# Chapter 4

# Sequences and Series–First View

Recall that, for any set $A$, a sequence of elements of $A$ is a function $f : \mathbb{J} \to A$. Rather than using the notation $f(n)$ for the elements that have been selected from $A$, since the domain is always the natural numbers, we use the notational convention $a_n = f(n)$ and denote sequences in any of the following forms:

$$\{a_n\}_{n=1}^{\infty}, \qquad \{a_n\}_{n\in\mathbb{J}}, \qquad or \qquad a_1, a_2, a_3, a_4, \ldots.$$

This is the only time that we use the set bracket notation $\{\quad\}$ in a different context. The distinction is made in the way that the indexing is communicated . For $a_n = \alpha$, the $\{a_n\}_{n=1}^{\infty}$ is the constant sequence that "lists the term $\alpha$ infinitely often," $\alpha, \alpha, \alpha, \alpha, \ldots$; while $\{a_n : n \in \mathbb{J}\}$ is the set consisting of one element $\alpha$. (When you read the last sentence, you should have come up with some version of "For '$a$ sub $n$' equal to $\alpha$, the sequence of '$a$ sub $n$' for $n$ going from one to infinity is the constant sequence that "lists the term $\alpha$ infinitely often," $\alpha, \alpha, \alpha, \ldots$; while the set consisting of '$a$ sub $n$' for $n$ in the set of positive integers is the set consisting of one element $\alpha$"; i.e., the point is that you should not have skipped over the $\{a_n\}_{n=1}^{\infty}$ and $\{a_n : n \in \mathbb{J}\}$.) Most of your previous experience with sequences has been with sequences of real numbers, like

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots,\ldots \qquad \left\{\frac{3}{n+1}\right\}_{n=1}^{\infty},$$

$$\left\{\frac{n^2+3n-5}{n+47}\right\}_{n=1}^{\infty}, \ \left\{\frac{n^3-1}{n^3+1} + (-1)^n\right\}_{n=1}^{\infty}, \ and \ \left\{\frac{\log n}{n} + \sin\left(\frac{n\pi}{8}\right)\right\}_{n=1}^{\infty}.$$

In this chapter, most of our sequences will be of elements in Euclidean $n$-space. In MAT127B, our second view will focus on sequence of functions.

As children, our first exposure to sequences was made in an effort to teach us to look for patterns or to develop an appreciation for patterns that occur naturally.

**Excursion 4.0.1** *For each of the following, find a description for the general term as a function of $n \in \mathbb{J}$ that fits the terms that are given.*

*1.* $\frac{2}{5}, \frac{4}{7}, \frac{8}{9}, \frac{16}{11}, \frac{32}{13}, \frac{64}{15}, \ldots$

*2.* $1, \frac{3}{5}, 9, \frac{7}{9}, 81, \frac{11}{13}, 729, \ldots$

***An equation that works for (1) is $(2^n)(2n+3)^{-1}$ while (2) needs a different formula for the odd terms and the even terms; one pair that works is $(2n-1)(2n+1)^{-1}$ for $n$ even and $3^{n-1}$ when $n$ is odd.***

As part of the bigger picture, pattern recognition is important in areas of mathematics or the mathematical sciences that generate and study models of various phenomena. Of course, the models are of value when they allow for analysis and/or making projections. In this chapter, we seek to build a deeper mathematical understanding of sequences and series; primary attention is on properties associated with convergence. After preliminary work with sequences in arbitrary metric spaces, we will restrict our attention to sequences of real and complex numbers.

## 4.1 Sequences and Subsequences in Metric Spaces

If you recall the definition of convergence from your frosh calculus course, you might notice that the definition of a limit of a sequence of points in a metric space merely replaces the role formerly played by absolute value with its generalization, distance.

**Definition 4.1.1** *Let* $\{p_n\}_{n=1}^{\infty}$ *denote a sequence of elements of a metric space* $(S, d)$ *and* $p_0$ *be an element of* $S$*. The* ***limit of*** $\{p_n\}_{n=1}^{\infty}$ *is* $p_0$ *as n tends to (goes to or approaches) infinity if and only if*

$$(\forall\varepsilon)\left[(\varepsilon \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow (\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow d(p_n, p_0) < \varepsilon))\right]$$

*We write either* $p_n \to p_0$ *or* $\lim_{n\to\infty} p_n = p_0$*.*

**Remark 4.1.2** *The description* $M = M(\varepsilon)$ *indicates that "limit of sequence proofs" require justification or specification of a means of prescribing how to find an M that "will work" corresponding to each* $\varepsilon > 0$*. A function that gives us a nice way to specify* $M(\varepsilon)$*'s is defined by*

$$\lceil x \rceil = \inf\{j \in \mathbb{Z} : x \le j\}$$

*and is sometimes referred to as the* ***ceiling function****. Note, for example, that* $\left\lceil \frac{1}{2} \right\rceil = 1$*,* $\lceil -2.2 \rceil = -2$*, and* $\lceil 5 \rceil = 5$*. Compare this to the greatest integer function, which is sometimes referred to as the* ***floor function****.*

**Example 4.1.3** *The sequence* $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ *has the limit* $0$ *in* $\mathbb{R}$*. We can take* $M(1) = 2$*,* $M\left(\frac{1}{100}\right) = 200$*, and* $M\left(\frac{3}{350}\right) = \left\lceil \frac{700}{3} \right\rceil = 234$*. Of course, three examples does not a proof make. In general, for* $\varepsilon > 0$*, let* $M(\varepsilon) = \left\lceil \frac{2}{\varepsilon} \right\rceil$*. Then* $n > M(\varepsilon)$ *implies that*

$$n > \left\lceil \frac{2}{\varepsilon} \right\rceil \ge \frac{2}{\varepsilon} > 0$$

*which, by Proposition 1.2.9 (#7) and (#5), implies that* $\frac{1}{n} < \frac{\varepsilon}{2}$ *and* $\frac{2}{n} = \left|\frac{2}{n}\right| < \varepsilon$*.*

Using the definition to prove that the limit of a sequence is some point in the metric space is an example of where our scratch work towards finding a proof might be quite different from the proof that is presented. This is because we are allowed to "work backwards" with our scratch work, but we can not present a proof that starts with the bound that we want to prove. We illustrate this with the following excursion.

**Excursion 4.1.4** *After reading the presented scratch work, fill in what is missing to complete the proof of the claim that* $\left\{\dfrac{1+in}{n+1}\right\}_{n=1}^{\infty}$ *converges to* $i$ *in* $\mathbb{C}$.

(a) ***Scratch work towards a proof.*** *Because* $i \in \mathbb{C}$, *it suffices to show that* $\lim\limits_{n\to\infty}\dfrac{1+in}{n+1} = i$. *Suppose* $\varepsilon > 0$ *is given. Then*

$$\left|\frac{1+in}{n+1} - i\right| = \left|\frac{1+in-i\,(n+1)}{n+1}\right| = \left|\frac{1-i}{n+1}\right| = \frac{\sqrt{2}}{n+1} < \frac{\sqrt{2}}{n} < \varepsilon$$

*whenever* $\dfrac{\sqrt{2}}{\varepsilon} < n$. *So taking* $M(\varepsilon) = \left\lceil \dfrac{\sqrt{2}}{\varepsilon} \right\rceil$ *will work.*

(b) ***A proof.*** *For* $\varepsilon > 0$, *let* $M(\varepsilon) = \underset{(1)}{\underline{\hspace{6em}}}$. *Then* $n \in \mathbb{J}$ *and* $n > M(\varepsilon)$ *implies that* $n > \dfrac{\sqrt{2}}{\varepsilon}$ *which is equivalent to* $\underset{(2)}{\underline{\hspace{6em}}} < \varepsilon$. *Because* $n+1 > n$ *and* $\sqrt{2} > 0$, *we also know that* $\dfrac{\sqrt{2}}{\underset{(3)}{\underline{\hspace{2em}}}} < \dfrac{\overset{(4)}{\underline{\hspace{2em}}}}{n}$. *Consequently, if* $n > M(\varepsilon)$, *then*

$$\left|\frac{1+in}{n+1} - i\right| = \left|\frac{1+in-i\,(n+1)}{n+1}\right| = \left|\frac{1-i}{n+1}\right| = \frac{\sqrt{2}}{n+1} < \underset{(5)}{\underline{\hspace{6em}}} < \varepsilon$$

*Since* $\varepsilon > 0$ *was arbitrary, we conclude that*

$$(\forall\varepsilon)\left[(\varepsilon > 0) \Rightarrow (\exists M(\varepsilon))\left(M \in \mathbb{J} \wedge (\forall n)\left(n > M \Rightarrow \left|\frac{1+in}{n+1} - i\right| < \varepsilon\right)\right)\right];$$

*i.e.,* $\underset{(6)}{\underline{\hspace{12em}}}$. *Finally,* $i = (0,1) \in \mathbb{C}$ *and* $\lim\limits_{n\to\infty}\dfrac{1+in}{n+1} = i$ *yields that* $\left\{\dfrac{1+in}{n+1}\right\}_{n=1}^{\infty}$ *converges to* $i$ *in* $\mathbb{C}$.

***Acceptable responses are (1) $\left\lceil \frac{\sqrt{2}}{\varepsilon} \right\rceil$, (2) $\frac{\sqrt{2}}{n}$, (3) $n+1$, (4) $\sqrt{2}$, (5) $\frac{\sqrt{2}}{n}$, (6) $\lim_{n\to\infty} \frac{1+in}{n+1} = i$.***

**Definition 4.1.5** *The sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space $S$ is said to* ***converge*** *(or* ***be convergent) in*** *$S$ if there is a point $p_0 \in S$ such that $\lim_{n\to\infty} p_n = p_0$; it is said to* ***diverge in*** *$S$ if it does not converge in $S$.*

**Remark 4.1.6** *Notice that a sequence in a metric space $S$ will be divergent in $S$ if its limit is a point that is not in $S$. In our previous example, we proved that $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ converges to $0$ in $\mathbb{R}$; consequently, $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ is convergent in Euclidean $1$-space. On the other hand, $\left\{\frac{2}{n}\right\}_{n=1}^{\infty}$ is divergent in $\left(\mathbb{R}^{+} = \{x \in \mathbb{R} : x > 0\}, d\right)$ where $d$ denotes the Euclidean metric on $\mathbb{R}$, $d(x,y) = |x-y|$.*

Our first result concerning convergent sequences is metric spaces assures us of the uniqueness of the limits when they exist.

**Lemma 4.1.7** *Suppose $\{p_n\}_{n=1}^{\infty}$ is a sequence of elements in a metric space $(S,d)$. Then*

$$(\forall p)(\forall q)\left(\left[p, q \in S \wedge \lim_{n\to\infty} p_n = p \wedge \lim_{n\to\infty} p_n = q\right] \Rightarrow q = p\right).$$

*Space for scratch work.*

**Excursion 4.1.8** *Fill in what is missing in order to complete the proof of the lemma.*

**Proof.** Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements in a metric space $(S,d)$ for which there exists $p$ and $q$ in $S$ such that $\lim_{n\to\infty} p_n = p$ and $\lim_{n\to\infty} p_n = q$. Suppose the $p \neq q$. Then $d(p,q) > 0$ and we let $\varepsilon = \frac{1}{2} d(p,q)$. Because $\lim_{n\to\infty} p_n = p$ and $\varepsilon > 0$, there exists a positive integer $M_1$ such that

$$n > M_1 \Rightarrow d(p_n, p) < \varepsilon;$$

similarly, $\lim_{n\to\infty} p_n = q$ yields the existence of a positive integer $M_2$ such that

$$\underset{(1)}{\underline{\hspace{6cm}}}.$$

Now, let $M = \max\{M_1, M_2\}$. It follows from the symmetry property and the triangular inequality for metrics that $n > M$ implies that

$$d(p,q) \leq d(p, p_n) + \underset{(2)}{\underline{\hspace{1cm}}} < \varepsilon + \varepsilon = 2\left(\underset{(3)}{\underline{\hspace{1cm}}}\right) = d(p,q)$$

which contradicts the trichotomy law. Since we have reached a contradiction, we conclude that $\underset{(4)}{\underline{\hspace{1.3cm}}}$ as needed. Therefore, the limit of any convergent sequence in a metric space is unique. ■

***Acceptable fill-ins are: (1) $n > M_2 \Rightarrow d(p_n, q) < \varepsilon$, (2) $d(p_n, q)$ (3) $\frac{1}{2} d(p,q)$, (4) $p = q$.***

**Definition 4.1.9** *The sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space $(S,d)$ is **bounded** if and only if*

$$(\exists M)(\exists x)\left[M > 0 \wedge x \in S \wedge (\forall n)(n \in \mathbb{J} \Rightarrow d(x, p_n) < M)\right].$$

Note that if the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space $S$ is an not bounded, then the sequence is divergent in $S$. On the other hand, our next result shows that convergence yields boundedness.

**Lemma 4.1.10** *If the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in a metric space $(S,d)$ is convergent in $S$, then it is bounded.*

*Space for scratch work.*

**Proof.** Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence of elements in a metric space $(S, d)$ that is convergent to $p_0 \in S$. Then, for $\varepsilon = 1$, there exists a positive integer $M = M(1)$ such that

$$n > M \Rightarrow d(p_n, p_0) < 1.$$

Because $\left\{d\left(p_j, p_0\right) : j \in \mathbb{J} \wedge 1 \leq j \leq M\right\}$ is a finite set of nonnegative real numbers, it has a largest element. Let

$$K = \max\left\{1, \max\left\{d\left(p_j, p_0\right) : j \in \mathbb{J} \wedge 1 \leq j \leq M\right\}\right\}.$$

Since $d(p_n, p_0) \leq K$, for each $n \in \mathbb{J}$, we conclude that $\{p_n\}_{n=1}^{\infty}$ is bounded. ■

**Remark 4.1.11** *To see that the converse of Lemma 4.1.10 is false, for $n \in \mathbb{J}$, let*

$$p_n = \begin{cases} \dfrac{1}{n^2} & , \quad \text{if } 2 \mid n \\[2ex] 1 - \dfrac{1}{n+3} & , \quad \text{if } 2 \nmid n \end{cases}.$$

*Then, for $d$ the Euclidean metric on $\mathbb{R}^1$, $d(0, p_n) = |0 - p_n| < 1$ for all $n \in \mathbb{J}$, but $\{p_n\}_{n=1}^{\infty}$ is not convergent in $\mathbb{R}$.*

**Excursion 4.1.12** *For each $n \in \mathbb{J}$, let $a_n = p_{2n}$ and $b_n = p_{2n-1}$ where $p_n$ is defined in Remark 4.1.11.*

*(a) Use the definition to prove that $\lim_{n \to \infty} a_n = 0$.*

(b) *Use the definition to prove that* $\lim_{n\to\infty} b_n = 1$.

***Note that $a_n = \frac{1}{(2n)^2}$ and $b_n = 1 - \frac{1}{(2n-1)+3} = 1 - \frac{1}{2(n+1)}$; if you used $\frac{1}{n^2}$ and $1 - \frac{1}{n+3}$, respectively, your choices for corresponding $M(\varepsilon)$ will be slightly off. The following are acceptable solutions, which of course are not unique; compare what you did for general sense and content. Make especially certain that you did not offer a proof that is "working backwards" from what you wanted to show. (a) For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \frac{1}{2\sqrt{\varepsilon}} \right\rceil$. Then $n > M$ implies that $n > \left(2\sqrt{\varepsilon}\right)^{-1}$ or $\frac{1}{2n} < \sqrt{\varepsilon}$. If follows that $\left|\frac{1}{(2n)^2} - 0\right| = \frac{1}{(2n)^2} = \frac{1}{2n} \cdot \frac{1}{2n} < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$ whenever $n > M$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{(2n)^2} = 0$. (b) For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \frac{1}{\varepsilon} \right\rceil$. Then $n > M$ implies that $n > (\varepsilon)^{-1}$ or $\frac{1}{n} < \varepsilon$. Note that, for $n \in \mathbb{J}$, $n \geq 1 > 0$ implies that $n + 2 > 0 + 2 = 2 > 0$ and $2n + 2 = n + (n+2) > 0 + n = n$. Thus, for $n \in \mathbb{J}$ and $n > M$, we have that

$$\left|\left(1 - \frac{1}{2n+2}\right) - 1\right| = \left|\frac{1}{2n+2}\right| = \frac{1}{2n+2} < \frac{1}{n} < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(1 - \frac{1}{2n+2}\right) = 1$.***

**Remark 4.1.13** *Hopefully, you spotted that there were some extra steps exhibited in our solutions to Excursion 4.1.12. I chose to show some of the extra steps that* *illustrated where we make explicit use of the ordered field properties that were discussed in Chapter 1. In particular, it is unnecessary for you to have explicitly demonstrated that* $2n + 2 > n$ *from the inequalities that were given in Proposition 1.2.9 or the step* $\frac{1}{2n} \cdot \frac{1}{2n} < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon}$ *that was shown in part (a). For the former you can just write things like* $2n + 2 > n$*; for the latter, you could just have written* $\frac{1}{(2n)^2} < \left(\sqrt{\varepsilon}\right)^2 = \varepsilon$.

What we just proved about the sequence given in Remark 4.1.11 can be translated to a statement involving subsequences.

**Definition 4.1.14** *Given a sequence* $\{p_n\}_{n=1}^{\infty}$ *of elements in a metric space* $X$ *and a sequence* $\{n_k\}_{k=1}^{\infty}$ *of positive integers such that of* $n_k < n_{k+1}$ *for each* $k \in \mathbb{J}$*, the sequence* $\left\{p_{n_k}\right\}_{k=1}^{\infty}$ *is called a* ***subsequence*** *of* $\{p_n\}_{n=1}^{\infty}$*. If* $\left\{p_{n_k}\right\}_{k=1}^{\infty}$ *converges in* $X$ *then its limit is called a* ***subsequential limit*** *of* $\{p_n\}_{n=1}^{\infty}$.

**Remark 4.1.15** *In our function terminology, a subsequence of* $f : \mathbb{J} \to X$ *is the restriction of* $f$ *to any infinite subset of* $\mathbb{J}$ *with the understanding that ordering is conveyed by the subscripts; i.e.,* $n_j < n_{j+1}$ *for each* $j \in \mathbb{J}$.

From Excursion 4.1.12, we know that the sequence $\{p_n\}_{n=1}^{\infty}$ given in Remark 4.1.11 has two subsequential limits; namely, 0 and 1. The uniqueness of the limit of a convergent sequence leads us to observe that every subsequence of a convergent sequence must also be convergent to the same limit as the original sequence. Consequently, the existence of two distinct subsequential limits for $\{p_n\}_{n=1}^{\infty}$ is an alternative means of claiming that $\{p_n\}_{n=1}^{\infty}$ is divergent. In fact, it follows from the definition of the limit of a sequence that infinitely many terms outside of any neighborhood of a point in the metric space from which the sequence is chosen will eliminate that point as a possible limit. A slight variation of this observation is offered in the following characterization of convergence of a sequence in metric space.

**Lemma 4.1.16** *Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of elements from a metric space* $(X, d)$*. Then* $\{p_n\}_{n=1}^{\infty}$ *converges to* $p \in X$ *if and only if every neighborhood of* $p$ *contains all but finitely many of the terms of* $\{p_n\}_{n=1}^{\infty}$.

*Space for scratch work.*

**Proof.** Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of elements from a metric space $(X, d)$.

Suppose that $\{p_n\}_{n=1}^{\infty}$ converges to $p \in X$ and $V$ is a neighborhood of $p$. Then there exists a positive real number $r$ such that $V = N_r(p)$. From the definition of a limit, there exists a positive integer $M = M(r)$ such that $n > M$ implies that $d(p, p_n) < r$; i.e., for all $n > M$, $p_n \in V$. Consequently, at most the $\{p_k : k \in \mathbb{J} \wedge 1 \leq k \leq M\}$ is excluded from $V$. Since $V$ was arbitrary, we conclude that every neighborhood of $p$ contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$.

Suppose that every neighborhood of $p$ contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$. For any $\varepsilon > 0$, $N_\varepsilon(p)$ contains all but finitely many of the terms of $\{p_n\}_{n=1}^{\infty}$. Let $M = \max\{k \in \mathbb{J} : p_k \notin N_\varepsilon(p)\}$. Then $n > M$ implies that $p_n \in N_\varepsilon(p)$ from which it follows that $d(p_n, p) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that, for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $n > M$ implies that $d(p_n, p) < \varepsilon$; that is, $\{p_n\}_{n=1}^{\infty}$ converges to $p \in X$. ■

It will come as no surprise that limit point of subsets of metric spaces can be related to the concept of a limit of a sequence. The approach used in the proof of the next theorem should look familiar.

**Theorem 4.1.17** *A point $p_0$ is a limit point of a subset $A$ of a metric space $(X, d)$ if and only if there is a sequence $\{p_n\}_{n=1}^{\infty}$ with $p_n \in A$ and $p_n \neq p_0$ for every $n$ such that $p_n \to p_0$ as $n \to \infty$.*

**Proof.** ($\Leftarrow$) Suppose that there is a sequence $\{p_n\}_{n=1}^{\infty}$ such that $p_n \in A$, $p_n \neq p_0$ for every $n$, and $p_n \to p_0$. For $r > 0$, consider the neighborhood $N_r(p_0)$. Since $p_n \to p_0$, there exists a positive integer $M$ such that $d(p_n, p_0) < r$ for all $n > M$. In particular, $p_{M+1} \in A \cap N_r(p_0)$ and $p_{M+1} \neq p_0$. Since $r > 0$ was arbitrary, we conclude that $p_0$ is a limit point of the set $A$.

($\Rightarrow$) Suppose that $p_0 \in X$ is a limit point of $A$. (Finish this part by first making judicious use of the real sequence $\left\{\frac{1}{j}\right\}_{j=1}^{\infty}$ to generate a useful sequence $\{p_n\}_{n=1}^{\infty}$

followed by using the fact that $\frac{1}{j} \to 0$ as $j \to \infty$ to show that $\{p_n\}_{n=1}^{\infty}$ converges to $p_0$.)

■

**Remark 4.1.18** *Since Theorem 4.1.17 is a characterization for limit points, it gives us an alternative definition for such. When called upon to prove things related to limit points, it can be advantageous to think about which description of limit points would be most fruitful; i.e., you can use the definition or the characterization interchangeably.*

We close this section with two results that relate sequences with the metric space properties of being closed or being compact.

**Theorem 4.1.19** *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in $X$ and $X$ is a compact subset of a metric space $(S, d)$, then there exists a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent in $X$.*

Space for scratch work.

**Proof.** Suppose that $\{p_n\}_{n=1}^{\infty}$ is a sequence in $X$ and $X$ is a compact subset of a metric space $(S, d)$. Let $P = \{p_n : n \in \mathbb{J}\}$. If $P$ is finite, then there is at least one $k$ such that $p_k \in P$ and, for infinitely many $j \in \mathbb{J}$, we have that $p_j = p_k$.

Consequently, we can choose a sequence $\left\{n_j\right\}_{j=1}^{\infty}$ such that $n_j < n_{j+1}$ and $p_{n_j} \equiv p_k$ for each $j \in \mathbb{J}$. It follows that $\left\{p_{n_j}\right\}_{j=1}^{\infty}$ is a (constant) subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent to $p_k \in X$. If $P$ is infinite, then $P$ is an infinite subset of a compact set. By Theorem 3.3.46, it follows that $P$ has a limit point $p_0$ in $X$. From Theorem 4.1.17, we conclude that there is a sequence $\{q_k\}_{k=1}^{\infty}$ with $q_k \in P$ and $q_k \neq p_0$ for every $k$ such that $q_k \to p_0$ as $k \to \infty$; that is, $\{q_k\}_{k=1}^{\infty}$ is a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent to $p_0 \in X$. ■

**Theorem 4.1.20** *If $\{p_n\}_{n=1}^{\infty}$ is a sequence in a metric space $(S, d)$, then the set of all subsequential limits of $\{p_n\}_{n=1}^{\infty}$ is a closed subset of $S$.*

Space for scratch work.

**Proof.** Let $E^*$ denote the set of all subsequential limits of the sequence $\{p_n\}_{n=1}^{\infty}$ of elements in the metric space $(S, d)$. If $E^*$ is finite, then it is closed. Thus, we can assume that $E^*$ is infinite. Suppose that $w$ is a limit point of $E^*$. Then, corresponding to $r = 1$, there exists $x \neq w$ such that $x \in N_1(w) \cap E^*$. Since $x \in E^*$, we can find a subsequence of $\{p_n\}_{n=1}^{\infty}$ that converges to $x$. Hence, we can choose $n_1 \in \mathbb{J}$ such that $p_{n_1} \neq w$ and $d\left(p_{n_1}, w\right) < 1$. Let $\delta = d\left(p_{n_1}, w\right)$. Because $\delta > 0$, $w$ is a limit point of $E^*$, and $E^*$ is infinite, there exists $y \neq w$ that is in $N_{\delta/4}(w) \cap E^*$. Again, $y \in E^*$ leads to the existence of a subsequence of $\{p_n\}_{n=1}^{\infty}$ that converges to $y$. This allows us to choose $n_2 \in \mathbb{J}$ such that $n_2 > n_1$ and $d\left(p_{n_2}, y\right) < \dfrac{\delta}{4}$. From the triangular inequality, $d\left(w, p_{n_2}\right) \leq d(w, y) + d\left(y, p_{n_2}\right) < \dfrac{\delta}{2}$. We can repeat this process. In general, if we have chosen the increasing finite sequence $n_1, n_2, ..., n_j$, then there exists a $u$ such that $u \neq w$ and $u \in N_{r_j}(w) \cap E^*$ where $r_j = \dfrac{\delta}{2^{j+1}}$. Since $u \in E^*$, $u$ is the limit of a subsequence of $\{p_n\}_{n=1}^{\infty}$. Thus, we can find $n_{j+1}$ such that $d\left(p_{n_{j+1}}, u\right) < r_j$ from which it follows that

$$d\left(w, p_{n_{j+1}}\right) \leq d(w, u) + d\left(u, p_{n_{j+1}}\right) < 2r_j = \frac{\delta}{2^j}.$$

The method of selection of the subsequence $\left\{p_{n_j}\right\}_{j=1}^{\infty}$ ensures that it converges to $w$. Therefore, $w \in E^*$. Because $w$ was arbitrary, we conclude that $E^*$ contains all of its limit points; i.e., $E^*$ is closed. ■

## 4.2 Cauchy Sequences in Metric Spaces

The following view of "proximity" of terms in a sequence doesn't isolate a point to serve as a limit.

**Definition 4.2.1** *Let* $\{p_n\}_{n=1}^{\infty}$ *be an infinite sequence in a metric space* $(S, d)$*. Then* $\{p_n\}_{n=1}^{\infty}$ *is said to be a* ***Cauchy sequence*** *if and only if*

$$(\forall \varepsilon)\left[\varepsilon > 0 \Rightarrow (\exists M = M(\varepsilon))\,(M \in \mathbb{J} \wedge (\forall m)\,(\forall n)\,(n, m > M \Rightarrow d(p_n, p_m) < \varepsilon))\right].$$

Another useful property of subsets of a metric space is the diameter. In this section, the term leads to a characterization of Cauchy sequences as well as a sufficient condition to ensure that the intersection of a sequence of nested compact sets will consist of exactly one element.

**Definition 4.2.2** *Let* $E$ *be a subset of a metric space* $(X, d)$*. Then the* ***diameter of*** $E$*, denoted by* $\operatorname{diam}(E)$ *is*

$$\sup\{d(p, q) : p \in E \wedge q \in E\}.$$

**Example 4.2.3** *Let* $A = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\right\}$ *and*

$$B = \left\{(x_1, x_2) \in \mathbb{R}^2 : \max\{|x_1|, |x_2|\} \leq 1\right\}.$$

*Then, in Euclidean* 2*-space,* $\operatorname{diam}(A) = 2$ *and* $\operatorname{diam}(B) = 2\sqrt{2}$.

Note that, for the sets $A$ and $B$ given in Example 4.2.3, $\operatorname{diam}\left(\overline{A}\right) = 2 = \operatorname{diam}(A)$ and $\operatorname{diam}\left(\overline{B}\right) = 2\sqrt{2} = \operatorname{diam}(B)$. These illustrate the observation that is made with the next result.

**Lemma 4.2.4** *If $E$ is any subset of a metric space $X$, then* $\operatorname{diam}(E) = \operatorname{diam}\left(\overline{E}\right)$.

**Excursion 4.2.5** *Use the space provided to fill in a proof of the lemma. (If you get stuck, a proof can be found on page 53 of our text.*

The property of being a Cauchy sequence can be characterized nicely in terms of the diameter of particular subsequence.

**Lemma 4.2.6** *If $\{p_n\}_{n=1}^{\infty}$ is an infinite sequence in a metric space $X$ and $E_M$ is the subsequence $p_M, p_{M+1}, p_{M+2}, \ldots$, then $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence if and only if* $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$.

**Proof.** Corresponding to the infinite sequence $\{p_n\}_{n=1}^{\infty}$ in a metric space $(X, d)$ let $E_M$ denote the subsequence $p_M, p_{M+1}, p_{M+2}, \ldots$ .

Suppose that $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence. For $j \in \mathbb{J}$, there exists a positive integer $M_j^* = M_j^*(\varepsilon)$ such that $n, m > M_j^*$ implies that $d(p_n, p_m) < \dfrac{1}{j}$. Let $M_j = M_j^* + 1$. Then, for any $u, v \in E_{M_j}$, it follows that $d(u, v) < \dfrac{1}{j}$. Hence, $\sup\left\{d(u, v) : u \in E_{M_j} \wedge v \in E_{M_j}\right\} \leq \dfrac{1}{j}$; i.e., $\operatorname{diam}\left(E_{M_j}\right) \leq \dfrac{1}{j}$. Now given any $\varepsilon > 0$, there exists $M'$ such that $j > M'$ implies that $\frac{1}{j} < \varepsilon$. For $M = \max\left\{M_j, M'\right\}$ and $j > M$, $\operatorname{diam}\left(E_{M_j}\right) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$.

Suppose that $\lim_{M\to\infty} \operatorname{diam}(E_M) = 0$ and let $\varepsilon > 0$. Then there exists a positive integer $K$ such that $m > K$ implies that $\operatorname{diam}(E_m) < \varepsilon$; i.e.,

$$\sup\{d(u, v) : u \in E_m \wedge v \in E_m\} < \varepsilon.$$

In particular, for $n, j > m$ we can write $n = m + x$ and $j = m + y$ for some positive integers $x$ and $y$ and it follows that

$$d\left(p_n, p_j\right) \le \sup\{d(u, v) : u \in E_m \wedge v \in E_m\} < \varepsilon.$$

Thus, we have shown that, for any $\varepsilon > 0$, there exists a positive integer $m$ such that $n, j > m$ implies that $d\left(p_n, p_j\right) < \varepsilon$. Therefore, $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence. ■

With Corollary 3.3.44, we saw that any nested sequence of nonempty compact sets has nonempty intersection. The following slight modification results from adding the hypothesis that the diameters of the sets shrink to 0.

**Theorem 4.2.7** *If $\{K_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty compact subsets of a metric space $X$ such that*

$$\lim_{n\to\infty} \operatorname{diam}(K_n) = 0,$$

*then $\bigcap_{n\in\mathbb{J}} K_n$ consists of exactly one point.*

*Space for scratch work.*

**Proof.** Suppose that $\{K_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty compact subsets of a metric space $(X, d)$ such that $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$. From Corollary 3.3.44, $\{K_n\}_{n=1}^{\infty}$ being a nested sequence of nonempty compact subsets implies that $\bigcap_{n\in\mathbb{J}} K_n \neq \emptyset$.

If $\bigcap_{n\in\mathbb{J}} K_n$ consists of more that one point, then there exists points $x$ and $y$ in $X$ such that $x \in \bigcap_{n\in\mathbb{J}} K_n$, $y \in \bigcap_{n\in\mathbb{J}} K_n$ and $x \neq y$. But this yields that

$$0 < d(x, y) \leq \sup\{d(p, q) : p \in K_n \wedge q \in K_n\}$$

for all $n \in \mathbb{J}$; i.e., $\operatorname{diam}(E_M) \not< d(x, y)$ for any $M \in \mathbb{J}$. Hence, $\lim_{n\to\infty} \operatorname{diam}(E_M) \neq 0$. Because $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$, it follows immediately that $\bigcap_{n\in\mathbb{J}} K_n$ consists of exactly one point. ■

**Remark 4.2.8** *To see that a Cauchy sequence in an arbitrary metric space need not converge to a point that is in the space, consider the metric space* $(S, d)$ *where* $S$ *is the set of rational numbers and* $d(a, b) = |a - b|$.

On the other hand, a sequence that is convergent in a metric space is Cauchy there.

**Theorem 4.2.9** *Let* $\{p_n\}_{n=1}^{\infty}$ *be an infinite sequence in a metric space* $(S, d)$. *If* $\{p_n\}_{n=1}^{\infty}$ *converges in* $S$*, then* $\{p_n\}_{n=1}^{\infty}$ *is Cauchy.*

**Proof.** Let $\{p_n\}_{n=1}^{\infty}$ be an infinite sequence in a metric space $(S, d)$ that converges in $S$ to $p_0$. Suppose $\varepsilon > 0$ is given. Then, there exists an $M \in \mathbb{J}$ such that $n > M \Rightarrow d(p_n, p_0) < \frac{\varepsilon}{2}$. From the triangular inequality, if $n > M$ and $m > M$, then

$$d(p_n, p_m) \leq d(p_n, p_0) + d(p_0, p_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\{p_n\}_{n=1}^{\infty}$ is Cauchy. ■

As noted by Remark 4.2.8, the converse of Theorem 4.2.9 is not true. However, if we restrict ourselves to sequences of elements from compact subsets of a metric space, we obtain the following partial converse. Before showing this, we will make some us

**Theorem 4.2.10** *Let* $A$ *be a compact subset of a metric space* $(S, d)$ *and* $\{p_n\}_{n=1}^{\infty}$ *be a sequence in* $A$. *If* $\{p_n\}_{n=1}^{\infty}$ *is Cauchy, then there exists a point* $p_0 \in A$ *such that* $p_n \to p_0$ *as* $n \to \infty$.

**Proof.** Let $A$ be a compact subset of a metric space $(S, d)$ and suppose that $\{p_n\}_{n=1}^{\infty}$ of elements in $A$ is Cauchy. Let $E_M$ be the subsequence $\left\{p_{M+j}\right\}_{j=0}^{\infty}$. Then $\left\{\overline{E}_M\right\}_{m=1}^{\infty}$ is a nested sequence of closed subsets of $A$ and $\left\{\overline{E}_M \cap A\right\}_{m=1}^{\infty}$ is a nested sequence of compact subsets of $S$ for which $\lim_{M\to\infty} \operatorname{diam}\left(\overline{E}_M \cap A\right) = 0$. By Theorem 4.2.7, there exists a unique $p$ such that $p \in \overline{E}_M \cap A$ for all $M$.

Now justify that $p_n \to p$ as $n \to \infty$.

■

## 4.3 Sequences in Euclidean $k$-space

When we restrict ourselves to Euclidean space we get several additional results including the equivalence of sequence convergence with being a Cauchy sequence. The first result is the general version of the one for Euclidean $n$-space that we discussed in class.

**Lemma 4.3.1** *On* $(\mathbb{R}^k, d)$, *where d denotes the Euclidean metric, let*

$$p_n = (x_{1n}, x_{2n}, x_{3n}, ..., x_{kn}).$$

*Then the sequence* $\{p_n\}_{n=1}^{\infty}$ *converges to* $P = (p_1, p_2, p_3, ..., p_k)$ *if and only if* $x_{jn} \to p_j$ *for each* $j$, $1 \le j \le k$ *as sequences in* $\mathbb{R}^1$.

**Proof.** The result follows from the fact that, for each $m$, $1 \le m \le k$,

$$|x_{mn} - p_m| = \sqrt{(x_{mn} - p_m)^2} \le \sqrt{\sum_{j=1}^{k} \left(x_{jn} - p_j\right)^2} \le \sum_{j=1}^{k} \left|x_{jn} - p_j\right|.$$

Suppose that $\varepsilon > 0$ is given. If $\{p_n\}_{n=1}^{\infty}$ converges to $P = (p_1, p_2, p_3, ..., p_k)$, then there exists a positive real number $M = M(\varepsilon)$ such than $n > M$ implies that

$$d(p_n, P) = \sqrt{\sum_{j=1}^{k} \left(x_{jn} - p_j\right)^2} < \varepsilon.$$

Hence, for each $m$, $1 \leq m \leq k$, and for all $n > M$,

$$|x_{mn} - p_m| \leq d\left(p_n, P\right) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\lim_{n\to\infty} x_{mn} = p_m$. Conversely, suppose that $x_{jn} \to p_j$ for each $j$, $1 \leq j \leq k$ as sequences in $\mathbb{R}^1$. Then, for each $j$, $1 \leq j \leq k$, there exists a positive integer $M_j = M_j(\varepsilon)$ such that $n > M_j$ implies that $|x_{mn} - p_m| < \dfrac{\varepsilon}{k}$. Let $M = \max_{1\leq j\leq k} M_j$. It follows that, for $n > M$,

$$d\left(p_n, P\right) \leq \sum_{j=1}^{k} \left|x_{jn} - p_j\right| < k\left(\frac{\varepsilon}{k}\right) = \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we have that $\lim_{n\to\infty} p_n = P$. ■

Once we are restricted to the real field we can relate sequence behavior with algebraic operations involving terms of given sequences. The following result is one of the ones that allows us to find limits of given sequences from limits of sequences that we know or have already proved elsewhere.

**Theorem 4.3.2** *Suppose that $\{z_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that $\lim_{n\to\infty} z_n = S$ and $\lim_{n\to\infty} \zeta_n = T$. Then*

*(a)* $\lim_{n\to\infty} (z_n + \zeta_n) = S + T$;

*(b)* $\lim_{n\to\infty} (cz_n) = cS$, *for any constant* $c$;

*(c)* $\lim_{n\to\infty} (z_n\zeta_n) = ST$;

*(d)* $\lim_{n\to\infty} \left(\dfrac{z_n}{\zeta_n}\right) = \dfrac{S}{T}$, *provided that* $(\forall n)\left[n \in \mathbb{J} \Rightarrow \zeta_n \neq 0\right] \wedge T \neq 0$.

**Excursion 4.3.3** *For each of the following, fill in either the proof in the box on the left of scratch work (notes) that support the proof that is given. If you get stuck, proofs can be found on pp 49-50 of our text.*

**Proof.** Suppose that $\{z_n\}_{n=1}^{\infty}$ and $\{\zeta_n\}_{n=1}^{\infty}$ are sequences of complex numbers such that $\lim_{n\to\infty} z_n = S$ and $\lim_{n\to\infty} \zeta_n = T$.

| | | |
|---|---|---|
| (a) | | *Space for scratch work.*<br>Need look at<br>$\lvert (z_n + \zeta_n) - (S + T) \rvert$<br>–Know we can make<br>$\lvert z_m - S \rvert < \frac{\varepsilon}{2}$ for $m > M_1$<br>& $\lvert \zeta_n - T \rvert < \frac{\varepsilon}{2}$ for $n > M_2$<br>–Go for $M = \max\{M_1, M_2\}$<br>and use Triangular Ineq. |

| | | |
|---|---|---|
| (b) | | *Space for scratch work.*<br>Need look at<br>$\lvert (cz_n) - cS \rvert = \lvert c \rvert \lvert z_n - S \rvert$<br>–Know we can make<br>$\lvert z_m - S \rvert < \frac{\varepsilon}{\lvert c \rvert}$ for $m > M_1$<br>–for $c \neq 0$, mention $c = 0$<br>——as separate case. |

(c)

Since $z_n \to S$, there exists $M_1 \in \mathbb{J}$ such that $n > M_1$ implies that $\lvert z_n - S \rvert < 1$. Hence, $\lvert z_n \rvert - \lvert S \rvert < 1$ or $\lvert z_n \rvert < 1 + \lvert S \rvert$ for all $n > M_1$. Suppose that $\varepsilon > 0$ is given. If $T = 0$, then $\zeta_n \to 0$ as $n \to \infty$ implies that there exists $M^* \in \mathbb{J}$ such that $\lvert \zeta_n \rvert < \dfrac{\varepsilon}{1 + \lvert S \rvert}$ whenever $n > M^*$. For $n > \max\{M_1, M^*\}$, it follows that

$$\lvert (z_n \zeta_n) - ST \rvert = \lvert z_n \zeta_n \rvert < (1 + \lvert S \rvert) \left( \frac{\varepsilon}{1 + \lvert S \rvert} \right) = \varepsilon.$$

Thus, $\lim_{n \to \infty} z_n \zeta_n = 0$.

If $T \neq 0$, then $\zeta_n \to T$ as $n \to \infty$ yields that there exists $M_2 \in \mathbb{J}$ such that $\lvert \zeta_n \rvert < \dfrac{\varepsilon}{2(1 + \lvert S \rvert)}$ whenever $n > M_2$. From $z_n \to S$, there exists $M_3 \in \mathbb{J}$ such that $n > M_3 \Rightarrow \lvert z_n - T \rvert < \dfrac{\varepsilon}{2\lvert T \rvert}$. Finally, for any $n > \max\{M_1, M_2, M_3\}$,

$$\begin{aligned} &\lvert (z_n \zeta_n) - ST \rvert \\ &= \lvert (z_n \zeta_n) - z_n T + z_n T - ST \rvert \leq \lvert z_n \rvert \lvert \zeta_n - T \rvert + \lvert \zeta_n \rvert \lvert z_n - S \rvert \\ &\qquad < (1 + \lvert S \rvert) \frac{\varepsilon}{2(1 + \lvert S \rvert)} + \lvert T \rvert \frac{\varepsilon}{2\lvert T \rvert} = \varepsilon. \end{aligned}$$

*Space for scratch work.*

| | | |
|---|---|---|
| (d) | | *Space for scratch work.* $\left(\frac{z_n}{\zeta_n}\right) = z_n\left(\frac{1}{\zeta_n}\right)$ —we can just apply the result from (c). ■ |

The following result is a useful tool for proving the limits of given sequences in $\mathbb{R}^1$.

**Lemma 4.3.4 (The Squeeze Principle)** *Suppose that* $\{x_n\}_{n=1}^{\infty}$ *and* $\{y_n\}_{n=1}^{\infty}$ *are sequences of real numbers such that* $\lim_{n\to\infty} x_n = S$ *and* $\lim_{n\to\infty} y_n = S$. *If* $\{u_n\}_{n=1}^{\infty}$ *is a sequence of real numbers such that, for some positive integer* $K$

$$x_n \le u_n \le y_n, \text{ for all } n > K,$$

*then* $\lim_{n\to\infty} u_n = S$.

**Excursion 4.3.5** *Fill in a proof for The Squeeze Principle.*

**Theorem 4.3.6 (Bolzano-Weierstrass Theorem)** *In* $\mathbb{R}^k$, *every bounded sequence contains a convergent subsequence.*

**Proof.** Suppose that $\{p_n\}_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^k$. Then $P \underset{def}{=} \{p_n : n \in \mathbb{J}\}$ is bounded. Since $\overline{P}$ is a closed and bounded subset of $\mathbb{R}^k$, by the Heine-Borel Theorem, $\overline{P}$ is compact. Because $\{p_n\}_{n=1}^{\infty}$ is a sequence in $\overline{P}$ a compact subset of a metric space, by Theorem 4.1.19, there exists a subsequence of $\{p_n\}_{n=1}^{\infty}$ that is convergent in $\overline{P}$. ■

**Theorem 4.3.7 ($\mathbb{R}^k$ Completeness Theorem)** *In $\mathbb{R}^k$, a sequence is convergent if and only if it is a Cauchy sequence.*

**Excursion 4.3.8** *Fill in what is missing in order to complete the following proof of the $\mathbb{R}^k$ Completeness Theorem.*

**Proof.** Since we are in Euclidean $k$-space, by Theorem $\underset{(1)}{\underline{\hspace{3cm}}}$, we know that any sequence that is convergent in $\mathbb{R}^k$ is a Cauchy sequence. Consequently, we only need to prove the converse.

Let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^k$. Then corresponding to $\varepsilon = 1$, there exists $M = M(1) \in \mathbb{J}$ such that $m, n > M$ implies that $\underset{(2)}{\underline{\hspace{6cm}}}$ where $d$ denotes the Euclidean metric. In particular, $d(p_n, p_{M+1}) < 1$ for all $n > M$. Let

$$B = \max\left\{1, \max_{1 \le j \le M} d\left(p_j, d_{M+1}\right)\right\}.$$

Then, for each $j \in \mathbb{J}$, $d\left(p_j, d_{M+1}\right) \le B$ and we conclude that $\{p_n\}_{n=1}^{\infty}$ is a $\underset{(3)}{\underline{\hspace{3cm}}}$ sequence in $\mathbb{R}^k$. From the $\underset{(4)}{\underline{\hspace{6cm}}}$ Theorem, $\overline{\{p_n : n \in \mathbb{J}\}}$ is a compact subset of $\mathbb{R}^k$. Because $\{p_n\}_{n=1}^{\infty}$ is a Cauchy sequence in a compact metric space, by Theorem 4.2.10, there exists a $p_0 \in \overline{\{p_n : n \in \mathbb{J}\}}$ such that $p_n \to p_0$ as $n \to \infty$. Since $\{p_n\}_{n=1}^{\infty}$ was arbitrary, we concluded that $\underset{(5)}{\underline{\hspace{9cm}}}$. ∎

***Acceptable responses are: (1) 4.2.9, (2) $d(p_n, p_m) < 1$, (3) bounded, (4) Heine-Borel, and (5) every Cauchy sequence in $\mathbb{R}^k$ is convergent.***

From Theorem 4.3.7, we know that for sequences in $\mathbb{R}^k$, being Cauchy is equivalent to being convergent. Since the equivalence can not be claimed over arbitrary metric spaces, the presence of that property receives a special designation.

**Definition 4.3.9** *A metric space $X$ is said to be **complete** if and only if for every sequence in $X$, the sequence being Cauchy is equivalent to it being convergent in $X$.*

**Remark 4.3.10** *As noted earlier, $\mathbb{R}^k$ is complete. In view of Theorem 4.2.10, any compact metric space is complete. Finally because every closed subset of a metric space contains all of its limit points and the limit of a sequence is a limit point, we also have that every closed subset of a complete metric space is complete.*

It is always nice to find other conditions that ensure convergence of a sequence without actually having the find its limit. We know that compactness of the metric space allows us to deduce convergence from being Cauchy. On the other hand, we know that, in $\mathbb{R}^k$, compactness is equivalent to being closed and bounded. From the Bolzano-Weierstrass Theorem, boundedness of a sequence gives us a convergent subsequence. The sequence $\{i^n\}_{n=1}^{\infty}$ of elements in $\mathbb{C}$ quickly illustrates that boundedness of a sequence is not enough to give us convergence of the whole sequence. The good news is that, in $\mathbb{R}^1$, boundedness coupled with being increasing or decreasing will do the job.

**Definition 4.3.11** *A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is*

*(a)* ***monotonically increasing*** *if and only if $(\forall n)\,(n \in \mathbb{J} \Rightarrow x_n \le x_{n+1})$ and*

*(b)* ***monotonically decreasing*** *if and only if $(\forall n)\,(n \in \mathbb{J} \Rightarrow x_n \ge x_{n+1})$.*

**Definition 4.3.12** *The **class of monotonic sequences** consists of all the sequences in $\mathbb{R}^1$ that are either monotonically increasing or monotonically decreasing.*

**Example 4.3.13** *For each $n \in \mathbb{J}$, $\left(\dfrac{n+1}{n}\right)^n \ge 1 = \dfrac{(n+1)!}{(n+1)\,n!}$. It follows that*

$$\frac{n!}{n^n} \ge \frac{(n+1)!}{(n+1)\,(n+1)^n}.$$

*Consequently, $\left\{\dfrac{n!}{n^n}\right\}_{n=1}^{\infty}$ is monotonically decreasing.*

**Theorem 4.3.14** *Suppose that $\{x_n\}_{n=1}^{\infty}$ is monotonic. Then $\{x_n\}_{n=1}^{\infty}$ converges if and only if $\{x_n\}_{n=1}^{\infty}$ is bounded.*

**Excursion 4.3.15** *Fill in what is missing in order to complete the following proof for the case when $\{x_n\}_{n=1}^{\infty}$ is monotonically decreasing.*

***Proof.*** *By Lemma 4.1.10, if $\{x_n\}_{n=1}^{\infty}$ converges, then $\underset{(1)}{\underline{\hspace{6cm}}}$.*

*Now suppose that $\{x_n\}_{n=1}^{\infty}$ is monotonically decreasing and bounded. Let $P = \{x_n : n \in \mathbb{J}\}$. If $P$ is finite, then there is at least one $k$ such that $x_k \in P$ and, for infinitely many $j \in \mathbb{J}$, we have that $x_j = x_k$. On the other hand we have that $x_{k+m} \ge x_{(k+m)+1}$ for all $m \in \mathbb{J}$. It follows that $\{x_n\}_{n=1}^{\infty}$ is eventually a constant sequence which is convergent to $x_k$. If $P$ is infinite and bounded, then from the* *greatest lower bound property of the reals, we can let* $g = \inf(P)$. *Because g is the greatest lower bound,*

$$(\forall n)\left(n \in \mathbb{J} \Rightarrow \underset{(2)}{\underline{\hspace{6em}}}\right).$$

*Suppose that* $\varepsilon > 0$ *is given. Then there exists a positive integer M such that* $g \le x_M < g + \varepsilon$*; otherwise,* $\underset{(3)}{\underline{\hspace{20em}}}$.

*Because* $\{x_n\}_{n=1}^{\infty}$ *is* $\underset{(4)}{\underline{\hspace{7em}}}$*, the transitivity of less than or equal to yields that, for all* $n > M$, $g \le x_n < g + \varepsilon$. *Hence,* $n > M \Rightarrow |x_n - g| < \varepsilon$. *Since* $\varepsilon > 0$ *was arbitrary, we conclude that* $\underset{(5)}{\underline{\hspace{12em}}}$. ■

*******Acceptable responses are: (1) it is bounded, (2) $g \le x_n$, (3) $g + \varepsilon$ would be a lower bound that is greater than $g$, (4) decreasing, and (5) $\lim_{n\to\infty} x_n = g$.*******

### 4.3.1 Upper and Lower Bounds

Our next definition expands the limit notation to describe sequences that are tending to infinity or negative infinity.

**Definition 4.3.16** *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence of real numbers. Then*

*(a)* $x_n \to \infty$ *as* $n \to \infty$ *if and only if*

$$(\forall K)\left(K \in \mathbb{R}^1 \Rightarrow (\exists M)\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow x_n \ge K))\right)$$

*and*

*(b)* $x_n \to -\infty$ *as* $n \to \infty$ *if and only if*

$$(\forall K)\left(K \in \mathbb{R}^1 \Rightarrow (\exists M)\,(M \in \mathbb{J} \wedge (\forall n)\,(n > M \Rightarrow x_n \le K))\right).$$

*In the first case, we write* $\lim_{n\to\infty} x_n = \infty$ *and in the second case we write* $\lim_{n\to\infty} x_n = -\infty$.

**Definition 4.3.17** *For $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers, let $E$ denote the set of all subsequential limits in the extended real number system (this means that $\infty$ and/or $-\infty$ are included if needed). Then the* ***limit superior*** *of $\{x_n\}_{n=1}^{\infty}$ is $x^* = \sup(E)$ and the* ***limit inferior*** *of $\{x_n\}_{n=1}^{\infty}$ is $x_* = \inf(E)$.*

We will use $\limsup_{n\to\infty} x_n$ to denote the limit superior and $\liminf_{n\to\infty} x_n$ to denote the limit inferior of $\{x_n\}_{n=1}^{\infty}$.

**Example 4.3.18** *For each $n \in \mathbb{J}$, let $a_n = 1 + (-1)^n + \frac{1}{2^n}$. Then the $\limsup_{n\to\infty} a_n = 2$ and $\liminf_{n\to\infty} a_n = 0$.*

**Excursion 4.3.19** *Find the limit superior and the limit inferior for each of the following sequences.*

1. $\left\{ s_n = \dfrac{n + (-1)^n (2n+1)}{n} \right\}_{n=1}^{\infty}$

2. $\left\{ s_n = (-1)^{n+1} + \sin \dfrac{\pi n}{4} \right\}_{n=1}^{\infty}$

3. $\left\{ s_n = \left(1 + \dfrac{1}{n}\right)\left(1 + \sin \dfrac{\pi n}{2}\right) \right\}_{n=1}^{\infty}$

4. $\left\{s_n = -\frac{n}{4} + \left\lceil \frac{n}{4} \right\rceil + (-1)^n\right\}_{n=1}^{\infty}$

***For (1), we have two convergent subsequences to consider; $s_{2n} \to 3$ while $s_{2n-1} \to -1$ and you should have concluded that $\limsup_{n\to\infty} s_n = 3$ and $\liminf_{n\to\infty} s_n = -1$. In working on (2), you should have gotten 5 subsequential limits: $s_{4k} \to -1$, $\{s_{4k+1}\}$ and $\{s_{4k+3}\}$ give two subsequential limits, $1+\frac{\sqrt{2}}{2}$ for k even and $1-\frac{\sqrt{2}}{2}$ for k odd; $\{s_{4k+2}\}$ also gives two subsequential limits, $-2$ for k odd and 0 for k even. Comparison of the 5 subsequential limits leads to the conclusion that $\limsup_{n\to\infty} s_n = 1+\frac{\sqrt{2}}{2}$ and $\liminf_{n\to\infty} s_n = -2$. The sequence given in (3) leads to three subsequential limits, namely, $0, 1,$, and 2 which leads to the conclusion that $\limsup_{n\to\infty} s_n = 2$ and $\liminf_{n\to\infty} s_n = 0$. Finally, for (4), the subsequences $\{s_{4k}\}, \{s_{4k+1}\}, \{s_{4k+2}\}$, and $\{s_{4k+3}\}$ give limits of $1, -\frac{1}{4}, \frac{3}{2}$, and $-\frac{3}{4}$, respectively; hence, $\limsup_{n\to\infty} s_n = \frac{3}{2}$ and $\liminf_{n\to\infty} s_n = -\frac{3}{4}$.***

**Theorem 4.3.20** *Let $\{s_n\}_{n=1}^{\infty}$ be a sequence of real numbers and $E$ be the set of (finite) subsequential limits of the sequence plus possibly $+\infty$ and $-\infty$. Then*

*(a)* $\limsup_{n\to\infty} s_n \in E$, *and*

*(b)* $(\forall x)\left(\left(x > \limsup_{n\to\infty} s_n\right) \Rightarrow (\exists M)\,(n > M \Rightarrow s_n < x)\right)$.

*Moreover,* $\limsup_{n\to\infty} s_n$ *is the only real number that has these two properties.*

**Excursion 4.3.21** *Fill in what is missing in order to complete the following proof of the theorem.*

***Proof.*** *For the sequence of real numbers* $\{s_n\}_{n=1}^{\infty}$*, let* $E$ *denote the set of subsequential limits of the sequence, adjoining* $+\infty$ *and/or* $-\infty$ *if needed, and* $s^* = \limsup_{n\to\infty} s_n$.

*Proof of part (a): If* $s^* = \infty$*, then* $E$ *is unbounded. Thus* $\{s_n\}_{n=1}^{\infty}$ *is not bounded above and we conclude that there is a subsequence* $\left\{s_{n_k}\right\}_{k=1}^{\infty}$ *of* $\{s_n\}_{n=1}^{\infty}$ *such that* $\lim_{k\to\infty} s_{n_k} = \infty$.

*If* $s^* = -\infty$*, then* $\{s_n\}_{n=1}^{\infty}$ *has no finite subsequential limits; i.e.,* $-\infty$ *is the only element of* $E$*. It follows that* $\lim_{n\to\infty} s_n = -\infty$.

*Suppose that* $s^* \in \mathbb{R}$*. Then* $E$ *is bounded above and contains at least one element. By CN Theorem 4.1.20, the set* $E$ *is* $\underset{(1)}{\underline{\hspace{3cm}}}$*. It follows from CN Theorem* $\underset{(2)}{\underline{\hspace{3cm}}}$ *that* $s^* = \sup(E) \in \overline{E} = E$.

*Proof of part (b): Suppose that there exists* $x \in \mathbb{R}$ *such that* $x > s^*$ *and* $s_n \geq x$ *for infinitely many natural numbers* $n$*. Then there exists a subsequence of* $\{s_n\}_{n=1}^{\infty}$ *that converges to some real number* $y$ *such that* $\underset{(3)}{\underline{\hspace{3cm}}}$*. From the triangular inequality,* $y > s^*$*. But* $y \in E$ *and* $y > s^*$ *contradicts the definition of* $\underset{(4)}{\underline{\hspace{3cm}}}$*. It follows that, for any* $x > s^*$ *there are at most finitely many* $n \in \mathbb{J}$ *for which* $\underset{(5)}{\underline{\hspace{3cm}}}$*. Hence, for any* $x > s^*$ *there exists a positive integer* $M$ *such that* $n > M$ *implies that* $s_n < x$.

*Proof of uniqueness. Suppose that* $p$ *and* $q$ *are distinct real numbers that satisfy property (b). Then*

$$(\forall x)\,((x > p) \Rightarrow (\exists M)\,(n > M \Rightarrow s_n < x))$$

*and*

$$(\forall x)\,((x > q) \Rightarrow (\exists K)\,(n > K \Rightarrow s_n < x)).$$

*Without loss of generality we can assume that* $p < q$*. Then there exists* $w \in \mathbb{R}$ *such that* $p < w < q$*. Since* $w > p$ *there exists* $M \in \mathbb{J}$ *such that* $n > M$ *implies that* $s_n < w$*. In particular, at most finitely many of the* $s_k$ *satisfy* $\underset{(6)}{\underline{\hspace{3cm}}}$*. Therefore,* $q$ *cannot be the limit of any subsequence of* $\{s_n\}_{n=1}^{\infty}$ *from which it follows that* $q \notin E$*; i.e.,* $q$ *does not satisfy property (a).* ■

***Acceptable responses are: (1) closed, (2) 3.3.26, (3) $y \geq x$, (4) $\sup E$, (5) $s_n \geq x$, and (6) $q > s_k > w$.***

**Remark 4.3.22** *Note that, if $\{s_n\}_{n=1}^{\infty}$ is a convergent sequence of real numbers, say $\lim s_n = s_0$, then the set of subsequential limits is just $\{s_0\}$ and it follows that*

$$\limsup_{n\to\infty} s_n = \liminf_{n\to\infty} s_n.$$

**Theorem 4.3.23** *If $\{s_n\}_{n=1}^{\infty}$ and $\{t_n\}_{n=1}^{\infty}$ are sequences of real numbers and there exists a positive integer $M$ such that $n > M$ implies that $s_n \leq t_n$, then*

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n \quad and \quad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n.$$

**Excursion 4.3.24** *Offer a well presented justification for Theorem 4.3.23.*

## 4.4 Some Special Sequences

This section offers some limits for sequences with which you should become familiar. Space is provided so that you can fill in the proofs. If you get stuck, proofs can be found on page 58 of our text.

**Lemma 4.4.1** *For any fixed positive real number, $\lim_{n\to\infty} \frac{1}{n^p} = 0$.*

**Proof.** For $\varepsilon > 0$, let $M = M(\varepsilon) = \left\lceil \left(\frac{1}{\varepsilon}\right)^{1/p} \right\rceil$.

■

**Lemma 4.4.2** *For any fixed complex number $x$ such that $|x| < 1$, $\lim_{n\to\infty} x^n = 0$.*

**Proof.** If $x = 0$, then $x^n = 0$ for each $n \in \mathbb{J}$ and $\lim_{n\to\infty} x^n = 0$. Suppose that $x$ is a fixed complex number such that $0 < |x| < 1$. For $\varepsilon > 0$, let

$$M = M(\varepsilon) = \begin{cases} 1 & , \quad \text{for } \varepsilon \geq 1 \\ \left\lceil \dfrac{\ln(\varepsilon)}{\ln|x|} \right\rceil & , \quad \text{for } \varepsilon < 1 \end{cases} .$$

■

The following theorem makes use of the Squeeze Principle and the Binomial Theorem. The special case of the latter that we will use is that, for $n \in \mathbb{J}$ and

$\zeta \in \mathbb{R}-\{-1\}$,

$$(1+\zeta)^n = \sum_{k=0}^{n} \binom{n}{k} \zeta^k, \text{ where } \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

In particular, if $\zeta > 0$ we have that $(1+\zeta)^n \geq 1 + n\zeta$ and $(1+\zeta)^n > \binom{n}{k}\zeta^k$ for each $k$, $1 \leq k \leq n$.

**Theorem 4.4.3** *(a) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.*

*(b) We have that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.*

*(c) If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \dfrac{n^\alpha}{(1+p)^n} = 0$.*

**Proof of (a).** We need prove the statement only for the case of $p > 1$; the result for $0 < p < 1$ will follow by substituting $\frac{1}{p}$ in the proof of the other case. If $p > 1$, then set $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$ and from the Binomial Theorem,

$$1 + nx_n \leq (1+x_n)^n = p$$

and

$$0 < x_n \leq \frac{p-1}{n}.$$

■

**Proof of (b).** Let $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$ and, from the Binomial Theorem,

$$n = (1+x_n)^n \geq \frac{n(n-1)}{2} x_n^2.$$

■

**Proof of (c).** Let $k$ be a positive integer such that $k > \alpha$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n\,(n-1)\,(n-1)\cdots(n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}$$

and

$$0 < \frac{n^\alpha}{(1+p)^n} < \underline{\hspace{6cm}}.$$

■

## 4.5 Series of Complex Numbers

For our discussion of series, we will make a slight shift is subscripting; namely, it will turn out to be more convenient for us to have our initial subscript be 0 instead of 1. Given any sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$, we can associate (or derive) a related sequence $\{S_n\}_{n=0}^{\infty}$ where $S_n = \sum_{k=0}^{n} a_k$ called the sequence of $nth$ partial sums. The associated sequence allows us to give precise mathematical meaning to the idea of "finding an infinite sum."

**Definition 4.5.1** *Given a sequence of complex numbers $\{a_k\}_{k=0}^{\infty}$, the symbol $\sum_{k=0}^{\infty} a_k$ is called an **infinite series** or simply a **series**. The symbol is intended to suggest an infinite summation*

$$a_0 + a_1 + a_2 + a_3 + \cdots$$

*and each $a_n$ is called a **term** in the series. For each $n \in \mathbb{J} \cup \{0\}$, let $S_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n$. Then $\{S_n\}_{n=0}^{\infty}$ is called the **sequence of** $nth$ **partial sums** for $\sum_{k=0}^{\infty} a_k$.*

On the surface, the idea of adding an infinite number of numbers has no real meaning which is why the series has been defined just as a symbol. We use the associated sequence of $nth$ partial sums to create an interpretation for the symbol that is tied to a mathematical operation that is well defined.

**Definition 4.5.2** *An infinite series $\sum_{k=0}^{\infty} a_k$ is said to be **convergent** to the complex number $S$ if and only if the sequence of $nth$ partial sums $\{S_n\}_{n=0}^{\infty}$ is convergent to $S$; when this occurs, we write $\sum_{k=0}^{\infty} a_k = S$. If $\{S_n\}_{n=0}^{\infty}$ does not converge, we say that the series is **divergent**.*

**Remark 4.5.3** *The way that convergence of series is defined, makes it clear that we really aren't being given a brand new concept. In fact, given any sequence $\{S_n\}_{n=0}^{\infty}$, there exists a sequence $\{a_k\}_{k=0}^{\infty}$ such that $S_n = \sum_{k=1}^{n} a_k$ for every $k \in \mathbb{J} \cup \{0\}$: To see this, simply choose $a_0 = S_0$ and $a_k = S_k - S_{k-1}$ for $k \geq 1$. We will treat sequences and series as separate ideas because it is convenient and useful to do so.*

The remark leads us immediately to the observation that for a series to converge it is necessary that the terms go to zero.

**Lemma 4.5.4 (kth term test)** *If the series $\sum_{k=0}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.*

**Proof.** Suppose that $\sum_{k=0}^{\infty} a_k = S$. Then $\lim_{k\to\infty} S_k = S$ and $\lim_{k\to\infty} S_{k-1} = S$. Hence, by Theorem 4.3.2(a),

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = S - S = 0.$$

■

**Remark 4.5.5** *To see that the converse is not true, note that the harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

*is divergent which is a consequence of the following excursion.*

**Excursion 4.5.6** *Use the Principle of Mathematical Induction to prove that, for* $\sum_{k=1}^{\infty} \frac{1}{k}$, $S_{2^n} > 1 + \dfrac{n}{2}$.

**Excursion 4.5.7** *Use the definition of convergence (divergence) to discuss the following series.*

*(a)* $\sum_{k=1}^{\infty} \sin \frac{\pi k}{4}$

*(b)* $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$

***The first example can be claimed as divergent by inspection, because the $nth$ term does not go to zero. The key to proving that the second one converges is noticing that $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$; in fact, the given problem is an example of what is known as a telescoping sum.***

The following set of lemmas are just reformulations of results that we proved for sequences.

**Lemma 4.5.8 (Cauchy Criteria for Series Convergence)** *The series (of complex numbers) $\sum_{k=0}^{\infty} a_k$ is convergent if and only if for every $\varepsilon > 0$ there exists a positive integer $M = M(\varepsilon)$ such that $(\forall m)(\forall n)(m, n > M \Rightarrow |S_m - S_n| < \epsilon)$.*

**Proof.** The lemma holds because the complex sequence of $nth$ partial sums $\{S_n\}_{n=0}^{\infty}$ is convergent if and only if it is Cauchy. This equivalence follows from the combination of Theorem 4.2.9 and Theorem 4.3.6(b). ■

**Remark 4.5.9** *We will frequently make use of the following alternative formulation for the sequence of $nth$ partial sums being Cauchy. Namely, $\{S_n\}_{n=0}^{\infty}$ is Cauchy if and only if for every $\epsilon > 0$, there exists a positive integer $M$ such that $n > M$ implies that $|S_{n+p} - S_n| = \left|\sum_{k=n+1}^{n+p} a_k\right| < \epsilon$, for $p = 1, 2, \ldots$*

**Lemma 4.5.10** *For the series (of complex numbers) $\sum_{k=0}^{\infty} a_k$, let $\operatorname{Re}\ a_k = x_k$ and $\operatorname{Im}\ a_k = y_k$. Then $\sum_{k=0}^{\infty} a_k$ is convergent if and only if $\sum_{k=0}^{\infty} x_k$ and $\sum_{k=0}^{\infty} y_k$ are convergent (real) sequences.*

**Proof.** For the complex series $\sum_{k=0}^{\infty} a_k$,

$$S_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} x_k + i \sum_{k=0}^{n} y_k = \left( \sum_{k=0}^{n} x_k, \sum_{k=0}^{n} y_k \right).$$

Consequently, the result is simply a statement of Lemma 4.3.1 for the case $n = 2$. ■

**Lemma 4.5.11** *Suppose that $\sum_{k=0}^{\infty} a_k$ is a series of nonnegative real numbers. Then $\sum_{k=0}^{\infty} a_k$ is convergent if and only if its sequence of $nth$ partial sums is bounded.*

**Proof.** Suppose that $\sum_{k=0}^{\infty} a_k$ is a series of nonnegative real numbers. Then $\{S_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence. Consequently, the result follows from Theorem 4.3.14. ■

**Lemma 4.5.12** *Suppose that $\sum_{k=0}^{\infty} u_k$ and $\sum_{k=0}^{\infty} v_k$ are convergent to $U$ and $V$, respectively., and $c$ is a nonzero constant. Then*

*1.* $\sum_{k=0}^{\infty}(u_k \pm v_k) = U \pm V$ *and*

*2.* $\sum_{k=0}^{\infty} cu_k = cU$.

Most of our preliminary discussion of series will be with series for which the terms are positive real numbers. When not all of the terms are positive reals, we first check for absolute convergence.

**Definition 4.5.13** *The series $\sum_{j=0}^{\infty} a_j$ is said to be **absolutely convergent** if and only if $\sum_{j=0}^{\infty} |a_j|$ converges. If $\sum_{j=0}^{\infty} a_j$ converges and $\sum_{j=0}^{\infty} |a_j|$ diverges, then the series $\sum_{j=0}^{\infty} a_j$ is said to be **conditionally convergent**.*

After the discussion of some tests for absolute convergence, we will see that absolute convergence implies convergence. Also, we will justify that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

## 4.5.1 Some (Absolute) Convergence Tests

While the definition may be fun to use, we would like other means to determine convergence or divergence of a given series. This leads us to a list of tests, only a few of which are discussed in this section.

**Theorem 4.5.14 (Comparison Test)** *Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of complex numbers).*

*(a) If there exists a positive integer $M$ such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \leq c_k)$ for real constants $c_k$ and $\sum_{k=0}^{\infty} c_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges absolutely.*

*(b) If there exists a positive integer $M$ such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \geq d_k \geq 0)$ for real constants $d_k$ and $\sum_{k=0}^{\infty} d_k$ diverges, then $\sum_{k=0}^{\infty} |a_k|$ diverges.*

**Proof of (a).** Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of complex numbers), there exists a positive integer $M$ such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \leq c_k)$, and $\sum_{k=0}^{\infty} c_k$ converges. For fixed $\varepsilon > 0$, there exists a positive integer $K$ such that $n > K$ and $p \in \mathbb{J}$ implies that

$$\left|\sum_{k=n+1}^{n+p} c_k\right| = \sum_{k=n+1}^{n+p} c_k < \epsilon.$$

For $n > M^* = \max\{M, K\}$ and any $p \in \mathbb{J}$, it follows from the triangular inequality that

$$\left|\sum_{k=n+1}^{n+p} a_k\right| \leq \sum_{k=n+1}^{n+p} |a_k| \leq \sum_{k=n+1}^{n+p} c_k < \epsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\sum_{k=0}^{\infty} a_k$ converges. ■

**Proof of (b).** Suppose that $\sum_{k=0}^{\infty} a_k$ is a series (of real numbers), there exists a positive integer $M$ such that $(\forall k)\,(k \geq M \Rightarrow |a_k| \geq d_k \geq 0)$, and $\sum_{k=0}^{\infty} d_k$ diverges. From Lemma 4.5.11, $\left\{\sum_{k=0}^{n} d_k\right\}_{n=0}^{\infty}$ is an unbounded sequence. Since

$$\sum_{k=M}^{n} |a_k| \geq \sum_{k=M}^{n} d_k$$

for each $n > M$, it follows that $\left\{\sum_{k=0}^{n} |a_k|\right\}_{n=0}^{\infty}$ is an unbounded. Therefore, $\sum_{k=0}^{\infty} |a_k|$ diverges. ■

In order for the Comparison Tests to be useful, we need some series about which convergence or divergence behavior is known. The best known (or most famous) series is the Geometric Series.

**Definition 4.5.15** *For a nonzero constant $a$, the series $\sum_{k=0}^{\infty} ar^k$ is called a* ***geometric series****. The number $r$ is the* ***common ratio****.*

**Theorem 4.5.16 (Convergence Properties of the Geometric Series)** *For $a \neq 0$, the geometric series $\sum_{k=0}^{\infty} ar^k$ converges to the sum $\dfrac{a}{1-r}$ whenever $0 < |r| < 1$ and diverges whenever $|r| \geq 1$.*

**Proof.** The claim will follow upon showing that, for each $n \in \mathbb{J} \cup \{0\}$,

$$\sum_{k=0}^{n} ar^k = \frac{a\left(1 - r^{n+1}\right)}{1 - r}.$$

■

The proof of the next result makes use of the "regrouping" process that was applied to our study of the harmonic series.

**Theorem 4.5.17** *If $\left\{a_j\right\}_{j=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers, then the series $\sum_{j=0}^{\infty} a_j$ is convergent if and only if $\sum_{j=0}^{\infty} 2^j a_{2^j}$ converges.*

**Excursion 4.5.18** *Fill in the two blanks in order to complete the following proof of Theorem 4.5.17.*

**Proof.** Suppose that $\left\{a_j\right\}_{j=0}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers. For each $n, k \in \mathbb{J} \cup \{0\}$, let

$$S_n = \sum_{j=0}^{n} a_j \quad and \quad T_k = \sum_{j=0}^{k} 2^j a_{2^j}.$$

Note that, because $\left\{a_j\right\}_{j=0}^{\infty}$ is a monotonically decreasing sequence, for any $j \in \mathbb{J} \cup \{0\}$ and $m \in \mathbb{J}$,

$$(m+1)\, a_j \geq a_j + a_{j+1} + \cdots + a_{j+m} \geq (m+1)\, a_{j+m},$$

while $\left\{a_j\right\}_{j=0}^{\infty}$ a sequence of nonnegative real numbers yields that $\{S_n\}$ and $\{T_k\}$ are monotonically decreasing sequences. For $n < 2^k$,

$$S_n \leq a_0 + a_1 + \underbrace{(a_2 + a_3)}_{2^1\ terms} + \underbrace{(a_4 + a_5 + a_6 + a_7)}_{2^2\ terms} + \cdots + \underbrace{\left(a_{2^k} + \cdots + a_{2^{k+1}-1}\right)}_{2^k\ terms}$$

from which it follows that

(1)

For $n > 2^k$,

$$S_n \geq a_0 + a_1 + a_2 + \underbrace{(a_3 + a_4)}_{2^1 \; terms} + \underbrace{(a_5 + a_6 + a_7 + a_8)}_{2^2 \; terms} \cdots + \underbrace{\left(a_{2^{k-1}+1} + \cdots + a_{2^k}\right)}_{2^{k-1} \; terms}$$

from which it follows that

(2)

The result now follows because we have that $\{S_n\}$ and $\{T_k\}$ are simultaneously bounded or unbounded. ■
***For (1), the grouping indicated leads to $S_n \leq a_1 + a_0 + 2a_2 + 4a_4 + \cdots + 2^k a_{2^k} = a_1 + T_k$, while the second regrouping yields that $S_n \geq a_0 + a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{k-1} a_{2^k} = \frac{1}{2} a_0 + a_1 + \frac{1}{2} T_k$.***

As an immediate application of this theorem, we obtain a family of real series for which convergence and divergence can be claimed by inspection.

**Theorem 4.5.19 (Convergence Properties of $p$-series)** *The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges whenever $p > 1$ and diverges whenever $p \leq 1$.*

**Proof.** If $p \leq 0$, the $p-$series diverges by the kth term test. If $p > 0$, then $\left\{a_n = \frac{1}{n^p}\right\}_{n=1}^{\infty}$ is a monotonically decreasing sequence of nonnegative real numbers. Note that

$$\sum_{j=0}^{\infty} 2^j a_{2^j} = \sum_{j=0}^{\infty} 2^j \frac{1}{\left(2^j\right)^p} = \sum_{j=0}^{\infty} \left(2^{(1-p)}\right)^j .$$

Now use your knowledge of the geometric series to finish the discussion.

■

A similar argument yields the following result with is offered without proof. It is discussed on page 63 of our text.

**Lemma 4.5.20** *The series* $\sum_{j=2}^{\infty} \dfrac{1}{j\,(\ln j)^p}$ *converges whenever* $p > 1$ *and diverges whenever* $p \leq 1$.

**Excursion 4.5.21** *Discuss the convergence (or divergence) of each of the following.*

*(a)* $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

*(b)* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

*(c)* $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

(d) $\sum_{n=1}^{\infty} \frac{3}{n^2+3n-1}$

***Notice that all of the series given in this excursion are over the positive reals; thus, checking for absolute convergence is the same as checking for convergence. At this point, we only the $n^{th}$ term test, Comparison, recognition as a $p$-series, or rearrangement in order to identify the given as a geometric series. For (a), noticing that, for each $n \in \mathbb{J}$, $\frac{n}{n^2+1} \geq \frac{n}{n^2+n} = \frac{1}{n+1}$ allows us to claim divergence by comparison with the "shifted" harmonic series. The series given in (b) is convergent as a $p$-series for $p = 3$. Because $\lim \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$ the series given in (c) diverges by the $n^{th}$ term test. Finally, since $3n - 1 > 0$ for each $n \in \mathbb{J}$, $\frac{3}{n^2+3n-1} \leq \frac{3}{n^2}$ which allows us to claim convergence of the series given in (d) by comparison with $\sum_{n=1}^{\infty} \frac{3}{n^2}$ which is convergent as a constant multiple times the $p$-series with $p = 2$.***

When trying to make use of the Comparison Test, it is a frequent occurrence that we know the nature of the series with which to make to comparison almost by inspection though the exact form of a beneficial comparison series requires some creative algebraic manipulation. In the last excursion, part (a) was a mild example of this phenomenon. A quick comparison of the degrees of the rational functions that form the term suggest divergence by association with the harmonic series, but when we see that $\frac{n}{n^2+1} \not\geq \frac{1}{n}$ we have to find some way to manipulate the expression $\frac{n}{n^2+1}$ more creatively. I chose to illustrate the "throwing more in the denominator" argument; as an alternative, note that for any natural number $n$, $n^2 \geq 1 \Rightarrow 2n^2 \geq n^2 + 1 \Rightarrow \frac{n}{n^2+1} \geq \frac{1}{2n}$ which would have justified divergence by comparison with a constant multiple of the harmonic series. We have a nice variation of the comparison test that can enable us to bypass the need for the algebraic manipulations. We state here and leave its proof as exercise.

**Theorem 4.5.22 (Limit Comparison Test)** *Suppose that* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are* *such that* $a_n \geq 0$*,* $b_n \geq 0$ *for each* $n \in \mathbb{J} \cup \{0\}$*, and* $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$*. Then either* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge or both diverge.*

We have two more important and well known tests to consider at this point.

**Theorem 4.5.23 (Ratio Test)** *The series* $\sum_{k=0}^{\infty} a_k$

*(i) converges absolutely if* $\limsup_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$*;*

*(ii) diverges if there exists a nonnegative integer* $M$ *such that* $k > M$ *implies that* $\left| \frac{a_{k+1}}{a_k} \right| \geq 1$.

**Proof.** Suppose that the series $\sum_{k=0}^{\infty} a_k$ is such that $\limsup_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$. It follows that we can find a positive real number $\beta$ such that $\beta < 1$ and there exists an $M \in \mathbb{J}$ such that $n > M$ implies that $\left| \frac{a_{k+1}}{a_k} \right| < \beta$. It can be shown by induction that, for each $p \in \mathbb{J}$ and $n > M$, $\left| a_{n+p} \right| < \beta^p \left| a_n \right|$. In particular, for $n \geq M+1$ and $p \in \mathbb{J} \cup \{0\}$, $\left| a_{n+p} \right| < \beta^p \left| a_{M+1} \right|$. Now, the series $\sum_{p=1}^{\infty} \left| a_{M+1} \right| \beta^p$ is convergent as a geometric series with ratio less than one. Hence, $\sum_{j=M+1}^{\infty} a_j = \sum_{p=1}^{\infty} a_{M+p}$ is absolutely convergent by comparison from which it follows that $\sum_{k=0}^{\infty} a_k$ is absolutely convergent.

Suppose that the series $\sum_{k=0}^{\infty} a_k$ is such that there exists a nonnegative integer $M$ for which $k > M$ implies that $\left| \frac{a_{k+1}}{a_k} \right| \geq 1$. Briefly justify that this yields divergence as a consequence of the $n^{th}$ term test.

■

**Remark 4.5.24** *Note that* $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = 1$ *leads to no conclusive information concerning the convergence or divergence of* $\sum_{k=0}^{\infty} a_k$.

**Example 4.5.25** *Use the Ratio Test to discuss the convergence of each of the following:*

1. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(n-1)!}$

   *For* $a_n = \frac{1}{(n-1)!}$, $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{1}{n!}(n-1)!\right| = \frac{1}{n} \to 0$ *as* $n \to \infty$*. Hence,* $\limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$ *and we conclude that the series is (absolutely) convergent from the ratio test.*

2. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2}{2^n}$

   *Let* $a_n = \frac{n^2}{2^n}$*. Then* $\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right| = \frac{1}{2}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{2}$ *as* $n \to \infty$*. Thus,* $\limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{2} < 1$ *and we conclude that the given series is (absolutely) convergent.*

**Theorem 4.5.26 (Root Test)** *For* $\sum_{k=0}^{\infty} a_k$*, let* $\alpha = \limsup_{k\to\infty}\sqrt[k]{|a_k|}$,

*(i) if* $0 \le \alpha < 1$*, then* $\sum_{k=0}^{\infty} a_k$ *converges absolutely;*

*(ii) if* $\alpha > 1$*, then* $\sum_{k=0}^{\infty} a_k$ *diverges; and*

*(iii) if* $\alpha = 1$*, then no information concerning the convergence or divergence of* $\sum_{k=0}^{\infty} a_k$ *can be claimed.*

**Proof.** For $\sum_{k=0}^{\infty} a_k$, let $\alpha = \limsup_{k\to\infty}\sqrt[k]{|a_k|}$. If $\alpha < 1$, there exists a real number $\beta$ such that $\alpha < \beta < 1$. Because $\alpha$ is a supremum of subsequential limits and $\alpha < \beta < 1$, by Theorem 4.3.20, there exists a positive integer $M$ such that $n > M$ implies that $\sqrt[n]{|a_n|} < \beta$; i.e., $|a_n| < \beta^n$ for all $n > M$. Since $\displaystyle\sum_{j=M+1}^{\infty}\beta^j$ is convergent as a geometric series (that sums to $\dfrac{\beta^{m+1}}{1-\beta}$), we conclude that $\sum_{k=0}^{\infty}|a_k|$ converges; that is, $\sum_{k=0}^{\infty} a_k$ converges absolutely.

Briefly justify that $\alpha > 1$ leads to divergence of $\sum_{k=0}^{\infty} a_k$ as a consequence of the $n^{th}$ term test.

Finally, since $\alpha = \limsup_{k\to\infty}\sqrt[k]{|a_k|} = 1$ for the $p$-series, we see that no conclusion can be drawn concerning the convergence of divergence of the given series.

■

**Example 4.5.27** *Use the Root Test, to establish the convergence of* $\displaystyle\sum_{n=1}^{\infty}\frac{n}{2^{n-1}}$.

*From Theorem 4.4.3(a) and (b),* $\lim_{n\to\infty}\sqrt[n]{2n} = 1$. *Hence,*

$$\limsup_{k\to\infty}\sqrt[k]{|\frac{k}{2^{k-1}}|} = \lim_{k\to\infty}\sqrt[k]{2\left(\frac{k}{2^k}\right)} = \lim_{k\to\infty}\frac{\sqrt[k]{2k}}{2} = \frac{1}{2} < 1$$

*from which we claim (absolute) convergence of the given series.*

Thus far our examples of applications of the Ratio and Root test have led us to exam sequences for which $\limsup_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$ or $\limsup_{k\to\infty}\sqrt[k]{|a_k|} = \lim_{k\to\infty}\sqrt[k]{|a_k|}$. This relates back to the form of the tests that you should have seen with your first exposure to series tests, probably in frosh (or AP) calculus. Of course, the point of offering the more general statements of the tests is to allow us to study the absolute convergence of series for which appeal to the limit superior is necessary. The next two excursion are in the vein; the parts that are described seek to help you to develop more comfort with the objects that are examined in order to make use of the Ratio and Root tests.

**Excursion 4.5.28** *For $n \in \mathbb{J} \cup \{0\}$, let* $a_j = \begin{cases} \left(\frac{1+i}{2}\right)^j & , \quad if\, 2 \mid j \\ \left(\frac{2}{5}\right)^j & , \quad if\, 2 \nmid j \end{cases}$.

1. *Find the first four terms of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$.

2. *Find the first four terms of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$.

3. *Find $E_1$ the set of subsequential limits of* $\left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

4. *Find $E_2$ the set of subsequential limits of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

5. *Find each of the following:*

   *(a)* $\limsup_{j \to \infty} \left\{ \left| \frac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

*(b)* $\liminf\limits_{j\to\infty} \left\{ \left| \dfrac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

*(c)* $\limsup\limits_{j\to\infty} \left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

*(d)* $\liminf\limits_{j\to\infty} \left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

6. *Discuss the convergence of* $\sum\limits_{j=0}^{\infty} a_j$

***For (1), we are looking at $\left\{ \dfrac{2}{5}, \dfrac{5}{4}, \dfrac{16}{125}, \dfrac{125}{32}, \cdots \right\}$ while (2) is $\left\{ \dfrac{2}{5}, \dfrac{\sqrt{2}}{2}, \dfrac{2}{5}, \dfrac{\sqrt{2}}{2}, \cdots \right\}$; for (3), if $c_j = \left| \dfrac{a_{j+1}}{a_j} \right|$, then the possible subsequential limits are given by looking at $\{c_{2j}\}$ and $\{c_{2j-1}\}$ and $E_1 = \{0, \infty\}$; if in (4) we let $d_j = \sqrt[j]{|a_j|}$, then consideration of $\{d_{2j}\}$ and $\{d_{2j-1}\}$ leads to $E_2 = \left\{ \dfrac{\sqrt{2}}{2}, \dfrac{2}{5} \right\}$; For (3) and (4), we conclude that the requested values are $\infty$, 0, $\dfrac{\sqrt{2}}{2}$, and $\dfrac{2}{5}$, respectively. For the discussion of (6), note that The Ratio Test yields no information because neither (a) nor (b) is satisfied; in the other hand, from (5c), we see that $\limsup\limits_{j\to\infty} \left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty} = \dfrac{\sqrt{2}}{2} < 1$, from which we conclude that the given series is absolutely convergent. (As an aside, examination of $\{S_{2n}\}$ and $\{S_{2n-1}\}$ corresponding to $\sum_{j=0}^{\infty} a_j$ even allows us to conclude that the sum of the given series is $\dfrac{4}{1+2i} + \dfrac{10}{21} = \dfrac{134-168i}{105}$.)***

**Excursion 4.5.29** *For $n \in \mathbb{J} \cup \{0\}$, let* $a_j = \begin{cases} \left(\dfrac{2}{3}\right)^{j+1} & , \quad if\, 2 \mid j \\ \left(\dfrac{2}{3}\right)^{j-1} & , \quad if\, 2 \nmid j \end{cases}$.

1. *Find the first four terms of* $\left\{ \left| \dfrac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$.

2. *Find the first four terms of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$.

3. *Find $E_1$ the set of subsequential limits of* $\left\{ \left| \dfrac{a_{j+1}}{a_j} \right| \right\}_{j=0}^{\infty}$

4. *Find $E_2$ the set of subsequential limits of* $\left\{ \sqrt[j]{|a_j|} \right\}_{j=1}^{\infty}$

5. *Find each of the following:*

*(a)* $\limsup_{j\to\infty} \left\{\left|\frac{a_{j+1}}{a_j}\right|\right\}_{j=0}^{\infty}$

*(b)* $\liminf_{j\to\infty} \left\{\left|\frac{a_{j+1}}{a_j}\right|\right\}_{j=0}^{\infty}$

*(c)* $\limsup_{j\to\infty} \left\{\sqrt[j]{|a_j|}\right\}_{j=1}^{\infty}$

*(d)* $\liminf_{j\to\infty} \left\{\sqrt[j]{|a_j|}\right\}_{j=1}^{\infty}$

6. *Discuss the convergence of* $\sum_{j=1}^{\infty} a_j$

*******Response this time are: (1) $\left\{\frac{3}{2}, \left(\frac{2}{3}\right)^3, \frac{3}{2}, \left(\frac{2}{3}\right)^3, \cdots\right\}$, (2) $\left\{1, \left(\frac{2}{3}\right)^{3/2} = \frac{2}{3}\sqrt{\frac{2}{3}}, \left(\frac{2}{3}\right)^{2/3} = \sqrt[3]{\frac{4}{9}}, \left(\frac{2}{3}\right)^{5/4} = \frac{4}{9}\sqrt[4]{\frac{2}{3}}, \cdots\right\}$; (3) $E_1 = \left\{\frac{3}{2}, \frac{4}{9}\right\}$, (4) $E_2 = \left\{\frac{2}{3}\right\}$ where this comes from separate consideration of $\lim_{j\to\infty} \left\{\sqrt[2j]{|a_{2j}|}\right\}$ and $\lim_{j\to\infty} \left\{\sqrt[2j-1]{|a_{2j-1}|}\right\}$, (5) the values are $\frac{3}{2}, \frac{4}{9}, \frac{2}{3}$ and $\frac{2}{3}$, respectively. Finally, the Ratio Test fails to offer information concerning convergence, however, the Root

Test yields that $\sum_{j=1}^{\infty} a_j$ is absolutely convergent. (Again, if we choose to go back to the definition, examination of the $nth$ partial sums allows us to conclude that the series converges to 3.)***

**Remark 4.5.30** *Note that, if* $\limsup_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| > 1$ *for a series* $\sum_{k=0}^{\infty} a_k$*, the ratio test yields no information concerning the convergence of the series.*

### 4.5.2 Absolute Convergence and Cauchy Products

When the terms in the generating sequence for a series are not all nonnegative reals, we pursue the possibility of different forms of convergence.

The next result tells us that absolute convergence is a stronger condition than convergence

**Lemma 4.5.31** *If* $\{a_j\}_{j=1}^{\infty}$ *is a sequence of complex numbers and* $\sum_{j=1}^{\infty} |a_j|$ *converges, then* $\sum_{j=0}^{\infty} a_j$ *converges and* $\left|\sum_{j=0}^{\infty} a_j\right| \le \sum_{j=0}^{\infty} |a_j|$.

**Proof. (if we were to restrict ourselves to real series)** The following argument that is a very slight variation of the one offered by the author of our text applies only to series over the reals; it is followed by a general argument that applies to series of complex terms. Suppose $\{a_j\}_{j=1}^{\infty}$ is a sequence of real numbers such that $\sum_{j=1}^{\infty} |a_j|$ converges and define

$$v_j = \frac{|a_j| + a_j}{2} \text{ and } w_j = \frac{|a_j| - a_j}{2}.$$

Then $v_j - w_j = a_j$ while $v_j + w_j = |a_j|$. Furthermore,

$$a_j \ge 0 \text{ implies that } \quad v_j = a_j = |a_j| \text{ and } \quad w_j = 0$$

while

$$a_j < 0 \text{ implies that } \quad v_j = 0 \text{ and } \quad w_j = -a_j = |a_j|.$$

Consequently, $0 \le v_n \le |a_n|$ and $0 \le u_n \le |a_n|$ and, from the Comparison Test, it follows that $\sum_{j=0}^{\infty} v_j$ and $\sum_{j=0}^{\infty} w_j$ converge. By Lemma 4.5.12, $\sum_{j=0}^{\infty} (v_j - w_j)$ converges. Finally, since

$$-\left(v_j + w_j\right) \le \left(v_j - w_j\right) \le \left(v_j + w_j\right),$$

we see that

$$-\sum_{j=1}^{\infty}\left(v_j+w_j\right)\leq\sum_{j=1}^{\infty}\left(v_j-w_j\right)\leq\sum_{j=1}^{\infty}\left(v_j+w_j\right);$$

i.e., $-\sum_{j=1}^{\infty}\left|a_j\right|\leq\sum_{j=1}^{\infty}a_j\leq\sum_{j=1}^{\infty}\left|a_j\right|$. Hence $0\leq\left|\sum_{j=1}^{\infty}a_j\right|\leq\sum_{j=1}^{\infty}\left|a_j\right|$ ■

The following proof of the lemma in general makes use of the Cauchy Criteria for Convergence.

**Proof.** Suppose that $\left\{a_j\right\}_{j=1}^{\infty}$ is a sequence of complex numbers such that $\sum_{j=1}^{\infty}|a_j|$ converges and $\varepsilon>0$ is given. Then there exists a positive integer $M=M(\varepsilon)$ such that $(\forall m)(\forall n)(m,n>M\Rightarrow|S_m-S_n|<\epsilon)$ where $S_m=\sum_{j=1}^{m}|a_j|$. In particular, for any $p\in\mathbb{J}$ and $n>M$, $\sum_{j=n+1}^{n+p}|a_j|=\left|\sum_{j=n+1}^{n+p}|a_j|\right|<\varepsilon$. From the triangular inequality, it follows that $\left|\sum_{j=n+1}^{n+p}a_j\right|\leq\sum_{j=n+1}^{n+p}|a_j|<\varepsilon$ for any $p\in\mathbb{J}$ and $n>M$. Since $\varepsilon>0$ was arbitrary, we conclude that $\sum_{j=1}^{\infty}a_j$ converges by the Cauchy Criteria for Convergence. ■

**Remark 4.5.32** *A re-read of the comparison, root and ratio tests reveals that they are actually tests for absolute convergence.*

Absolute convergence offers the advantage of allowing us to treat the absolutely convergence series much as we do finite sums. We have already discussed the term by term sums and multiplying by a constant. There are two kinds of product that come to mind: The first is the one that generalizes what we do with the distributive law (multiplying term-by-term and collecting terms), the second just multiplies the terms with the matching subscripts.

**Definition 4.5.33 (The Cauchy Product)** *For $\sum_{j=0}^{\infty}a_j$ and $\sum_{j=0}^{\infty}b_j$, set*

$$C_k=\sum_{j=0}^{k}a_jb_{k-j}\text{ for each }k\in\mathbb{J}\cup\{0\}.$$

*Then $\sum_{k=0}^{\infty}C_k$ is called the Cauchy product of the given series.*

**Definition 4.5.34 (The Hadamard Product)** *For $\sum_{j=0}^{\infty}a_j$ and $\sum_{j=0}^{\infty}b_j$, the series $\sum_{j=0}^{\infty}a_jb_j$ is called the Hadamard product of the given series.*

The convergence of two given series does not automatically lead to the convergence of the Cauchy product. The example given in our text (pp 73-74) takes

$$a_j = b_j = \frac{(-1)^j}{\sqrt{j+1}}.$$

We will see in the next section that $\sum_{j=0}^{\infty} a_j$ converges (conditionally). On the other hand, $C_k = \sum_{j=0}^{k} a_j b_{k-j} = (-1)^k \sum_{j=0}^{k} \dfrac{1}{\sqrt{(k-j+1)\,(j+1)}}$ is such that

$$|C_k| \geq \sum_{j=0}^{k} \frac{2}{k+2} = (k+1)\,\frac{2}{k+2}$$

which does not go to zero as $k$ goes to infinity.

If one of the given series is absolutely convergent and the other is convergent we have better news to report.

**Theorem 4.5.35 (Mertens Theorem)** *For $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$, if (i) $\sum_{j=0}^{\infty} a_j$ converges absolutely, (ii) $\sum_{j=0}^{\infty} a_j = A$, and $\sum_{j=0}^{\infty} b_j = B$, then the Cauchy product of $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ is convergent to $AB$.*

**Proof.** For $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$, let $\{A_n\}$ and $\{B_n\}$ be the respective sequences of $nth$ partial sums. Then

$$C_n = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} a_j b_{n-j} \right) = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} \cdots + a_n b_0)$$

which can be rearranged–using commutativity, associativity and the distributive laws–to

$$a_0(b_0 + b_1 + \cdots + b_n) + a_1(b_0 + b_1 + \cdots + b_{n-1}) + \cdots + a_n b_0.$$

Thus,

$$C_n = a_0 B_n + a_1 B_{n-1} + \cdots + a_{n-1} B_1 + a_n B_0.$$

Since $\sum_{j=0}^{\infty} b_j = B$, for $\beta_n \underset{def}{=} B - B_n$ we have that $\lim_{n\to\infty} \beta_n = 0$. Substitution in the previous equation yields that

$$C_n = a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_{n-1}(B + \beta_1) + a_n(B + \beta_0)$$

which simplifies to

$$C_n = A_n B + (a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-1}\beta_1 + a_n\beta_0).$$

Let

$$\gamma_n = a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-1}\beta_1 + a_n\beta_0$$

Because $\lim_{n\to\infty} A_n = A$, we will be done if we can show that $\lim_{n\to\infty} \gamma_n = 0$. In view of the absolute convergence of $\sum_{j=0}^{\infty} a_j$, we can set $\sum_{j=0}^{\infty} |a_j| = \alpha$.

Suppose that $\varepsilon > 0$ is given. From the convergence of $\{\beta_n\}$, there exists a positive integer $M$ such that $n > M$ implies that $|\beta_n| < \varepsilon$. For $n > M$, it follows that

$$|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1} + a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0|$$

From the convergence of $\{\beta_n\}$ and $\sum_{j=0}^{\infty} |a_j|$, we have that

$$|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1}| < \varepsilon\alpha$$

while $M$ being a fixed number and $a_k \to 0$ as $k \to \infty$ yields that

$$|a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0| \to 0 \text{ as } n \to \infty.$$

Hence, $|\gamma_n| =$
$|a_0\beta_n + a_1\beta_{n-1} + \cdots + a_{n-M-1}\beta_{M+1} + a_{n-M}\beta_M + \cdots + a_{n-1}\beta_1 + a_n\beta_0|$ implies that $\limsup_{n\to\infty} |\gamma_n| \le \varepsilon\alpha$. Since $\varepsilon > 0$ was arbitrary, it follows that $\lim_{n\to\infty} |\gamma_n| = 0$ as needed. ■

The last theorem in this section asserts that if the Cauchy product of two given convergent series is known to converge and its limit must be the product of the limits of the given series.

**Theorem 4.5.36** *If the series* $\sum_{j=0}^{\infty} a_j$, $\sum_{j=0}^{\infty} b_j$, *and* $\sum_{j=0}^{\infty} c_j$ *are known to converge,* $\sum_{j=0}^{\infty} a_j = A$, $\sum_{j=0}^{\infty} b_j = B$, *and* $\sum_{j=0}^{\infty} c_j$ *is the Cauchy product of* $\sum_{j=0}^{\infty} a_j$ *and* $\sum_{j=0}^{\infty} b_j$, *then* $\sum_{j=0}^{\infty} c_j = AB$.

### 4.5.3 Hadamard Products and Series with Positive and Negative Terms

Notice that $\sum_{j=1}^{\infty} \dfrac{1}{j\,(j+3)^3}$ can be realized as several different Hadamard products; letting $a_j = \dfrac{1}{j}$, $b_j = \dfrac{1}{(j+3)^3}$, $c_j = \dfrac{1}{j\,(j+3)}$ and $d_j = \dfrac{1}{(j+3)^2}$, gives us $\sum_{j=1}^{\infty} \dfrac{1}{j\,(j+3)^3}$ as the Hadamard product of $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ as well as the Hadamard product of $\sum_{j=1}^{\infty} c_j$ and $\sum_{j=1}^{\infty} d_j$. Note that only $\sum_{j=1}^{\infty} a_j$ diverges.

The following theorem offers a useful tool for studying the $nth$ partial sums for Hadamard products.

**Theorem 4.5.37 (Summation-by-Parts)** *Corresponding to the sequences* $\{a_j\}_{j=0}^{\infty}$, *let*

$$A_n = \sum_{j=0}^{n} a_j \text{ for } n \in \mathbb{J} \cup \{0\}, \text{ and } A_{-1} = 0.$$

*Then for the sequence* $\{b_j\}_{j=0}^{\infty}$ *and nonnegative integers* $p$ *and* $q$ *such that* $0 \le p \le q$,

$$\sum_{j=p}^{q} a_j b_j = \sum_{j=p}^{q-1} A_j \left(b_j - b_{j+1}\right) + A_q b_q - A_{p-1} b_p$$

**Excursion 4.5.38** *Fill in a proof for the claim.*

As an immediate application of this formula, we can show that the Hadamard product of a series whose $nth$ partial sums form a bounded sequence with a series that is generated from a monotonically decreasing sequence of nonnegative terms is convergent.

**Theorem 4.5.39** *Suppose that the series $\sum_{j=0}^{\infty} a_j$ and $\sum_{j=0}^{\infty} b_j$ are such that*

*(i)* $\left\{\sum_{j=0}^{n} a_j\right\}_{n=0}^{\infty}$ *is a bounded sequence,*

*(ii)* $\left\{b_j\right\}_{j=0}^{\infty}$ *is a monotonically decreasing sequence of nonnegative reals, and*

*(iii)* $\lim_{j\to\infty} b_j = 0$.

*Then $\sum_{j=0}^{\infty} a_j b_j$ is convergent.*

**Proof.** For each $n \in \mathbb{J}$, let $A_n = \sum_{j=0}^{n} a_j$. Then there exists a positive integer $M$ such that $|A_n| \leq M$ for all $n$. Suppose that $\varepsilon > 0$ is given. Because $\left\{b_j\right\}_{j=0}^{\infty}$ is monotonically decreasing to zero, there exists a positive integer $K$ for which $b_K < \frac{\varepsilon}{2K}$. Using summation-by-parts, for any integers $p$ and $q$ satisfying $K \leq q \leq p$, it follows that

$$
\begin{aligned}
\left|\sum_{j=p}^{q} a_j b_j\right| &= \left|\sum_{j=p}^{q-1} A_j\left(b_j - b_{j+1}\right) + A_q b_q - A_{p-1} b_p\right| \\
&\leq \left|\sum_{j=p}^{q-1} A_j\left(b_j - b_{j+1}\right)\right| + \left|A_q b_q\right| + \left|A_{p-1} b_p\right| \\
&\leq \sum_{j=p}^{q-1} \left|A_j\right|\left(b_j - b_{j+1}\right) + \left|A_q\right| b_q + \left|A_{p-1}\right| b_p \\
&\leq M\left(\sum_{j=p}^{q-1}\left(b_j - b_{j+1}\right) + b_q + b_p\right) \\
&= M\left(\left(b_p - b_q\right) + b_q + b_p\right) = 2Mb_p \\
&\leq 2Mb_K < \varepsilon
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\left\{\sum_{j=0}^{n} a_j b_j\right\}_{n=0}^{\infty}$ is a Cauchy sequence of complex numbers. Therefore, it is convergent. ■

A nice application of this result, gives us an "easy to check" criteria for convergence of series that are generated by sequences with alternating positive and negative terms.

**Theorem 4.5.40 (Alternating Series Test)** *Suppose that the sequence $\{u_j\}_{j=1}^{\infty} \subset \mathbb{R}$ satisfies the following conditions:*

*(i)* sgn $(u_j) = -$sgn $(u_{j+1})$ *for each* $j \in \mathbb{J} \cup \{0\}$, *where* sgn *denotes "the sign of";*

*(ii)* $|u_{j+1}| \leq |u_j|$ *for every* $j$*; and*

*(iii)* $\lim_{j\to\infty} u_j = 0$.

*Then* $\sum_{j=1}^{\infty} u_j$ *is convergent. Furthermore, if the sum is denoted by* $S$*, then* $S_n \leq S \leq S_{n+1}$ *for each n where* $\{S_n\}_{n=0}^{\infty}$ *is the sequence of* $nth$ *partial sums.*

The result is an immediate consequence of Theorem 4.5.39; it follows upon setting $a_j = (-1)^j$ and $b_j = |c_j|$. As an illustration of how "a regrouping argument" can get us to the conclusion, we offer the following proof for your reading pleasure.

**Proof.** Without loss in generality, we can take $u_0 > 0$. Then $u_{2k+1} < 0$ and $u_{2k} > 0$ for $k = 0, 1, 2, 3, \ldots$ Note that for each $n \in \mathbb{J} \cup \{0\}$,

$$S_{2n} = (u_0 + u_1) + (u_2 + u_3) + \cdots + (u_{2n-2} + u_{2n-1}) + u_{2n}$$

which can be regrouped as

$$S_{2n} = u_0 + (u_1 + u_2) + (u_3 + u_4) + \cdots + (u_{2n-1} + u_{2n}).$$

The first arrangement justifies that $\{S_{2n}\}_{n=0}^{\infty}$ is monotonically increasing while the second yields that $S_{2n} < u_0$ for each $n$. By Theorem 4.3.14, the sequence $\{S_{2n}\}_{n=0}^{\infty}$ is convergent. For $\lim_{n\to\infty} S_{2n} = S$, we have that $S_{2n} \leq S$ for each $n$.

Since $S_{2n-1} = S_{2n} - u_{2n}$, $S_{2n-1} > S_{2n}$ for each $n \in \mathbb{J}$. On the other hand,

$$S_{2n+1} = S_{2n-1} + (u_{2n} + u_{2n+1}) < S_{2n-1}.$$

These inequalities combined with $S_{2n} > S_2 = u_1 + u_2$, yield that the sequence $\{S_{2n-1}\}_{n=1}^{\infty}$ is a monotonically decreasing sequence that is bounded below. Again, by Theorem 4.3.14, $\{S_{2n-1}\}_{n=1}^{\infty}$ is convergent. From (iii), we deduce that $S_{2n-1} \to S$ also. We have that $S_{2n-1} \geq S$ because $\{S_{2n-1}\}_{n=1}^{\infty}$ is decreasing. Pulling this together, leads to the conclusion that $\{S_n\}$ converges to $S$ where $S \leq S_k$ for $k$ odd and $S \geq S_k$ when $k$ is even. ■

**Remark 4.5.41** *Combining the Alternating Series Test with Remark 4.5.5 leads to the quick observation that the alternating harmonic series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *is conditionally convergent.*

### 4.5.4 Discussing Convergence

When asked to discuss the convergence of a given series, there is a system that we should keep in mind. Given the series $\sum_{n=0}^{\infty} u_n$:

1. Check whether or not $\lim\limits_{n\to\infty} u_n = 0$. If not, claim divergence by the $kth$ term test; if yes, proceed to the next step.

2. Check for absolute convergence by testing $\sum_{n=0}^{\infty} |u_n|$. Since $\sum_{n=0}^{\infty} |u_n|$ is a series having nonnegative terms, we have several tests of convergence at our disposal–Comparison, Limit Comparison, Ratio, and Root–in addition to the possibility of recognizing the given series as directly related to a geometric or a $p$-series. Practice with the tests leads to a better ability to discern which test to use. If $\sum_{n=0}^{\infty} |u_n|$ converges, by any of the our tests, then we conclude that $\sum_{n=0}^{\infty} u_n$ converges absolutely and we are done. If $\sum_{n=0}^{\infty} |u_n|$ diverges by either the Ratio Test or the Root Test, then we conclude that $\sum_{n=0}^{\infty} u_n$ diverges and we are done.

3. If $\sum_{n=0}^{\infty} |u_n|$ diverges by either the Comparison Test or the Limit Comparison Test , then test $\sum_{n=1}^{\infty} u_n$ for conditional convergence–using the Alternating Series Test if it applies. If the series involves nonreal complex terms, try checking the corresponding series of real and imaginary parts.

**Excursion 4.5.42** *Discuss the Convergence of each of the following:*

*1.* $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n-1}}{n^2+1}$

*2.* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n \ln n}{e^n}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$

4. $\sum_{n=1}^{\infty} \frac{1}{1+\alpha^n}$ , $\alpha > -1$

5. $\sum_{n=1}^{\infty} \frac{\cos(n\alpha)}{n^2}$

*******The ratio test leads to the divergence of the first one. The second one is absolutely convergent by the root test. The third one diverges due to failure to pass the $kth$ term test. The behavior of the fourth one depends on $\alpha$: it diverges for $|\alpha| < 1$ and converges for $|\alpha| > 1$ from the ratio test. Finally the last one converges by comparison.*******

### 4.5.5 Rearrangements of Series

Given any series $\sum_{j=o}^{\infty} a_j$ and a function $f : \mathbb{J} \cup \{0\} \overset{1-1}{\twoheadrightarrow} \mathbb{J} \cup \{0\}$, the series $\sum_{j=o}^{\infty} a_{f(j)}$ is a **rearrangement** of the original series. Given a series $\sum_{j=o}^{\infty} a_j$ and a rearrangement $\sum_{j=o}^{\infty} a_{f(j)}$, the corresponding sequence of $nth$ partial sums may be completely different. There is no reason to expect that they would have the same limit. The commutative law that works so well for finite sums tells us nothing about what may happen with infinite series. It turns out that if the original series is absolutely convergent, then all rearrangements are convergent to the same limit. In the last section of Chapter 3 in our text, it is shown that the situation is shockingly different for conditionally convergent real series. We will state the result that is proved on pages 76-77 or our text.

**Theorem 4.5.43** *Let* $\sum_{j=0}^{\infty} a_j$ *be a real series that converges conditionally. Then for any elements in the extended real number system such that* $-\infty \le \alpha \le \beta \le +\infty$*, there exists a rearrangement of the given series* $\sum_{j=0}^{\infty} a_{f(j)}$ *such that*

$$\liminf_{n\to\infty} \sum_{j=0}^{n} a_{f(j)} = \alpha \qquad \textit{and} \qquad \limsup_{n\to\infty} \sum_{j=0}^{n} a_{f(j)} = \beta.$$

***Theorem 4.5.44*** *Let* $\sum_{j=0}^{\infty} a_j$ *be a series of complex numbers that converges absolutely. Then every rearrangement of* $\sum_{j=0}^{\infty} a_j$ *converges and each rearrangement converges to the same limit.*

## 4.6 Problem Set D

1. Use the definition to prove each of the following claims. Your arguments must be well written and make use of appropriate approaches to proof.

   (a) $\lim_{n\to\infty} \frac{n^2 + in}{n^2 + 1} = 1$

   (b) $\lim_{n\to\infty} \frac{3n^2 + i}{2n^3} = 0$

   (c) $\lim_{n\to\infty} \frac{3n + 2}{2n - 1} = \frac{3}{2}$

   (d) $\lim_{n\to\infty} \frac{3n + 1 + 2ni}{n + 3} = 3 + 2i$

   (e) $\lim_{n\to\infty} \frac{1 + 3n}{1 + in} = -3i$

2. Find the limits, if they exist, of the following sequences in $\mathbb{R}^2$. Show enough work to justify your conclusions.

   (a) $\left\{\left(\frac{(-1)^n}{n}, \frac{\cos n}{n}\right)\right\}_{n=1}^{\infty}$

   (b) $\left\{\left(\frac{3n+1}{4n-1}, \frac{2n^2+3}{n^2+2}\right)\right\}_{n=1}^{\infty}$

   (c) $\left\{\left(\frac{(-1)^n n^2+5}{2n^2}, \frac{1+3n}{1+2n}\right)\right\}_{n=1}^{\infty}$

   (d) $\left\{\left(\frac{(\sin n)^n}{n}, \frac{1}{n^2}\right)\right\}_{n=1}^{\infty}$

   (e) $\left\{\left(\frac{\cos n\pi}{n}, \frac{\sin(n\pi + (\pi/2))}{n}\right)\right\}_{n=1}^{\infty}$

3. Suppose that $\{x_n\}_{n=1}^{\infty}$ converges to $x$ in Euclidean $k$-space. Show that $A = \{x_n : n \in \mathbb{J}\} \cup \{x\}$ is closed.

4. For $j, n \in \mathbb{J}$, let $f_j(n) = \dfrac{n^2 \sin\left(\frac{\pi j}{4}\right) + 3n}{4j^2n^2 + 2jn + 1}$. Find the limit of the following sequence in $\mathbb{R}^5$, showing enough work to carefully justify your conclusions: $\{(f_1(n), f_2(n), f_3(n), f_4(n), f_5(n))\}_{n=1}^{\infty}$.

5. Find the limit superior and the limit inferior for each of the following sequences.

   (a) $\left\{n \cos \frac{n\pi}{2}\right\}_{n=1}^{\infty}$

   (b) $\left\{\dfrac{1+\cos\frac{n\pi}{2}}{(-1)^n n^2}\right\}_{n=1}^{\infty}$

   (c) $\left\{\frac{1}{2^n} + (-1)^n \cos\frac{n\pi}{4} + \sin\frac{n\pi}{2}\right\}_{n=1}^{\infty}$

   (d) $\left\{2^{(-1)^n}\left(1+\frac{1}{n^2}\right) + 3^{(-1)^{n+1}}\right\}_{n=1}^{\infty}$

6. If $\{a_n\}_{n=0}^{\infty}$ is a bounded sequence of complex numbers and $\{b_n\}_{n=0}^{\infty}$ is a sequence of complex numbers that converges to 0, prove that $\lim_{n\to\infty} a_n b_n = 0$.

7. If $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers with the property that $|a_n - a_{n+1}| \leq \frac{1}{2^n}$ for each $n \in \mathbb{J} \cup \{0\}$, prove that $\{a_n\}_{n=0}^{\infty}$ converges.

8. If $\{a_n\}_{n=0}^{\infty}$ is a monotonically increasing sequence such that $a_{n+1} - a_n \leq \frac{1}{n}$ for each $n \in \mathbb{J} \cup \{0\}$, must $\{a_n\}_{n=0}^{\infty}$ converge? Carefully justify your response.

9. Discuss the convergence of each of the following. If the given series is convergence and it is possible to find the sum, do so.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(n+2)}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n n}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{2n+3}{n^3}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^n}$

10. Prove the **Limit Comparison Test**.

    Suppose that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are such that $a_n \geq 0$, $b_n \geq 0$ for each $n \in \mathbb{J} \cup \{0\}$, and $\lim_{n\to\infty} a_n\,(b_n)^{-1} = L > 0$. Then either $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ both converge or both diverge.

    [Hint: For sufficiently large $n$, justify that $\frac{1}{2}L < \frac{a_n}{b_n} < \frac{3}{2}L$.]

11. Suppose that $a_n \geq 0$ for each $n \in \mathbb{J} \cup \{0\}$.

(a) If $\sum_{n=1}^{\infty} a_n$ converges and $b_n = \sum_{k=n}^{\infty} a_k$, prove that $\sum_{n=1}^{\infty} \left(\sqrt{b_n} - \sqrt{b_{n+1}}\right)$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges and $S_n = \sum_{k=1}^{n} a_k$, prove that $\sum_{n=1}^{\infty} \left(\sqrt{S_{n+1}} - \sqrt{S_n}\right)$ diverges.

12. For each of the following **use our tests** for convergence to check for absolute convergence and, when needed, conditional convergence.

(a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n + i3^n}{5 \cdot 4^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \sin\left(\frac{(2n-1)\pi}{2}\right)}{n^2+1}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt{2n^2+1} - \sqrt{2n^2-1}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^4}{(n+1)!}$

(e) $\displaystyle\sum_{n=2}^{\infty} (\cos(\pi n)) \left(1 + \frac{1}{n}\right)^{-n^2}$

(f) $\displaystyle\sum_{n=2}^{\infty} \frac{(1+i)^{n+3}}{3^{2n+1} \cdot 4^n}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\left(\frac{(-1)^n+1}{2}\right)\left(\frac{1+2i}{5}\right)^n + \left(\frac{(-1)^{n+1}+1}{2}\right)\left(\frac{2}{3}\right)^n\right)$

13. Justify that $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\right)^p$ is absolutely convergent for $p > 2$, conditionally convergent for $0 < p \leq 2$, and divergent for $p \leq 0$.

14. Let $\ell_2$ be the collection of infinite sequences $\{x_n\}_{n=1}^{\infty}$ of reals such that $\sum_{n=1}^{\infty} x_n^2$

converges and define $d(x, y) = \sqrt{\sum_{n=1}^{\infty} (x_n - y_n)^2}$ for each $x = \{x_n\}_{n=1}^{\infty}$, $y = \{y_n\}_{n=1}^{\infty} \in \ell_2$. Show that $(\ell_2, d)$ is a metric space.

15. A sequence $\{x_n\}_{n=1}^{\infty}$ of reals is bounded if and only if there is a number $m$ such that $|x_n| \leq m$ for each $n \in \mathbb{J}$. Let $M$ denote the collection of all bounded sequences, and defined $d(x, y) = \sup_{n \in \mathbb{J}} |x_n - y_n|$. Show that $(M, d)$ is a metric space.

16. Let $B$ be the collection of all absolutely convergent series and define $d(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|$. Show that $(B, d)$ is a metric space.