightarrow |\mathbf{x} - \mathbf{y}| \le \delta\right]$. Suppose that I_0 is not compact. Then there exists an open cover $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ of I_0 for which no finite subcollection covers I_0. Now we will describe the construction of a nested sequence of n-cells each member of which is not compact. Use the space provided to sketch appropriate pictures for $n = 1$, $n = 2$, and $n = 3$ that illustrate the described construction.

For each j, $1 \le j \le n$, let $c_j = \dfrac{a_j + b_j}{2}$. The sets of intervals

$$\left\{\left(a_j, c_j\right) : 1 \le j \le n\right\} \qquad \textit{and} \qquad \left\{\left(c_j, b_j\right) : 1 \le j \le n\right\}$$

can be used to determine or generate 2^n new n-cells, $I_k^{(1)}$ for $1 \le k \le 2^n$. For example, each of

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (j \in \mathbb{J}) \left(1 \le j \le n \Rightarrow a_j \le x_j \le c_j\right)\right\},$$

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J}) \left(1 \le j \le n \Rightarrow c_j \le x_j \le b_j\right)\right\},$$

and

$$\left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : a_j \le x_j \le c_j \text{ if } 2 \mid j \text{ and } c_j \le x_j \le b_j \text{ if } 2 \nmid j\right\}$$

is an element of $\left\{I_k^{(1)} : 1 \le k \le 2^n\right\}$. For each $k \in \mathbb{J}$, $1 \le k \le 2^n$, $I_k^{(1)}$ is a subset (sub-n-cell) of I_0 and $\bigcup_{k=1}^{2^n} I_k^{(1)} = I_0$. Consequently, $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for each of the 2^n sub-n-cells. Because I_0 is such that no finite subcollection from $\mathcal{G}$ covers I_0, it follows that at least one of the elements of $\left\{I_k^{(1)} : 1 \le k \le 2^n\right\}$ must also satisfy that property. Let I_1 denote an element of $\left\{I_k^{(1)} : 1 \le k \le 2^n\right\}$ for which no finite subcollection from $\mathcal{G}$ covers I_1. For $(x_1, x_2, ..., x_n) \in I_1$ we have that either $a_j \le x_j \le c_j$ or $c_j \le x_j \le b_j$ for each j, $1 \le j \le n$. Since

$$\frac{c_j - a_j}{2} = \frac{b_j - c_j}{2} = \frac{b_j - a_j}{2},$$

it follows that, for $\mathbf{x} = (x_1, x_2, ..., x_n)$, $\mathbf{y} = (y_1, y_2, ..., y_n) \in I_1$

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{n} \left(y_j - x_j\right)^2} \le \sqrt{\sum_{j=1}^{n} \frac{\left(b_j - a_j\right)^2}{2^2}} = \frac{\delta}{2};$$

i.e., the diam (I_1) is $\frac{\delta}{2}$.

The process just applied to I_0 to obtain I_1 can not be applied to obtain a sub-n-cell of I_1 that has the transferred properties. That is, if

$$I_1 = \left\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : (\forall j \in \mathbb{J}) \left(1 \le j \le n \Rightarrow a_j^{(1} \le x_j \le b_j^{(1)}\right)\right\},$$

letting $c_j^{(1)} = \frac{a_j^{(1)} + b_j^{(1)}}{2}$ generates two set of intervals

$$\left\{\left(a_j^{(1)}, c_j^{(1)}\right) : 1 \le j \le n\right\} \quad \text{and} \quad \left\{\left(c_j^{(1)}, b_j^{(1)}\right) : 1 \le j \le n\right\}$$

that will determine 2^n new n-cells, $I_k^{(2)}$ for $1 \le k \le 2^n$, that are sub-n-cells of I_1. Now, since $\mathcal{G}$ is an open cover for I_1 such that no finite subcollection from $\mathcal{G}$ covers I_1 and $\bigcup_{k=1}^{2^n} I_k^{(2)} = I_1$, it follows that there is at least one element of $\left\{I_k^{(2)} : 1 \leq k \leq 2^n\right\}$ that cannot be covered with a finite subcollection from $\mathcal{G}$; choose one of those elements and denote it by I_2. Now the choice of $c_j^{(1)}$ allows us to show that $\operatorname{diam}(I_2) = \dfrac{\operatorname{diam}(I_1)}{2} = \dfrac{\delta}{2^2}$. Continuing this process generates $\{I_k\}_{k=0}^{\infty}$ that satisfies each of the following properties:

- $\{I_k\}_{k=0}^{\infty}$ is a nested sequence of n-cells,
- for each $k \in \mathbb{J}$, no finite subfamily of $\mathcal{G}$ covers I_k, and
- $(\forall \mathbf{x})(\forall \mathbf{y})\left[\mathbf{x}, \mathbf{y} \in I_k \Rightarrow |\mathbf{x} - \mathbf{y}| \leq 2^{-k}\delta\right]$.

From the Nested n-cells Theorem, $\bigcap_{k=0}^{\infty} I_k \neq \emptyset$. Let $\zeta \in \bigcap_{k=0}^{\infty} I_k$. Because $\mathcal{G} = \{G_\alpha : \alpha \in \Delta\}$ is an open cover for I_0 and $\bigcap_{k=0}^{\infty} I_k \subset I_0$, there exists $G \in \mathcal{G}$ such that $\zeta \in G$. Since G is open, we there is a positive real number r such that $N_r(\zeta) \subset G$. Now $\operatorname{diam}(N_r(\zeta)) = 2r$ and, for $n \in \mathbb{J}$ large enough, $\operatorname{diam}(I_n) = 2^{-n}\delta < 2r$. Now, $\zeta \in I_k$ for all $k \in \mathbb{J}$ assures that $\zeta \in I_k$ for all $k \geq n$. Hence, for all $k \in \mathbb{J}$ such that $k \geq n$, $I_k \subset N_r(\zeta) \subset G$. In particular, each I_k, $k \geq n$, can be covered by one element of $\mathcal{G}$ which contradicts the method of choice that is assured if I_0 is not compact. Therefore, I_0 is compact. ■

The next result is a classical result in analysis. It gives us a characterization for compactness in real n-space that is simple; most of the "hard work" for the proof was done in when we proved Theorem 3.3.50.

Theorem 3.3.51 (The Heine-Borel Theorem) *Let A be a subset of Euclidean n-space. Then A is compact if and only if A is closed and bounded.*

Proof. Let A be a subset of Euclidean n-space $(\mathbb{R}^n, d)$

Suppose that A is closed and bounded. Then there exists an n-cell I such that $A \subset I$. For example, because A is bounded, there exists $M > 0$ such that $A \subset N_M\left(\overrightarrow{\mathbf{0}}\right)$; for this case, the n-cell

$$I = \left\{(x_1, x_2, ..., x_n) \in R^n : \max_{1 \leq j \leq n} \left|x_j\right| \leq M + 1\right\}$$

satisfies the specified condition. From Theorem 3.3.50, I is compact. Since $A \subset I$ and A is closed, it follows from Theorem 3.3.37 that A is compact.

Suppose that A is a compact subset of Euclidean n-space. From Theorem 3.3.35, we know that A is closed. Assume that A is not bounded and let $p_1 \in A$. Corresponding to each $m \in \mathbb{J}$, choose a p_m in A such that $p_m \neq p_k$ for $k = 1, 2, \ldots, (m-1)$ and $d(p_1, p_m) > m - 1$. As an infinite subset of the compact set A, by Theorem 3.3.46, $\{p_m : m \in \mathbb{J}\}$ has a limit point in A. Let $q \in A$ be a limit point for $\{p_m : m \in \mathbb{J}\}$. Then, for each $t \in \mathbb{J}$, there exists $p_{m_t} \in \{p_m : m \in \mathbb{J}\}$ such that $d\left(p_{m_t}, q\right) < \dfrac{1}{t+1}$. From the triangular inequality, it follows that for any $p_{m_t} \in \{p_m : m \in \mathbb{J}\}$,

$$d\left(p_{m_t}, p_1\right) \leq d\left(p_{m_t}, q\right) + d(q, p_1) < \frac{1}{1+t} + d(q, p_1) < 1 + d(q, p_1).$$

But $1+d(q, p_1)$ is a fixed real number, while p_{m_t} was chosen such that $d\left(p_{m_t}, p_1\right) > m_t - 1$ and $m_t - 1$ goes to infinity as t goes to infinity. Thus, we have reached a contradiction. Therefore, A is bounded. ■

The next theorem gives us another characterization for compactness. It can be shown to be valid over arbitrary metric spaces, but we will show it only over real n-space.

Theorem 3.3.52 *Let A be a subset of Euclidean n-space. Then A is compact if and only if every infinite subset of A has a limit point in A.*

Excursion 3.3.53 *Fill in what is missing in order to complete the following proof of Theorem 3.3.52.*

Proof. If A is a compact subset of Euclidean n-space, then every infinite subset of A has a limit point in A by Theorem 3.3.46.

Suppose that A is a subset of Euclidean n-space for which every infinite subset of A has a limit point in A. We will show that this assumption implies that A is closed and bounded. Suppose that w is a limit point of A. Then, for each $n \in \mathbb{J}$, there exists an x_n such that

$$x_n \in N_{\frac{1}{n}}(w) - \{w\}.$$

Let $S = \{x_n : n \in \mathbb{J}\}$. Then S is an $\underset{(1)}{\underline{\hspace{5cm}}}$ of A. Consequently, S has $\underset{(2)}{\underline{\hspace{5cm}}}$ in A. But S has only one limit point;

namely $\underset{(3)}{\underline{\hspace{2cm}}}$. Thus, $w \in A$. Since w was arbitrary, we conclude that A contains all of its limit point; i.e., $\underset{(4)}{\underline{\hspace{5cm}}}$.

Suppose that A is not bounded. Then, for each $n \in \mathbb{J}$, there exists y_n such that $|y_n| > n$. Let $S = \{y_n : n \in \mathbb{J}\}$. Then S is an $\underset{(5)}{\underline{\hspace{5cm}}}$ of A that has no finite limit point in A. Therefore,

$$A \text{ not bounded} \Rightarrow (\exists S)\left(S \subset A \wedge S \text{ is infinite } \wedge S \cap A' = \emptyset\right);$$

taking the contrapositive and noting that $\neg (P \wedge Q \wedge M)$ is logically equivalent to $[(P \wedge Q) \Rightarrow M]$ for any propositions P, Q and M, we conclude that

$$(\forall S)\left[\left(\underset{(6)}{\underline{\hspace{5cm}}}\right) \Rightarrow S \cap A' \neq \emptyset\right] \Rightarrow \underset{(7)}{\underline{\hspace{5cm}}}.$$

■

Acceptable completions include: (1) infinite subset, (2) a limit point, (3) w, (4) A is closed, (5) infinite subset, (6) $S \subset A \wedge S$ is infinite, and (7) A is bounded.

As an immediate consequence of Theorems 3.3.50 and 3.3.46, we have the following result that is somewhat of a generalization of the Least Upper Bound Property to n-space.

Theorem 3.3.54 (Weierstrass) *Every bounded infinite subset of Euclidean n-space has a limit point in* $\mathbb{R}^n$.

3.3.5 Connected Sets

With this section we take a brief look at one mathematical description for a subset of a metric space to be "in one piece." This is one of those situations where "we recognize it when we see it," at least with simply described sets in $\mathbb{R}$ and $\mathbb{R}^2$. The concept is more complicated than it seems since it needs to apply to all metric spaces and, of course, the mathematical description needs to be precise. Connectedness is defined in terms of the absence of a related property.

Definition 3.3.55 *Two subsets A and B of a metric space X are* ***separated*** *if and only if*

$$A \cap \overline{B} = \emptyset \wedge \overline{A} \cap B = \emptyset.$$

Definition 3.3.56 *A subset E of a metric space X is* ***connected*** *if and only if E is not the union of two nonempty separated sets.*

Example 3.3.57 *To justify that $A = \{x \in \mathbb{R} : 0 < x < 2 \vee 2 < x \leq 3\}$ is not connected, we just have to note that $B_1 = \{x \in \mathbb{R} : 0 < x < 2\}$ and $B_2 = \{x \in \mathbb{R} : 2 < x \leq 3\}$ are separated sets in $\mathbb{R}$ such that $A = B_1 \cup B_2$.*

Example 3.3.58 *In Euclidean 2-space, if $C = D_1 \cup D_2$ where*

$$D_1 = \left\{(x_1, x_2) \in \mathbb{R}^2 : d\left((1,0), (x_1, x_2) \leq 1\right)\right\}$$

and

$$D_2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : d\left((-1,0), (x_1, x_2) < 1\right)\right\},$$

then C is a connected subset of $\mathbb{R}^2$.

Remark 3.3.59 *The following is a symbolic description for a subset E of a metric space X to be connected:*

$$(\forall A)(\forall B)\left[(A \subset X \wedge B \subset X \wedge E = A \cup B) \Rightarrow \left(A \cap \overline{B} \neq \emptyset \vee \overline{A} \cap B \neq \emptyset \vee A = \emptyset \vee B = \emptyset\right)\right].$$

The statement is suggestive of the approach that is frequently taken when trying to prove sets having given properties are connected; namely, the direct approach would take an arbitrary set E and let $E = A \cup B$. This would be followed by using other information that is given to show that one of the sets must be empty.

The good news is that connected subsets of $\mathbb{R}^1$ can be characterized very nicely.

Theorem 3.3.60 *Let E be a subset of $\mathbb{R}^1$. Then E is connected in $\mathbb{R}^1$ if and only if*

$$(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right].$$

Excursion 3.3.61 *Fill in what is missing in order to complete the following proof of the Theorem.*

Proof. Suppose that E is a subset of $\mathbb{R}^1$ with the property that there exist real numbers x and y with $x < y$ such that $x, y \in E$ and, for some $z \in \mathbb{R}^1$,

$$z \in (x, y) \text{ and } z \notin E.$$

Let $A_z = E \cap (-\infty, z)$ and $B_z = E \cap (z, \infty)$. Since $z \notin E$, $E = A_z \cup B_z$. Because $x \in A_z$ and $y \in B_z$, both A_z and B_z are $\underset{(1)}{\underline{\hspace{3cm}}}$. Finally, $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$ yields that

$$\overline{A}_z \cap B_z = A_z \cap \overline{B}_z = \underset{(2)}{\underline{\hspace{1.5cm}}}.$$

Hence, E can be written as the union of two $\underset{(3)}{\underline{\hspace{3cm}}}$ sets; i.e., E is $\underset{(4)}{\underline{\hspace{6cm}}}$. Therefore, if E is connected, then $x, y \in E \wedge z \in \mathbb{R} \wedge x < z < y$ implies that $\underset{(5)}{\underline{\hspace{3cm}}}$.

To prove the converse, suppose that E is a subset of $\mathbb{R}^1$ that is not connected. Then there exist two nonempty separated subsets of $\mathbb{R}^1$, A and B, such that $E = A \cup B$. Choose $x \in A$ and $y \in B$ and assume that the set-up admits that $x < y$. Since $A \cap [x, y]$ is a nonempty subset of real numbers, by the least upper bound property, $z \underset{def}{=} \sup\left(A \cap [x, y]\right)$ exists and is real. From Theorem 3.3.26, $z \in \overline{A}$; then $\overline{A} \cap B = \emptyset$ yields that $z \notin B$. Now we have two possibilities to consider; $z \notin A$ and $z \in A$. If $z \notin A$, then $z \notin A \cup B = E$ and $x < z < y$. If $z \in A$, then $A \cap \overline{B} = \emptyset$ implies that $z \notin \overline{B}$ and we conclude that there exists w such that $z < w < y$ and $w \notin B$. From $z < w$, $w \notin A$. Hence, $w \notin A \cup B = E$ and $x < w < y$. In either case, we have that $\neg\,(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right]$. By the contrapositive $(\forall x)\,(\forall y)\,(\forall z)\left[\left(x, y \in E \wedge z \in \mathbb{R}^1 \wedge x < z < y\right) \Rightarrow z \in E\right]$ implies that E is connected. ■

*******Acceptable responses are: (1) nonempty, (2) $\emptyset$, (3) separated, (4) not connected, and (5) E is connected.*******

From the theorem, we know that, for a set of reals to be connected it must be either empty, all of $\mathbb{R}$, an interval, a segment, or a half open interval.

3.3.6 Perfect Sets

Definition 3.3.62 *A subset E of a metric space X is* ***perfect*** *if and only if E is closed and every point of E is a limit point of E.*

Alternatively, a subset E of a metric space X is perfect if and only if E is closed and contains no isolated points.

From Theorem 3.3.7, we know that any neighborhood of a limit point of a subset E of a metric space contains infinitely many points from E. Consequently, any nonempty perfect subset of a metric space is necessarily infinite; with the next theorem it is shown that, in Euclidean n-space, the nonempty perfect subsets are uncountably infinite.

Theorem 3.3.63 *If P is a nonempty perfect subset of Euclidean n-space, then P is uncountable.*

Proof. Let P be a nonempty perfect subset of $\mathbb{R}^n$. Then P contains at least one limit point and, by Theorem 3.3.6, P is infinite. Suppose that P is denumerable. It follows that P can be arranged as an infinite sequence; let

$$x_1, x_2, x_3, \ldots$$

represent the elements of P. First, we will justify the existence (or construction) of a sequence of neighborhoods $\left\{V_j\right\}_{j=1}^{\infty}$ that satisfies the following conditions:

(i) $(\forall j)\left(j \in \mathbb{J} \Rightarrow \overline{V}_{j+1} \subseteq V_j\right)$,

(ii) $(\forall j)\left(j \in \mathbb{J} \Rightarrow x_j \notin \overline{V}_{j+1}\right)$, and

(iii) $(\forall j)\left(j \in \mathbb{J} \Rightarrow V_j \cap P \neq \emptyset\right)$.

Start with an arbitrary neighborhood of x_1; i.e., let V_1 be any neighborhood of x_1. Suppose that $\left\{V_j\right\}_{j=1}^{n}$ has been constructed satisfying conditions (i)–(iii) for $1 \leq j \leq n$. Because P is perfect, every $x \in V_n \cap P$ is a limit point of P. Thus there are an infinite number of points of P that are in V_n and we may choose $y \in V_n \cap P$ such that $y \neq x_n$. Let V_{n+1} be a neighborhood of y such that $x_n \notin \overline{V}_{n+1}$ and $\overline{V}_{n+1} \subseteq V_n$. *Show that you can do this.*

Note that $V_{n+1} \cap P \neq \emptyset$ since $y \in V_{n+1} \cap P$. Thus we have a sequence $\{V_j\}_{j=1}^{n+1}$ satisfying (i)–(iii) for $1 \leq j \leq n+1$. By the Principle of Complete Induction we can construct the desired sequence.

Let $\{K_j\}_{j=1}^{\infty}$ be the sequence defined by $K_j = \overline{V_j} \cap P$ for each j. Since $\overline{V_j}$ and P are closed, K_j is closed. Since $\overline{V_j}$ is bounded, K_j is bounded. Thus K_j is closed and bounded and hence compact. Since $x_j \notin K_{j+1}$, no point of P lies in $\cap_{j=1}^{\infty} K_j$. Since $K_j \subseteq P$, this implies $\cap_{j=1}^{\infty} K_j = \emptyset$. But each K_j is nonempty by (iii) and $K_j \supseteq K_{j+1}$ by (i). This contradicts the Corollary 3.3.27. ■

Corollary 3.3.64 *For any two real numbers a and b such that $a < b$, the segment (a, b) is uncountable.*

The Cantor Set

The Cantor set is a fascinating example of a perfect subset of $\mathbb{R}^1$ that contains no segments. In Chapter 11 the idea of the measure of a set is studied; it generalizes the idea of length. If you take MAT127C, you will see the Cantor set offered as an example of a set that has measure zero even though it is uncountable.

The Cantor set is defined to be the intersection of a sequence of closed subsets of $[0, 1]$; the sequence of closed sets is defined recursively. Let $E_0 = [0, 1]$. For E_1 partition the interval E_0 into three subintervals of equal length and remove the middle segment (the interior of the middle section). Then

$$E_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

For E_2 partition each of the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ into three subintervals of equal length and remove the middle segment from each of the partitioned intervals; then

$$\begin{aligned} E_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right] \\ &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]. \end{aligned}$$

Continuing the process E_n will be the union of 2^n intervals. To obtain E_{n+1}, we partition each of the 2^n intervals into three subintervals of equal length and remove the middle segment, then E_{n+1} is the union of the 2^{n+1} intervals that remain.

Excursion 3.3.65 *In the space provided sketch pictures of* E_0, E_1, E_2, *and* E_3 *and find the sum of the lengths of the intervals that form each set.*

By construction $\{E_n\}_{n=1}^{\infty}$ is a nested sequence of compact subsets of $\mathbb{R}^1$.

Excursion 3.3.66 *Find a formula for the sum of the lengths of the intervals that* *form each set E_n.*

The Cantor set is defined to be $P = \bigcap_{n=1}^{\infty} E_n$.

Excursion 3.3.67 *Justify each of the following claims.*

(a) The Cantor set is compact.

(b) The $\{E_n\}_{n=1}^{\infty}$ satisfies the finite intersection property

Remark 3.3.68 *It follows from the second assertion that P is nonempty.*

Finally we want to justify the claims that were made about the Cantor set before we described its construction.

- *The Cantor set contains no segment from E_0.*

  To see this, we observe that each segment in the form of

  $$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right) \text{ for } k, m \in \mathbb{J}$$

  is disjoint from P. Given any segment (α, β) for $\alpha < \beta$, if $m \in \mathbb{J}$ is such that $3^{-m} < \dfrac{\beta - \alpha}{6}$, then (α, β) contains an interval of the form $\left(\dfrac{3k+1}{3^m}, \dfrac{3k+2}{3^m}\right)$ from which it follows that (α, β) is not contained in P.

- *The Cantor set is perfect.* For $x \in P$, let S be any segment that contains x. Since $x \in \bigcap_{n=1}^{\infty} E_n$, $x \in E_n$ for each $n \in \mathbb{J}$. Corresponding to each $n \in \mathbb{J}$, let I_n be the interval in E_n such that $x \in I_n$. Now, choose $m \in \mathbb{J}$ large enough to get $I_m \subset S$ and let x_m be an endpoint of I_m such that $x_m \neq x$. From the way that P was constructed, $x_m \in P$. Since S was arbitrary, we have shown that every segment containing x also contains at least one element from P. Hence, x is a limit point of P. That x was arbitrary yields that every element of P is a limit point of P.

3.4 Problem Set C

1. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let

   $$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{j=1}^{N} \left(x_j - y_j\right)^2}.$$

   Prove that $(\mathbb{R}^N, d)$ is a metric space.

2. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let

$$D(\mathbf{x}, \mathbf{y}) = \sum_{j=1}^{N} |x_j - y_j|.$$

Prove that $(\mathbb{R}^N, D)$ is a metric space.

3. For $\mathbf{x} = (x_1, x_2, ..., x_N)$ and $\mathbf{y} = (y_1, y_2, ..., y_N)$ in $\mathbb{R}^N$, let

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \le j \le N} \left| x_j - y_j \right|.$$

Prove that $(\mathbb{R}^N, d_\infty)$ is a metric space.

4. Show that the Euclidean metric d, given in problem #1, is equivalent to the metric d_∞, given in problem #3.

5. Suppose that (S, d) is a metric space. Prove that $\left(S, d'\right)$ is a metric space where

$$d'(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

[Hint: You might find it helpful to make use of properties of $h(\xi) = \dfrac{\xi}{1+\xi}$ for $\xi \ge 0$.]

6. If $a_1, a_2, ..., a_n$ are positive real numbers, is

$$d(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} a_k |x_k - y_k|$$

where $\mathbf{x} = (x_1, x_2, ..., x_n), \mathbf{y} = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, a metric on $\mathbb{R}^n$? Does your response change if the hypothesis is modified to require that $a_1, a_2, ..., a_n$ are nonnegative real numbers?

7. Is the metric D, given in problem #2, equivalent to the metric d_∞, given in problem #3? Carefully justify your position.

8. Are the metric spaces $(\mathbb{R}^N, d)$ and $\left(\mathbb{R}^N, d'\right)$ where the metrics d and d' are given in problems #1 and #5, respectively, equivalent? Carefully justify the position taken.

9. For (x_1, x_2) and $\left(x_1', x_2'\right)$ in $\mathbb{R}^2$,

$$d_3\left((x_1, x_2), \left(x_1', x_2'\right)\right) = \begin{cases} |x_2| + \left|x_2'\right| + \left|x_1 - x_1'\right| & , \text{if} \quad x_1 \neq x_1' \\ \left|x_2 - x_2'\right| & , \text{if} \quad x_1 = x_1' \end{cases}$$

Show that $\left(\mathbb{R}^2, d_3\right)$ is a metric space.

10. For $x, y \in \mathbb{R}^1$, let $d(x, y) = |x - 3y|$. Is $(\mathbb{R}, d)$ a metric space? Briefly justify your position.

11. For $\mathbb{R}^1$ with $d(x, y) = |x - y|$, give an example of a set which is neither open nor closed.

12. Show that, in Euclidean $n - space$, a set that is open in $\mathbb{R}^n$ has no isolated points.

13. Show that every finite subset of $\mathbb{R}^N$ is closed.

14. For $\mathbb{R}^1$ with the Euclidean metric, let $A = \{x \in \mathbb{Q} : 0 \le x \le 1\}$. Describe $\overline{A}$.

15. Prove each of the following claims that are parts of Theorem 3.3.13. Let S be a metric space.

    (a) The union of any family $\mathcal{F}$ of open subsets of S is open.

    (b) The intersection of any family $\mathcal{F}$ of closed subsets of S is closed.

    (c) If $A_1, A_2, \ldots, A_m$ is a finite family of closed subsets of S, then the union $\bigcup_{j=1}^{m} A_j$ is closed.

    (d) The space S is both open and closed.

    (e) The null set is both open and closed.

16. For $X = [-8, -4) \cup \{-2, 0\} \cup \left(\mathbb{Q} \cap \left(1, 2\sqrt{2}\right]\right)$ as a subset of $\mathbb{R}^1$, identify (describe or show a picture of) each of the following.

    (a) The interior of X, $\operatorname{Int}(X)$

    (b) The exterior of X, $\operatorname{Ext}(X)$

    (c) The closure of X, $\overline{X}$

(d) The boundary of X, ∂X

(e) The set of isolated points of X

(f) The set of lower bounds for X and the least upper bound of X, $\sup(X)$

17. As subsets of Euclidean 2-space, let

$$A = \left\{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1 + 1|, |x_2|\} \leq \frac{1}{2}\right\},$$

$B = \left\{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1 + 1|, |x_2|\} \leq 1\right\}$ and

$$Y = \left\{(x_1, x_2) \in \mathbb{R}^2 : (x_1, x_2) \in B - A \vee \left((x_1 - 1)^2 + x_2^2 < 1\right)\right\}.$$

(a) Give a nicely labelled sketch of Y on a representation for the Cartesian coordinate plane.

(b) Give a nicely labelled sketch of the exterior of Y, $\mathrm{Ext}(Y)$, on a representation for the Cartesian coordinate plane.

(c) Is Y open? Briefly justify your response.

(d) Is Y closed? Briefly justify your response.

(e) Is Y connected? Briefly justify your response.

18. Justify each of the following claims that were made in the Remark following Definition 3.3.15

(a) If A is a subset of a metric space (S, d), then $\mathrm{Ext}(A) = \mathrm{Int}(A^c)$.

(b) If A is a subset of a metric space (S, d), then

$$x \in \partial A \Leftrightarrow (\forall N_r(x))\left(N_r(x) \cap A \neq \emptyset \wedge N_r(x) \cap A^c \neq \emptyset\right).$$

19. For $\mathbb{R}^2$ with the Euclidean metric, show that the set

$$S = \left\{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\right\}$$

is open. Describe each of S^0, S', ∂S, $\overline{S}$, and S^c.

20. Prove that $\left\{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 < 1 \wedge 0 \leq x_2 \leq 1\right\}$ is not compact.

21. Prove that $\mathbb{Q}$, the set of rationals in $\mathbb{R}^1$, is not a connected subset of $\mathbb{R}^1$.

22. Let $\mathcal{F}$ be any family of connected subsets of a metric space X such that any two members of $\mathcal{F}$ have a common point. Prove that $\bigcup_{F \in \mathcal{F}} F$ is connected.

23. Prove that if S is a connected subset of a metric space, then $\overline{S}$ is connected.

24. Prove that any interval $I \subset \mathbb{R}^1$ is a connected subset of $\mathbb{R}^1$.

25. Prove that if A is a connected set in a metric space and $A \subset B \subset \overline{A}$, then B is connected.

26. Let $\{F_n\}_{n=1}^{\infty}$ be a nested sequence of compact sets, each of which is connected. Prove that $\bigcap_{n=1}^{\infty} F_n$ is connected.

27. Give an example to show that the compactness of the sets F_k given in problem #26 is necessary; i.e., show that a nested sequence of closed connected sets would not have been enough to ensure a connected intersection.