Chapter 7

Riemann-Stieltjes Integration

Calculus provides us with tools to study nicely behaved phenomena using small discrete increments for information collection. The general idea is to (intelligently) connect information obtained from examination of a phenomenon over a lot of tiny discrete increments of some related quantity to "close in on" or approximate something that behaves in a controlled (i.e., bounded, continuous, etc.) way. The "closing in on" approach is useful only if we can get back to information concerning the phenomena that was originally under study. The benefit of this approach is most beautifully illustrated with the elementary theory of integral calculus over $\mathbb{R}$. It enables us to adapt some "limiting" formulas that relate quantities of physical interest to study more realistic situations involving the quantities.

Consider three formulas that are encountered frequently in most standard physical science and physics classes at the pre-college level:

$$A = l \cdot w \qquad d = r \cdot t \qquad m = d \cdot l.$$

Use the space that is provided to indicate what you "know" about these formulas.

Our use of these formulas is limited to situations where the quantities on the right are constant. The minute that we are given a shape that is not rectangular, a velocity that varies as a function of time, or a density that is determined by our position in (or on) an object, at first, we appear to be "out of luck." However, when the quantities given are well enough behaved, we can obtain bounds on what we wish to study, by making certain assumptions and applying the known formulas incrementally.

Note that except for the units, the formulas are indistinguishable. Consequently, illustrating the “closing in on" or approximating process with any one of them carries over to the others, though the physical interpretation (of course) varies.

Let’s get this more down to earth! Suppose that you build a rocket launcher as part of a physics project. Your launcher fires rockets with an initial velocity of 25 ft/min, and, due to various forces, travels at a rate $\upsilon(t)$ given by

$$\upsilon(t) = 25 - t^2 \text{ ft/min}$$

where t is the time given in minutes. We want to know how far the rocket travels in the first three minutes after launch. The only formula that we have is $d = r \cdot t$, but to use it, we need a constant rate of speed. *We can make use of the formula to obtain bounds or estimates on the distance travelled.* To do this, we can take increments in the time from 0 minutes to 3 minutes and “pick a relevant rate” to compute a bound on the distance travelled in each section of time. For example, over the entire three minutes, the velocity of the rocket is never more that 25 ft/min.

What does this tell us about the product

$$(25 \text{ ft/min}) \cdot 3 \text{ min}$$

compared to the distance that we seek?

How does the product $(16 \text{ ft/min}) \cdot (3 \text{ min})$ *relate to the distance that we seek?*

We can improve the estimates by taking smaller increments (subintervals of 0 minutes to 3 minutes) and choosing a different “estimating velocity” on each subinterval. For example, using increments of 1.5 minutes and the maximum velocity that is achieved in each subinterval as the estimate for a constant rate through each subinterval, yields an estimate of

$$(25 \text{ ft/min}) \cdot (1.5 \text{ min}) + \left(\left(25 - \frac{9}{4}\right) \text{ ft/min}\right) \cdot (1.5 \text{ min}) = \frac{573}{8} \text{ ft.}$$

Excursion 7.0.1 *Find the estimate for the distance travelled taking increments of one minute (which is not small for the purposes of calculus) and using the minimum velocity achieved in each subinterval as the "estimating velocity."*

*******Hopefully, you obtained 61 feet.*******

Notice that none of the work done actually gave us the answer to the original problem. Using Calculus, we can develop the appropriate tools to solve the problem as an appropriate limit. This motivates the development of the very important and useful theory of integration. We start with some formal definitions that enable us to carry the "closing in on process" to its logical conclusion.

7.1 Riemann Sums and Integrability

Definition 7.1.1 *Given a closed interval $I = [a, b]$, a* ***partition*** *of I is any finite strictly increasing sequence of points $\mathcal{P} = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ such that $a = x_0$ and $b = x_n$. The* ***mesh of the partition*** *$\{x_0, x_1, \dots, x_{n-1}, x_n\}$ is defined by*

$$\operatorname{mesh} \mathcal{P} = \max_{1 \le j \le n} \left(x_j - x_{j-1}\right).$$

Each partition of I, $\{x_0, x_1, \dots, x_{n-1}, x_n\}$, decomposes I into n subintervals $I_j = \left[x_{j-1}, x_j\right]$, $j = 1, 2, ..., n$, such that $I_j \cap I_k = x_j$ if and only if $k = j + 1$ and is empty for $k \neq j$ or $k \neq (j + 1)$. Each such decomposition of I into subintervals is called a ***subdivision of*** **I**.

Notation 7.1.2 *Given a partition $\mathcal{P} = \{x_0, x_1, \dots, x_{n-1}, x_n\}$ of an interval $I = [a, b]$, the two notations Δx_j and $\ell\left(I_j\right)$ will be used for $\left(x_j - x_{j-1}\right)$, the length of the j^{th} subinterval in the partition. The symbol Δ or $\Delta\left(I\right)$ will be used to denote an arbitrary subdivision of an interval I.*

If f is a function whose domain contains the closed interval I and f is bounded on the interval I, we know that f has both a least upper bound and a greatest lower bound on I as well as on each interval of any subdivision of I.

Definition 7.1.3 *Given a function f that is bounded and defined on the interval I and a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ of I, let $I_j = \left[x_{j-1}, x_j\right]$, $M_j = \sup_{x\in I_j} f(x)$ and $m_j = \inf_{x\in I_j} f(x)$ for $j = 1, 2, ..., n$. Then the* ***upper Riemann sum of f with respect to the partition*** *$\mathcal{P}$, denoted by $U(\mathcal{P}, f)$, is defined by*

$$U(\mathcal{P}, f) = \sum_{j=1}^{n} M_j \Delta x_j$$

and the ***lower Riemann sum of f with respect to the partition*** *$\mathcal{P}$, denoted by $L(\mathcal{P}, f)$, is defined by*

$$L(\mathcal{P}, f) = \sum_{j=1}^{n} m_j \Delta x_j$$

where $\Delta x_j = \left(x_j - x_{j-1}\right)$.

Notation 7.1.4 *With the subdivision notation the upper and lower Riemann sums for f are denoted by $U(\Delta, f)$ and $L(\Delta, f)$, respectively.*

Example 7.1.5 *For $f(x) = 2x + 1$ in $I = [0, 1]$ and $\mathcal{P} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}$, $U(\mathcal{P}, f) = \frac{1}{4}\left(\frac{3}{2} + 2 + \frac{5}{2} + 3\right) = \frac{9}{4}$ and $L(\mathcal{P}, f) = \frac{1}{4}\left(1 + \frac{3}{2} + 2 + \frac{5}{2}\right) = \frac{7}{4}$.*

Example 7.1.6 *For* $g(x) = \begin{cases} 0 & , \textit{for} \quad x \in \mathbb{Q} \cap [0, 2] \\ 1 & , \textit{for} \quad x \notin \mathbb{Q} \cap [0, 2] \end{cases}$

$U(\Delta(I), g) = 2$ and $L(\Delta(I), g) = 0$ for any subdivision of $[0, 2]$.

To build on the motivation that constructed some Riemann sums to estimate a distance travelled, we want to introduce the idea of refining or adding points to partitions in an attempt to obtain better estimates.

Definition 7.1.7 *For a partition* $\mathcal{P}_k = \{x_0, x_1, ..., x_{k-1}, x_k\}$ *of an interval* $I = [a, b]$*, let* Δ_k *denote to corresponding subdivision of* $[a, b]$*. If* $\mathcal{P}_n$ *and* $\mathcal{P}_m$ *are partitions of* $[a, b]$ *having* $n + 1$ *and* $m + 1$ *points, respectively, and* $\mathcal{P}_n \subset \mathcal{P}_m$*, then* $\mathcal{P}_m$ *is a* ***refinement*** *of* $\mathcal{P}_n$ *or* Δ_m *is a* ***refinement*** *of* Δ_n*. If the partitions* $\mathcal{P}_n$ *and* $\mathcal{P}_m$ *are independently chosen, then the partition* $\mathcal{P}_n \cup \mathcal{P}_m$ *is a* ***common refinement of*** $\mathcal{P}_n$ ***and*** $\mathcal{P}_m$ *and the resulting* $\Delta(\mathcal{P}_n \cup \mathcal{P}_m)$ *is called a* ***common refinement of*** Δ_n ***and*** Δ_m.

Excursion 7.1.8 *Let* $\mathcal{P} = \left\{0, \frac{1}{2}, \frac{3}{4}, 1\right\}$ *and* $\mathcal{P}^* = \left\{0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, 1\right\}$.

(a) *If* Δ *and* Δ^* *are the subdivisions of* $I = [0, 1]$ *that correspond* $\mathcal{P}$ *and* $\mathcal{P}^*$*, respectively, then* $\Delta = \left\{\left[0, \frac{1}{2}\right], \left[\frac{1}{2}, \frac{3}{4}\right], \left[\frac{3}{4}, 1\right]\right\}$*. Find* Δ^*.

(b) *Set* $I_1 = \left[0, \frac{1}{2}\right]$, $I_2 = \left[\frac{1}{2}, \frac{3}{4}\right]$*, and* $I_3 = \left[\frac{3}{4}, 1\right]$*. For* $k = 1, 2, 3$*, let* $\Delta(k)$ *be the subdivision of* I_k *that consists of all the elements of* Δ^* *that are contained in* I_k*. Find* $\Delta(k)$ *for* $k = 1, 2,$ *and* 3.

(c) *For* $f(x) = x^2$ *and the notation established in parts (a) and (b), find each of the following.*

(i) $m = \inf_{x \in I} f(x)$

(ii) $m_j = \inf_{x \in I_j} f(x)$ *for* $j = 1, 2, 3$

(iii) $m_j^* = \inf \left\{ \inf_{x \in J} f(x) : J \in \Delta(j) \right\}$

(iv) $M = \sup_{x \in I} f(x)$

(v) $M_j = \sup_{x \in I_j} f(x)$ *for* $j = 1, 2, 3$

(vi) $M_j^* = \sup \left\{ \sup_{x \in J} f(x) : J \in \Delta(j) \right\}$

(d) Note how the values m, m_j, m_j^*, M, M_j, *and* M_j^* *compare. What you observed is a special case of the general situation. Let*

$$\mathcal{P} = \{x_0 = a, x_1, \ldots, x_{n-1}, x_n = b\}$$

be a partition of an interval $I = [a, b]$, Δ *be the corresponding subdivision of* $[a, b]$ *and* $\mathcal{P}^*$ *denote a refinement of* $\mathcal{P}$ *with corresponding subdivision denoted by* Δ^*. *For* $k = 1, 2, \ldots, n$, *let* $\Delta(k)$ *be the subdivision of* I_k *consisting of the elements of* Δ^* *that are contained in* I_k. *Justify each of the following claims for any function that is defined and bounded on* I.

(i) If $m = \inf_{x \in I} f(x)$ *and* $m_j = \inf_{x \in I_j} f(x)$, *then, for* $j = 1, 2, \ldots, n$, $m \leq m_j$ *and* $m_j \leq \inf_{x \in J} f(x)$ *for* $J \in \Delta(j)$.

(ii) If $M = \sup_{x \in I} f(x)$ *and* $M_j = \sup_{x \in I_j} f(x)$*, then, for* $j = 1, 2, ..., n$*,* $M_j \leq M$ *and* $M_j \geq \sup_{x \in J} f(x)$ *for* $J \in \Delta(j)$*.*

Our next result relates the Riemann sums taken over various subdivisions of an interval.

Lemma 7.1.9 *Suppose that* f *is a bounded function with domain* $I = [a, b]$*. Let* Δ *be a subdivision of* I*,* $M = \sup_{x \in I} f(x)$*, and* $m = \inf_{x \in I} f(x)$*. Then*

$$m(b-a) \leq L(\Delta, f) \leq U(\Delta, f) \leq M(b-a) \tag{7.1}$$

and

$$L(\Delta, f) \leq L\left(\Delta^*, f\right) \leq U\left(\Delta^*, f\right) \leq U(\Delta, f) \tag{7.2}$$

for any refinement Δ^* *of* Δ*. Furthermore, if* Δ_γ *and* Δ_λ *are any two subdivisions of* I*, then*

$$L\left(\Delta_\gamma, f\right) \leq U(\Delta_\lambda, f) \tag{7.3}$$

Excursion 7.1.10 *Fill in what is missing to complete the following proofs.*

Proof. Suppose that f is a bounded function with domain $I = [a, b]$, $M = \sup_{x \in I} f(x)$, and $m = \inf_{x \in I} f(x)$. For $\Delta = \{I_k : k = 1, 2, ..., n\}$ an arbitrary subdivision of I, let $M_j = \sup_{x \in I_j} f(x)$ and $m_j = \inf_{x \in I_j} f(x)$. Then $I_j \subset I$ for each $j = 1, 2, ..., n$, we have that

$$m \leq m_j \leq \underbrace{\qquad\qquad\qquad\qquad}_{(1)}, \text{ for each } j = 1, 2, ..., n.$$

Because $\Delta x_j = \left(x_j - x_{j-1}\right) \geq 0$ for each $j = 1, 2, ..., n$, it follows immediately that

$$\underbrace{\qquad\qquad\qquad}_{(2)} = m \sum_{j=1}^{n} \left(x_j - x_{j-1}\right) \leq \sum_{j=1}^{n} m_j \Delta x_j = L(\Delta, f)$$

and

$$\sum_{j=1}^{n} m_j \Delta x_j \leq \sum_{j=1}^{n} M_j \Delta x_j = U(\Delta, f) \leq \underset{(3)}{\underline{\hspace{3cm}}} = M(b-a).$$

Therefore, $m(b-a) \leq L(\Delta, f) \leq U(\Delta, f) \leq M(b-a)$ as claimed in equation (7.1).

Let Δ^* be a refinement of Δ and, for each $k = 1, 2, ..., n$, let $\Delta(k)$ be the subdivision of I_k that consists of all the elements of Δ^* that are contained in I_k. In view of the established conventions for the notation being used, we know that $(\forall J)(J \in \Delta^* \Rightarrow (\exists! k)(k \in \{1, 2, ..., n\} \wedge J \in \Delta(k)))$; also, for each $J \in \Delta(k)$, $J \subset I_k \Rightarrow m_k = \inf_{x \in I_k} f(x) \leq \inf_{x \in J} f(x)$ and $M_k = \sup_{x \in I_k} f(x) \geq \sup_{x \in J} f(x)$. Thus,

$$m_k \ell(I_k) \leq L(\Delta(k), f) \qquad and \qquad M_k \ell(I_k) \geq U(\Delta(k), f)$$

from which it follows that

$$L(\Delta, f) = \sum_{j=1}^{n} m_j \ell\left(I_j\right) \leq \sum_{j=1}^{n} L(\Delta(j), f) = L\left(\Delta^*, f\right)$$

and

$$U(\Delta, f) = \underset{(4)}{\underline{\hspace{3cm}}} \geq \sum_{j=1}^{n} U(\Delta(j), f) = \underset{(5)}{\underline{\hspace{3cm}}}.$$

From equation (7.1), $L(\Delta^*, f) \leq U(\Delta^*, f)$. Finally, combining the inequalities yields that

$$L(\Delta, f) \leq L\left(\Delta^*, f\right) \leq U\left(\Delta^*, f\right) \leq U(\Delta, f)$$

which completes the proof of equation (7.2).

Suppose that Δ_γ and Δ_λ are two subdivisions of I. Then $\Delta = \Delta_\gamma \cup \Delta_\lambda$ is $\underset{(6)}{\underline{\hspace{6cm}}}$ Δ_γ and Δ_λ. Because Δ is a refinement of Δ_γ, by the comparison of lower sums given in equation (7.2), $L\left(\Delta_\gamma, f\right) \leq L(\Delta, f)$. On the other hand, from Δ being a refinement of Δ_λ, it follows that $\underset{(7)}{\underline{\hspace{4.5cm}}}$.

Combining the inequalities with equation (7.1) leads to equation (7.3). ■

Acceptable responses are: (1) $M_j \leq M$, (2) $m\,(b-a)$, (3) $M \sum_{j=1}^{n} \left(x_j - x_{j-1}\right)$, (4) $\sum_{j=1}^{n} M_j \ell \left(I_j\right)$, (5) $U\left(\Delta^*,\, f\right)$, (6) the common refinement of, and (7) $U\left(\Delta,\, f\right) \leq U\left(f,\, \Delta_\lambda\right)$.

If f is a bounded function with domain $I = [a, b]$ and $\wp = \wp\,[a, b]$ is the set of all partitions of $[a, b]$, then the Lemma assures us that $\{L\left(\Delta,\, f\right) : \Delta \in \wp\}$ is bounded above by $(b-a) \sup\limits_{x \in I} f(x)$ and $\{U\left(\Delta,\, f\right) : \Delta \in \wp\}$ is bounded below by $(b-a) \inf\limits_{x \in I} f(x)$. Hence, by the least upper bound and greatest lower bound properties of the reals both $\sup\{L\left(\Delta,\, f\right) : \Delta \in \wp\}$ and $\inf\{U\left(\Delta,\, f\right) : \Delta \in \wp\}$ exist; to see that they need not be equal, note that–for the bounded function g given in Example 7.1–we have that $\sup\{L\left(\Delta, g\right) : \Delta \in \wp\} = 0$ while $\inf\{U\left(\Delta, g\right) : \Delta \in \wp\} = 2$.

Definition 7.1.11 *Suppose that f is a function on $\mathbb{R}$ that is defined and bounded on the interval $I = [a, b]$ and $\wp = \wp\,[a, b]$ is the set of all partitions of $[a, b]$. Then the* ***upper Riemann integral*** *and the* ***lower Riemann integral*** *are defined by*

$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P} \in \wp} U\left(\mathcal{P},\, f\right) \qquad \textit{and} \qquad \underline{\int_a^b} f(x)\,dx = \sup_{\mathcal{P} \in \wp} L\left(\mathcal{P},\, f\right),$$

respectively. If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then f is ***Riemann integrable****, or just* ***integrable****, on I, and the common value of the integral is denoted by $\displaystyle\int_a^b f(x)\,dx$.*

Excursion 7.1.12 *Let* $f(x) = \begin{cases} 5x + 3 & , \textit{for} \quad x \notin \mathbb{Q} \\ \\ 0 & , \textit{for} \quad x \in \mathbb{Q} \end{cases}$.

For each $n \in \mathbb{J}$, let Δ_n denote the subdivision of the interval $[1, 2]$ that consists of n segments of equal length. Use $\{\Delta_n : n \in \mathbb{J}\}$ to find an upper bound for

$\overline{\int_1^2} f(x)\,dx$. *[Hint: Recall that* $\sum_{k=1}^n k = \dfrac{n(n+1)}{2}$.*]*

*******Corresponding to each Δ_n you needed to find a useful form for $U(\Delta_n, f)$. Your work should have led you to a sequence for which the limit exists as $n \to \infty$. For $n \in \mathbb{J}$, the partition that gives the desired Δ_n is $\left\{1, 1+\dfrac{1}{n}, 1+\dfrac{2}{n}, ..., 1+\dfrac{n}{n}\right\}$. Then $\Delta_n = \{I_1, I_2, ..., I_n\}$ with $I_j = \left[1+\dfrac{j-1}{n}, 1+\dfrac{j}{n}\right]$ and $M_j = 8 + \dfrac{5j}{n}$ leads to $U(\Delta_n, f) = \dfrac{21}{2} + \dfrac{5}{2n}$. Therefore, you should proved that $\overline{\int_1^2} f(x)\,dx \le \dfrac{21}{2}$.*******

It is a rather short jump from Lemma 7.1.9 to upper and lower bounds on the Riemann integrals. They are given by the next theorem.

Theorem 7.1.13 *Suppose that f is defined on the interval $I = [a, b]$ and $m \le f(x) \le M$ for all $x \in I$. Then*

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a). \tag{7.4}$$

Furthermore, if f is Riemann integrable on I, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a). \tag{7.5}$$

Proof. Since equation (7.5), is an immediate consequence of the definition of the Riemann integral, we will prove only equation (7.4). Let $\mathcal{D}$ denote the set of all subdivisions of the interval $[a, b]$. By Lemma 7.1.9, we have that, for $\Delta^*, \Delta \in \mathcal{D}$,

$$m(b-a) \le L\left(f, \Delta^*\right) \le U(f, \Delta) \le M(b-a).$$

Since Δ is arbitrary,

$$m(b-a) \le L\left(f, \Delta^*\right) \le \inf_{\Delta \in \mathcal{D}} U(f, \Delta)$$

and $\inf\limits_{\Delta \in \mathcal{D}} U(f, \Delta) \le M(b-a)$; i.e.,

$$m(b-a) \le L\left(f, \Delta^*\right) \le \overline{\int_a^b} f(x)\,dx \le M(b-a).$$

Because Δ^* is also arbitrary, $m(b-a) \le \sup\limits_{\Delta^* \in \mathcal{D}} L(f, \Delta^*)$ and $\sup\limits_{\Delta^* \in \mathcal{D}} L(f, \Delta^*) \le \overline{\int_a^b} f(x)\,dx$; i.e.,

$$m(b-a) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

Combining the inequalities leads to equation (7.4). ■

Before getting into some of the general properties of upper and lower integrals, we are going to make a slight transfer to a more general set-up. A re-examination of the proof of Lemma 7.1.9 reveals that it relied only upon independent application of properties of infimums and supremums in conjunction with the fact that, for any partition $\{x_0, x_1, \ldots, x_{n-1}, x_n\}$, $x_j - x_{j-1} > 0$ and $\sum\limits_{j=1}^{n} \left(x_j - x_{j-1}\right) = x_n - x_0$. Now, given any function α that is defined and strictly increasing on an interval $[a, b]$, for any partition $\mathcal{P} = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ of $[a, b]$,

$$\alpha(\mathcal{P}) = \{\alpha(a) = \alpha(x_0), \alpha(x_1), \ldots, \alpha(x_{n-1}), \alpha(x_n) = \alpha(b)\} \subset \alpha([a, b]),$$

$\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right) > 0$ and $\sum\limits_{j=1}^{n} \left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right) = \alpha(b) - \alpha(a)$. Consequently, $\alpha(\mathcal{P})$ is a partition of

$$[\alpha(a), \alpha(b)] = \bigcap \{I : I = [c, d] \wedge \alpha(\mathcal{P}) \subset I\},$$

which is the "smallest" interval that contains $\alpha([a, b])$. The case $\alpha(t) = t$ returns us to the set-up for Riemann sums; on the other hand, $\alpha([a, b])$ need not be an interval because α need not be continuous.

Example 7.1.14 *Let* $I = [0,3]$ *and* $\alpha(t) = t^2 + \lfloor t \rfloor$. *Then* $\alpha(I) = [0,1) \cup [2,5) \cup [6,11) \cup \{12\}$. *For the partition* $\mathcal{P} = \left\{0, \frac{1}{2}, 1, \frac{5}{4}, 2, \frac{8}{3}, 3\right\}$ *of* I, $\alpha(\mathcal{P}) = \left\{0, \frac{1}{4}, 2, \frac{41}{16}, 6, \frac{82}{9}, 12\right\}$ *is a partition of* $[0,12]$ *which contains* $\alpha(I)$.

Definition 7.1.15 *Given a function* f *that is bounded and defined on the closed interval* $I = [a,b]$, *a function* α *that is defined and monotonically increasing on* I, *and a partition* $\mathcal{P} = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ *of* I *with corresponding subdivision* Δ, *let* $M_j = \sup_{x\in I_j} f(x)$ *and* $m_j = \inf_{x \in I_j} f(x)$, *for* $I_j = \left[x_{j-1}, x_j\right]$. *Then the* ***upper Riemann-Stieltjes sum of*** f ***over*** α ***with respect to the partition*** $\mathcal{P}$, *denoted by* $U(\mathcal{P}, f, \alpha)$ *or* $U(\Delta, f, \alpha)$, *is defined by*

$$U(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} M_j \Delta\alpha_j$$

and the ***lower Riemann-Stieltjes sum of*** f ***over*** α ***with respect to the partition*** $\mathcal{P}$, *denoted by* $L(\mathcal{P}, f, \alpha)$ *or* $L(\Delta, f, \alpha)$, *is defined by*

$$L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} m_j \Delta\alpha_j$$

where $\Delta\alpha_j = \left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right)$.

Replacing x_j with $\alpha\left(x_j\right)$ in the proof of Lemma 7.1.9 and Theorem 7.1.13 yields the analogous results for Riemann-Stieltjes sums.

Lemma 7.1.16 *Suppose that* f *is a bounded function with domain* $I = [a,b]$ *and* α *is a function that is defined and monotonically increasing on* I. *Let* $\mathcal{P}$ *be a partition of* I, $M = \sup_{x\in I} f(x)$, *and* $m = \inf_{x\in I} f(x)$. *Then*

$$m(\alpha(b) - \alpha(a)) \le L(\mathcal{P}, f, \alpha) \le U(\mathcal{P}, f, \alpha) \le M(\alpha(b) - \alpha(a)) \tag{7.6}$$

and

$$L(\mathcal{P}, f, \alpha) \le L\left(\mathcal{P}^*, f, \alpha\right) \le U\left(\mathcal{P}^*, f, \alpha\right) \le U(\mathcal{P}, f, \alpha) \tag{7.7}$$

for any refinement $\mathcal{P}^*$ *of* $\mathcal{P}$. *Furthermore, if* Δ_γ *and* Δ_λ *are any two subdivisions of* I, *then*

$$L\left(\Delta_\gamma, f, \alpha\right) \le U(\Delta_\lambda, f, \alpha) \tag{7.8}$$

The bounds given by Lemma 7.1.16 with the greatest lower and least upper bound properties of the reals the following definition.

Definition 7.1.17 *Suppose that f is a function on $\mathbb{R}$ that is defined and bounded on the interval $I = [a, b]$, $\wp = \wp[a, b]$ is the set of all partitions of $[a, b]$, and α is a function that is defined and monotonically increasing on I. Then the **upper Riemann-Stieltjes integral** and the **lower Riemann-Stieltjes integral** are defined by*

$$\overline{\int_a^b} f(x)\, d\alpha(x) = \inf_{\mathcal{P} \in \wp} U(\mathcal{P}, f, \alpha) \quad \text{and} \quad \underline{\int_a^b} f(x)\, d\alpha(x) = \sup_{\mathcal{P} \in \wp} L(\mathcal{P}, f, \alpha),$$

*respectively. If $\overline{\int_a^b} f(x)\, d\alpha(x) = \underline{\int_a^b} f(x)\, d\alpha(x)$, then f is **Riemann-Stieltjes integrable**, or **integrable with respect to α in the Riemann sense**, on I, and the common value of the integral is denoted by $\displaystyle\int_a^b f(x)\, d\alpha(x)$ or $\displaystyle\int_a^b f\, d\alpha$.*

Definition 7.1.18 *Suppose that α is a function that is defined and monotonically increasing on the interval $I = [a, b]$. Then the set of all functions that are integrable with respect to α in the Riemann sense is denoted by $\mathfrak{R}(\alpha)$.*

Because the proof is essentially the same as what was done for the Riemann upper and lower integrals, we offer the following theorem without proof.

Theorem 7.1.19 *Suppose that f is a bounded function with domain $I = [a, b]$, α is a function that is defined and monotonically increasing on I, and $m \le f(x) \le M$ for all $x \in I$. Then*

$$m(\alpha(b) - \alpha(a)) \le \underline{\int_a^b} f\, d\alpha \le \overline{\int_a^b} f\, d\alpha \le M(\alpha(b) - \alpha(a)). \tag{7.9}$$

Furthermore, if f is Riemann-Stieltjes integrable on I, then

$$m(\alpha(b) - \alpha(a)) \le \int_a^b f(x)\, d\alpha(x) \le M(\alpha(b) - \alpha(a)). \tag{7.10}$$

In elementary Calculus, we restricted our study to Riemann integrals of continuous functions. Even there we either glossed over the stringent requirement of needing to check all possible partitions or limited ourselves to functions where some trick could be used. Depending on how rigorous your course was, some examples of finding the integral from the definition might have been based on taking partitions of equal length and using some summation formulas (like was done in Excursion 7.1.12) or might have made use of a special bounding lemma that applied to x^n for each $n \in \mathbb{J}$.

It is not worth our while to grind out some tedious processes in order to show that special functions are integrable. Integrability will only be a useful concept if it is verifiable with a reasonable amount of effort. Towards this end, we want to seek some properties of functions that would guarantee integrability.

Theorem 7.1.20 (Integrability Criterion) *Suppose that f is a function that is bounded on an interval $I = [a, b]$ and α is monotonically increasing on I. Then $f \in \Re(\alpha)$ on I if and only if for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of I such that*

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon. \tag{7.11}$$

Excursion 7.1.21 *Fill in what is missing to complete the following proof.*

Proof. Let f be a function that is bounded on an interval $I = [a, b]$ and α be monotonically increasing on I.

Suppose that for every $\epsilon > 0$ there exists a partition $\mathcal{P}$ of I such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon. \tag{*}$$

From the definition of the Riemann-Stieltjes integral and Lemma 7.1.16, we have that

$$L(\mathcal{P}, f, \alpha) \leq \underline{\int_a^b} f(x)\, d\alpha(x) \leq \underbrace{\qquad\qquad\qquad}_{(1)} \leq \underbrace{\qquad\qquad\qquad}_{(2)}.$$

It follows immediately from (*) that

$$0 \leq \overline{\int_a^b} f(x)\, d\alpha(x) - \underline{\int_a^b} f(x)\, d\alpha(x) < \varepsilon.$$

Since ε was arbitrary and the upper and lower Riemann Stieltjes integrals are constants, we conclude that $\overline{\int_a^b} f(x)\, d\alpha(x) = \underline{\int_a^b} f(x)\, d\alpha(x)$; i.e., $\underbrace{\qquad\qquad\qquad\qquad}_{(3)}$.

Conversely, suppose that $f \in \Re(\alpha)$ and let $\varepsilon > 0$ be given. For $\wp = \wp[a,b]$ the set of all partitions of $[a,b]$, $\int_a^b f(x)\,d\alpha(x) = \inf_{\mathcal{P}\in\wp} U(\mathcal{P}, f, \alpha)$ and $\int_a^b f(x)\,d\alpha(x) = \sup_{\mathcal{P}\in\wp} L(\mathcal{P}, f, \alpha)$. Thus, $\frac{\varepsilon}{2} > 0$ implies that there exists a $\mathcal{P}_1 \in \wp[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) < U(\mathcal{P}_1, f, \alpha) < \int_a^b f(x)\,d\alpha(x) + \frac{\varepsilon}{2}$ and there exists $\mathcal{P}_2 \in \wp[a,b]$ such that $\int_a^b f(x)\,d\alpha(x) - \frac{\varepsilon}{2}$ $\underset{(4)}{\underline{\hspace{8cm}}}$.

Therefore,

$$U(\mathcal{P}_1, f, \alpha) - \int_a^b f(x)\,d\alpha(x) < \frac{\varepsilon}{2} \text{ and } \int_a^b f(x)\,d\alpha(x) - L(\mathcal{P}_2, f, \alpha) < \frac{\varepsilon}{2}. \tag{**}$$

Let $\mathcal{P}$ be the common refinement of $\mathcal{P}_1$ and $\mathcal{P}_2$. Lemma 7.1.16, equation (7.7) applied to (**) yields that

$$\underset{(5)}{\underline{\hspace{3cm}}} - \int_a^b f(x)\,d\alpha(x) < \frac{\varepsilon}{2} \text{ and } \int_a^b f(x)\,d\alpha(x) - \underset{(6)}{\underline{\hspace{3cm}}} < \frac{\varepsilon}{2}.$$

Thus

$$\begin{aligned}&(U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)) \\ &\qquad = \left(U(\mathcal{P}, f, \alpha) - \int_a^b f(x)\,d\alpha(x)\right) + \left(\underset{(7)}{\underline{\hspace{8cm}}}\right) < \varepsilon.\end{aligned}$$

■

Acceptable responses are: (1) $\overline{\int_a^b} f(x)\,d\alpha(x)$, (2) $U(\mathcal{P}, f, \alpha)$, (3) $f \in \Re(\alpha)$, (4) $< L(\mathcal{P}_2, f, \alpha) < \int_a^b f(x)\,d\alpha(x)$, (5) $U(\mathcal{P}, f, \alpha)$, (6) $L(\mathcal{P}, f, \alpha)$, and (7) $\int_a^b f(x)\,d\alpha(x) - L(\mathcal{P}, f, \alpha)$.

Theorem 7.1.20 will be useful to us whenever we have a way of closing the gap between functional values on the same intervals. The corollaries give us two "big" classes of integrable functions.

Corollary 7.1.22 *If f is a function that is continuous on the interval $I = [a,b]$, then f is Riemann-Stieltjes integrable on $[a,b]$.*

Proof. Let α be monotonically increasing on I and f be continuous on I. Suppose that $\varepsilon > 0$ is given. Then there exists an $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \varepsilon$. By the Uniform Continuity Theorem, f is uniformly continuous in $[a, b]$ from which it follows that there exists a $\delta > 0$ such that

$$(\forall u)(\forall v)\left[u, v \in I \wedge |u - v| < \delta \Rightarrow |f(u) - f(v)| < \varepsilon\right].$$

Let $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ for which mesh $\mathcal{P} < \delta$ and, for each j, $j = 1, 2, ..., n$, set $M_j = \sup\limits_{x_{j-1} \le x \le x_j} f(x)$ and $m_j = \inf\limits_{x_{j-1} \le x \le x_j} f(x)$. Then $M_j - m_j \le \eta$ and

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n}\left(M_j - m_j\right)\Delta\alpha_j \le \eta\sum_{j=1}^{n}\Delta\alpha_j = \eta[\alpha(b) - \alpha(a)] < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$(\forall\varepsilon)(\varepsilon > 0 \Rightarrow (\exists\mathcal{P})(\mathcal{P} \in \wp[a, b] \wedge U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon)).$$

In view of the Integrability Criterion, $f \in \Re(\alpha)$. Because α was arbitrary, we conclude that f is Riemann-Stieltjes Integrable (with respect to any monotonically increasing function on $[a, b]$). ■

Corollary 7.1.23 *If f is a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I, then $f \in \Re(\alpha)$.*

Proof. Suppose that f is a function that is monotonic on the interval $I = [a, b]$ and α is continuous and monotonically increasing on I. For $\varepsilon > 0$ given, let $n \in \mathbb{J}$, be such that

$$(\alpha(b) - \alpha(a))|f(b) - f(a)| < n\varepsilon.$$

Because α is continuous and monotonically increasing, we can choose a partition $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ of $[a, b]$ such that $\Delta\alpha_j = \left(\alpha\left(x_j\right) - \alpha\left(x_{j-1}\right)\right) = \dfrac{\alpha(b) - \alpha(a)}{n}$. If f is monotonically increasing in I, then, for each $j \in \{1, 2, ..., n\}$,

$M_j = \sup_{x_{j-1} \leq x \leq x_j} f(x) = f(x_j)$ and $m_j = \inf_{x_{j-1} \leq x \leq x_j} f(x) = f(x_{j-1})$ and

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \sum_{j=1}^{n} (M_j - m_j) \Delta\alpha_j \\ &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{j=1}^{n} \left(f(x_j) - f(x_{j-1}) \right) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \left((f(b) - f(a)) \right) < \varepsilon; \end{aligned}$$

while f monotonically decreasing yields that $M_j = f(x_{j-1})$, $m_j = f(x_j)$ and

$$\begin{aligned} U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{j=1}^{n} \left(f(x_{j-1}) - f(x_j) \right) \\ &= \frac{\alpha(b) - \alpha(a)}{n} \left((f(a) - f(b)) \right) < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have that

$$(\forall \varepsilon)\,(\varepsilon > 0 \Rightarrow (\exists \mathcal{P})\,(\mathcal{P} \in \wp[a,b] \wedge U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \varepsilon)).$$

In view of the Integrability Criterion, $f \in \Re(\alpha)$. ■

Corollary 7.1.24 *Suppose that f is bounded on $[a,b]$, f has only finitely many points of discontinuity in $I = [a,b]$, and that the monotonically increasing function α is continuous at each point of discontinuity of f. Then $f \in \Re(\alpha)$.*

Proof. Let $\varepsilon > 0$ be given. Suppose that f is bounded on $[a,b]$ and continuous on $[a,b] - E$ where $E = \{\zeta_1, \zeta_2, ..., \zeta_k\}$ is the nonempty finite set of points of discontinuity of f in $[a,b]$. Suppose further that α is a monotonically increasing function on $[a,b]$ that is continuous at each element of E. Because E is finite and α is continuous at each $\zeta_j \in E$, we can find k pairwise disjoint intervals $[u_j, v_j]$, $j = 1, 2, ..., k$, such that

$$E \subset \bigcup_{j=1}^{k} [u_j, v_j] \subsetneq [a,b] \qquad \textit{and} \qquad \sum_{j=1}^{k} (\alpha(v_j) - \alpha(u_j)) < \varepsilon^*$$

for any $\varepsilon^* > 0$; furthermore, the intervals can be chosen in such a way that each point $\zeta_m \in E \cap (a, b)$ is an element of the interior of the corresponding interval, $[u_m, v_m]$. Let

$$K = [a, b] - \bigcup_{j=1}^{k} \left(u_j, v_j\right).$$

Then K is compact and f continuous on K implies that f is uniformly continuous there. Thus, corresponding to $\varepsilon^* > 0$, there exists a $\delta > 0$ such that

$$(\forall s)\,(\forall t)\left(s, t \in K \wedge |s - t| < \delta \Rightarrow |f(s) - f(t)| < \varepsilon^*\right).$$

Now, let $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be a partition of $[a, b]$ satisfying the following conditions:

- $(\forall j)\left(j \in \{1, 2, ..., k\} \Rightarrow u_j \in \mathcal{P} \wedge v_j \in \mathcal{P}\right)$,
- $(\forall j)\left(j \in \{1, 2, ..., k\} \Rightarrow \left(u_j, v_j\right) \cap \mathcal{P} = \emptyset\right)$, and
- $(\forall p)\,(\forall j)\left[\left(p \in \{1, 2, ..., n\} \wedge j \in \{1, 2, ..., k\} \wedge x_{p-1} \neq u_j\right) \Rightarrow \Delta x_p < \delta\right]$.

Note that under the conditions established, $x_{q-1} = u_j$ implies that $x_q = v_j$. If $M = \sup_{x \in I} |f(x)|$, $M_p = \sup_{x_{p-1} \leq x \leq x_p} f(x)$ and $m_p = \inf_{x_{p-1} \leq x \leq x_p} f(x)$, then for each p, $M_p - m_p \leq 2M$. Furthermore, $M_p - m_p < \varepsilon^*$ as long as $x_{p-1} \neq u_j$. Using commutativity to regroup the summation according to the available bounds yields that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) = \sum_{j=1}^{n} \left(M_j - m_j\right) \Delta\alpha_j \leq [\alpha(b) - \alpha(a)]\,\varepsilon^* + 2M\varepsilon^* < \varepsilon$$

whenever $\varepsilon^* < \dfrac{\varepsilon}{2M + [\alpha(b) - \alpha(a)]}$. Since $\varepsilon > 0$ was arbitrary, from the Integrability Criterion we conclude that $f \in \Re(\alpha)$. ■

Remark 7.1.25 *The three Corollaries correspond to Theorems 6.8, 6.9, and 6.10 in our text.*

As a fairly immediate consequence of Lemma 7.1.16 and the Integrability Criterion we have the following Theorem which is Theorem 6.7 in our text.

Theorem 7.1.26 *Suppose that f is bounded on $[a, b]$ and α is monotonically increasing on $[a, b]$.*

(a) *If there exists an $\varepsilon > 0$ and a partition $\mathcal{P}^*$ of $[a, b]$ such that equation (7.11) is satisfied, then equation (7.11) is satisfied for every refinement $\mathcal{P}$ of $\mathcal{P}^*$.*

(b) *If equation (7.11) is satisfied for the partition $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ and, for each j, $j = 1, 2, ..., n$, s_j and t_j are arbitrary points in $\left[x_{j-1}, x_j\right]$, then*

$$\sum_{j=1}^{n} \left| f\left(s_j\right) - f\left(t_j\right)\right| \Delta\alpha_j < \varepsilon.$$

(c) *If $f \in \Re(\alpha)$, equation (7.11) is satisfied for the partition*

$$\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$$

and, for each j, $j = 1, 2, ..., n$, t_j is an arbitrary point in $\left[x_{j-1}, x_j\right]$, then

$$\left|\sum_{j=1}^{n} f\left(t_j\right) \Delta\alpha_j - \int_a^b f(x)\, d\alpha(x)\right| < \varepsilon.$$

Remark 7.1.27 *Recall the following definition of Riemann Integrals that you saw in elementary calculus: Given a function f that is defined on an interval $I = \{x : a \leq x \leq b\}$, the "R" sum for $\Delta = \{I_1, I_2, ..., I_n\}$ a subdivision of I is given by*

$$\sum_{j=1}^{j=n} f\left(\xi_j\right) \ell\left(I_j\right)$$

where ξ_j is any element of I_j. The point ξ_j is referred to as a sampling point. To get the "R" integral we want to take the limit over such sums as the mesh of the partitions associated with Δ goes to 0. In particular, if the function f is defined on $I = \{x : a \leq x \leq b\}$ and $\wp[a, b]$ denotes the set of all partitions $\{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ of the interval I, then f is said to be "R" integrable over I if and only if

$$\lim_{mesh\mathcal{P}[a,b]\to 0} \sum_{j=1}^{j=n} f\left(\xi_j\right)\left(x_j - x_{j-1}\right)$$

exists for any choices of $\xi_j \in [x_{j-1}, x_j]$. *The limit is called the "R" integral and is denoted by* $\int_a^b f(x)\,dx$.

Taking $\alpha(t) = t$ *in Theorem 7.1.26 justifies that the old concept of an "R" integrability is equivalent to a Riemann integrability as introduced at the beginning of this chapter.*

The following theorem gives a sufficient condition for the composition of a function with a Riemann-Stieltjes integrable function to be Riemann-Stieltjes integrable.

Theorem 7.1.28 *Suppose* $f \in \Re(\alpha)$ *on* $[a, b]$, $m \le f \le M$ *on* $[a, b]$, ϕ *is continuous on* $[m, M]$, *and* $h(x) = \phi(f(x))$ *for* $x \in [a, b]$. *Then* $h \in \Re(\alpha)$ *on* $[a, b]$.

Excursion 7.1.29 *Fill in what is missing in order to complete the proof.*

Proof. For $f \in \Re(\alpha)$ on $[a, b]$ such that $m \le f \le M$ on $[a, b]$ and ϕ a continuous function on $[m, M]$, let $h(x) = \phi(f(x))$ for $x \in [a, b]$. Suppose that $\varepsilon > 0$ is given. By the $\underset{(1)}{\underline{\hspace{6cm}}}$, ϕ is uniformly continuous on $[m, M]$. Hence, there exists a $\delta > 0$ such that $\delta < \varepsilon$ and

$$(\forall s)(\forall t)\,(s, t \in [m, M] \wedge |s - t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \varepsilon). \qquad (\star)$$

Because $\underset{(2)}{\underline{\hspace{3cm}}}$, there exists a $\mathcal{P} = \{x_0 = a, x_1, \ldots, x_n = b\} \in \wp[a, b]$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \delta^2. \qquad (\star\star)$$

For each $j \in \{1, 2, \ldots, n\}$, let $M_j = \sup_{x_{j-1} \le x \le x_j} f(x)$, $m_j = \inf_{x_{j-1} \le x \le x_j} f(x)$, $M_j^* = \sup_{x_{j-1} \le x \le x_j} h(x)$, and $m_j^* = \inf_{x_{j-1} \le x \le x_j} h(x)$. From the Trichotomy Law, we know that

$$A = \left\{ j : j \in \{1, 2, \ldots, n\} \wedge \left(M_j - m_j\right) < \delta \right\}$$

and

$$B = \left\{ j : j \in \{1, 2, \ldots, n\} \wedge \left(M_j - m_j\right) \ge \delta \right\}$$

are disjoint.

If $j \in A$, then $u, v \in \left[x_{j-1}, x_j\right] \Rightarrow |f(u) - f(v)| < \delta$. It follows from $(\star)$ that $\underset{(3)}{\underline{\hspace{3cm}}}$; i.e., $|h(u) - h(v)| < \varepsilon$. Hence, $M_j^* - m_j^* \le \varepsilon$. Since $B \subset \{1, 2, \ldots, j\}$, $(\star\star)$ implies that

$$\delta \sum_{j \in B} \Delta\alpha_j \le \sum_{j \in B} \left(M_j - m_j\right) \Delta\alpha_j \le \underset{(4)}{\underline{\hspace{6cm}}} < \delta^2.$$

Because $\delta < \varepsilon$ by choice, we conclude that $\sum\limits_{j \in B} \Delta\alpha_j < \varepsilon$. Consequently, for $K = \sup\limits_{m \le t \le M} |\phi(t)|$, we have that $\left(M_j^* - m_j^*\right) \le 2K$ for each $j \in \{1, 2, \ldots, n\}$ and $\sum\limits_{j \in B} \left(M_j^* - m_j^*\right) \Delta\alpha_j < 2K\varepsilon$. Combining the bounds yields that

$$\begin{aligned} & U(\mathcal{P}, h, \alpha) - L(\mathcal{P}, h, \alpha) \\ = \quad & \sum_{j=1}^{n} \left(M_j^* - m_j^*\right) \Delta\alpha_j \\ = \quad & \sum_{j \in A} \left(M_j^* - m_j^*\right) \Delta\alpha_j + \sum_{j \in B} \left(M_j^* - m_j^*\right) \Delta\alpha_j \\ \le \quad & \underset{(5)}{\underline{\hspace{3cm}}} + 2K\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, the Integrability Criterion allows us to conclude that $h \in \Re(\alpha)$. ■

*******Acceptable responses are: (1) Uniform Continuity Theorem, (2) $f \in \Re(\alpha)$, (3) $|\phi(f(u)) - \phi(f(v))| < \varepsilon$, (4) $U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha)$, (5) $\varepsilon[\alpha(b) - \alpha(a)]$.*******

7.1.1 Properties of Riemann-Stieltjes Integrals

This section offers a list of properties of the various Riemann-Stieltjes integrals. The first lemma allows us to draw conclusions concerning the upper and lower Riemann-Stieltjes sums of a constant times a bounded function in relationship to the upper and lower Riemann-Stieltjes sums of the function.

Lemma 7.1.30 *Suppose that f is a function that is bounded and defined on the interval $I = [a, b]$. For k a nonzero real number and $g = kf$, we have*

$$\inf_{x\in I} g(x) = \begin{cases} k \cdot \inf_{x\in I} f(x) & , \text{if } k > 0 \\ k \cdot \sup_{x\in I} f(x) & , \text{if } k < 0 \end{cases} \qquad \sup_{x\in I} g(x) = \begin{cases} k \cdot \sup_{x\in I} f(x) & , \text{if } k > 0 \\ k \cdot \inf_{x\in I} f(x) & , \text{if } k < 0 \end{cases}.$$

Proof. We will prove two of the four equalities. For f a function that is defined and bounded on the interval $I = [a, b]$ and k a nonzero real number, let $g(x) = kf(x)$.

Suppose that $k > 0$ and that $M = \sup_{x\in I} f(x)$. Then $f(x) \leq M$ for all $x \in I$ and

$$g(x) = kf(x) \leq kM \text{ for all } \quad x \in I.$$

Hence, kM is an upper bound for $g(x)$ on the interval I. If kM is not the least upper bound, then there exists an $\varepsilon > 0$ such that $g(x) \leq kM - \varepsilon$ for all $x \in I$. (Here, ε can be taken to be any positive real that is less than or equal to the distance between kM and $\sup_{x\in I} g(x)$.) By substitution, we have $kf(x) \leq kM - \varepsilon$ for all $x \in I$. Since k is positive, the latter is equivalent to

$$f(x) \leq M - \left(\frac{\varepsilon}{k}\right) \text{ for all } \ x \in I$$

which contradicts that M is the supremum of f over I. Therefore,

$$\sup_{x\in I} g(x) = kM = k \sup_{x\in I} f(x).$$

Next, suppose that $k < 0$ and that $M = \sup_{x\in I} f(x)$. Now, $f(x) \leq M$ for all $x \in I$ implies that $g(x) = kf(x) \geq kM$. Hence, kM is a lower bound for $g(x)$ on I. If kM is not a greatest lower bound, then there exists an $\varepsilon > 0$, such that $g(x) \geq kM + \varepsilon$ for all $x \in I$. But, from $kf(x) \geq kM + \varepsilon$ and $k < 0$, we conclude that $f(x) \leq M + (\varepsilon/k)$ for all $x \in I$. Since ε/k is negative, $M + (\varepsilon/k) < M$ which gives us a contradiction to M being the $\sup_{x\in I} f(x)$. Therefore,

$$\inf_{x\in I} g(x) = kM = k \sup_{x\in I} f(x).$$

■

Theorem 7.1.31 (Properties of Upper and Lower Riemann-Stieltjes Integrals) *Suppose that the functions f, f_1, and f_2 are bounded and defined on the closed interval $I = [a, b]$ and α is a function that is defined and monotonically increasing in I.*

(a) If $g = kf$ for $k \in \mathbb{R} - \{0\}$, then $\underline{\int_a^b} g\,d\alpha = \begin{cases} k\underline{\int_a^b} f(x)\,d\alpha(x) & \text{, if } k > 0 \\ k\overline{\int_a^b} f(x)\,d\alpha(x) & \text{, if } k < 0 \end{cases}$

and $\overline{\int_a^b} g\,d\alpha = \begin{cases} k\overline{\int_a^b} f(x)\,d\alpha(x) & \text{, if } k > 0 \\ k\underline{\int_a^b} f(x)\,d\alpha(x) & \text{, if } k < 0 \end{cases}.$

(b) If $h = f_1 + f_2$, then

(i) $\underline{\int_a^b} h(x)\,d\alpha(x) \geq \underline{\int_a^b} f_1(x)\,d\alpha(x) + \underline{\int_a^b} f_2(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^b} h(x)\,d\alpha(x) \leq \overline{\int_a^b} f_1(x)\,d\alpha(x) + \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(c) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then

(i) $\underline{\int_a^b} f_1(x)\,d\alpha(x) \leq \underline{\int_a^b} f_2(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^b} f_1(x)\,d\alpha(x) \leq \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(d) If $a < b < c$ and f is bounded on $I^* = \{x : a \leq x \leq c\}$ and α is monotonically increasing on I^*, then

(i) $\underline{\int_a^c} f(x)\,d\alpha(x) = \underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x)$, *and*

(ii) $\overline{\int_a^c} f(x)\,d\alpha(x) = \overline{\int_a^b} f(x)\,d\alpha(x) + \overline{\int_b^c} f(x)\,d\alpha(x)$.

Excursion 7.1.32 *Fill in what is missing in order to complete the following proof of part d(i).*

Proof. Suppose that $a < b < c$ and that the function f is bounded in the interval $I^* = [a, c]$. For any finite real numbers γ and λ, let $\mathcal{D}[\gamma, \lambda]$ denote the set of all subdivisions of the interval $[\gamma, \lambda]$. Suppose that $\epsilon > 0$ is given. Since

$$\underline{\int_a^b} f(x)\,d\alpha(x) = \sup_{\Delta \in \mathcal{D}[a,b]} L(\Delta, f, \alpha) \text{ and } \underline{\int_b^c} f(x)\,d\alpha(x) = \sup_{\Delta \in \mathcal{D}[b,c]} L(\Delta, f, \alpha),$$

there exists partitions P_n and P_m of $[a, b]$ and $[b, c]$, respectively, with corresponding subdivisions Δ_n and Δ_m, such that

$$L\left(\Delta_n, f, \alpha\right) \geq \underline{\int_a^b} f(x)\, d\alpha(x) - \frac{\epsilon}{2} \text{ and } L\left(\Delta_m, f, \alpha\right) \geq \underline{\int_b^c} f(x)\, d\alpha(x) - \frac{\epsilon}{2}.$$

For $P = P_n \cup P_m$, let Δ denote the corresponding subdivision of $[a, c]$. Then

$$\begin{aligned} \underline{\int_a^c} f(x)\, d\alpha(x) &\geq \underset{(1)}{\underline{\qquad\qquad\qquad}} \\ &= L\left(\Delta_n, f, \alpha\right) + L\left(\Delta_m, f, \alpha\right) \\ &> \underset{(2)}{\underline{\qquad\qquad\qquad\qquad\qquad\qquad}} \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\underline{\int_a^c} f(x)\, d\alpha(x) \geq \underline{\int_a^b} f(x)\, d\alpha(x) + \underline{\int_b^c} f(x)\, d\alpha(x).$$

Now, we want to show that the inequality can be reversed. Suppose that $\varepsilon > 0$ is given. Since

$$\underline{\int_a^c} f(x)\, d\alpha(x) = \sup_{\Delta \in \mathcal{D}[a,c]} L\left(\Delta, f, \alpha\right),$$

There exists a $\Delta' \in \mathcal{D}[a, c]$ such that

$$L\left(\Delta', f, \alpha\right) > \underline{\int_a^c} f(x)\, d\alpha(x) - \varepsilon.$$

For $\mathcal{P}'$ the partition of $[a, c]$ that corresponds to Δ', let $\mathcal{P}'' = \mathcal{P}' \cup \{b\}$ and Δ'' denote the refinement of Δ' that corresponds to $\mathcal{P}''$. From $\underset{(3)}{\underline{\qquad\qquad\qquad}}$

$$L\left(\Delta', f, \alpha\right) \leq L\left(\Delta'', f, \alpha\right).$$

Because Δ'' is the union of a subdivision of $[a, b]$ and a subdivision of $[b, c]$, it follows from the definition of the lower Riemann-Stieltjes integrals that

$$\underline{\int_a^b} f(x)\, d\alpha(x) + \underline{\int_b^c} f(x)\, d\alpha(x) \geq L\left(\Delta'', f, \alpha\right).$$

Therefore,

$$\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq L\left(\Delta'', f, \alpha\right) \geq L\left(\Delta', f, \alpha\right) > \underset{(4)}{\underline{\qquad\qquad\qquad\qquad}}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq \underline{\int_a^c} f(x)\,d\alpha(x).$$

In view of the Trichotomy Law, $\underline{\int_a^c} f(x)\,d\alpha(x) \geq \underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x)$ and $\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) \geq \underline{\int_a^c} f(x)\,d\alpha(x)$ yields the desired result. ■
*******Acceptable responses include: (1) $L(\Delta, f, \alpha)$,
(2) $\underline{\int_a^b} f(x)\,d\alpha(x) + \underline{\int_b^c} f(x)\,d\alpha(x) - \varepsilon$, (3) Lemma 7.1.16, (4) same as completion for (2).*******

Given Riemann-Stieltjes integrable functions, the results of Theorem 7.1.31 translate directly to some of the algebraic properties that are listed in the following Theorem.

Theorem 7.1.33 (Algebraic Properties of Riemann-Stieltjes Integrals) *Suppose that the functions $f, f_1, f_2 \in \mathfrak{R}(\alpha)$ on the interval $I = [a, b]$.*

(a) If $g(x) = kf(x)$ for all $x \in I$, then $g \in \mathfrak{R}(\alpha)$ and

$$\int_a^b g(x)\,d\alpha(x) = k\int_a^b f(x)\,d\alpha(x).$$

(b) If $h = f_1 + f_2$, then $f_1 + f_2 \in \mathfrak{R}(\alpha)$ and

$$\int_a^b h(x)\,d\alpha(x) = \int_a^b f_1(x)\,d\alpha(x) + \int_a^b f_2(x)\,d\alpha(x).$$

(c) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then

$$\int_a^b f_1(x)\,d\alpha(x) \leq \int_a^b f_2(x)\,d\alpha(x).$$

(d) *If the function* $f \in \Re(\alpha)$ *also on* $I^* = \{x : b \leq x \leq c\}$*, then* f *is Riemann-Stieltjes integrable on* $I \cup I^*$ *and*

$$\int_a^c f(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha(x) + \int_b^c f(x)\, d\alpha(x).$$

(e) *If* $|f(x)| \leq M$ *for* $x \in I$*, then*

$$\left| \int_a^b f(x)\, d\alpha(x) \right| \leq M[\alpha(b) - \alpha(a)].$$

(f) *If* $f \in \Re(\alpha^*)$ *on* I*, then* $f \in \Re(\alpha + \alpha^*)$ *and*

$$\int_a^b f d\left(\alpha + \alpha^*\right) = \int_a^b f(x)\, d\alpha(x) + \int_a^b f(x)\, d\alpha^*(x).$$

(g) *If* c *is any positive real constant, then* $f \in \Re(c\alpha)$ *and*

$$\int_a^b f d(c\alpha) = c\int_a^b f(x)\, d\alpha(x).$$

Remark 7.1.34 *As long as the integrals exist, the formula given in (d) of the Corollary holds regardless of the location of* b*; i.e., b need not be between* a *and* c*.*

Remark 7.1.35 *Since a point has no dimension (that is, has length* 0*), we note that*

$$\int_a^a f(x)\, d\alpha(x) = 0 \text{ for any function } f.$$

Remark 7.1.36 *If we think of the definition of the Riemann-Stieltjes integrals as taking direction into account (for example, with* $\int_a^b f(x)\, d\alpha(x)$ *we had* $a < b$ *and took the sums over subdivisions as we were going from* a *to* b*), then it makes sense to introduce the convention that*

$$\int_b^a f(x)\, d\alpha(x) = -\int_a^b f(x)\, d\alpha(x)$$

for Riemann-Stieltjes integrable functions f*.*

The following result follows directly from the observation that corresponding to each partition of an interval we can derive a partition of any subinterval and vice versa.

Theorem 7.1.37 (Restrictions of Integrable Functions) *If the function f is (Riemann) integrable on an interval I, then $f|_{I^*}$ is integrable on I^* for any subinterval I^* of I.*

Choosing different continuous functions for ϕ in Theorem 7.1.28 in combination with the basic properties of Riemann-Stieltjes integrals allows us to generate a set of Riemann-Stieltjes integrable functions. For example, because $\phi_1(t) = t^2$, $\phi_2(t) = |t|$, and $\phi_3(t) = \gamma t + \lambda$ for any real constants γ and λ, are continuous on $\mathbb{R}$, if $f \in \Re(\alpha)$ on an interval $[a, b]$, then each of $(f)^2$, $|f|$, and $\gamma f + \lambda$ will be Riemann-Stieltjes integrable with respect to α on $[a, b]$. The proof of the next theorem makes nice use of this observation.

Theorem 7.1.38 *If $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$, then $fg \in \Re(\alpha)$.*

Proof. Suppose that $f \in \Re(\alpha)$ and $g \in \Re(\alpha)$ on $[a, b]$. From the Algebraic Properties of the Riemann-Stieltjes Integral, it follows that $(f + g) \in \Re(\alpha)$ on $[a, b]$ and $(f - g) \in \Re(\alpha)$ on $[a, b]$. Taking $\phi(t) = t^2$ in Theorem 7.1.28 yields that $(f + g)^2$ and $(f - g)^2$ are also Riemann-Stieltjes integrable with respect to α on $[a, b]$. Finally, the difference

$$4fg = (f + g)^2 - (f - g)^2 \in \Re(\alpha) \text{ on } [a, b]$$

as claimed. ■

Theorem 7.1.39 *If $f \in \Re(\alpha)$ on $[a, b]$, then $|f| \in \Re(\alpha)$ and*

$$\left|\int_a^b f(x)\,d\alpha(x)\right| \le \int_a^b |f(x)|\,d\alpha(x).$$

Proof. Suppose $f \in \Re(\alpha)$ on $[a, b]$. Taking $\phi(t) = |t|$ in Theorem 7.1.28 yields that $|f| \in \Re(\alpha)$ on $[a, b]$. Choose $\gamma = 1$, if $\int f(x)\,d\alpha(x) \ge 0$ and $\gamma = -1$, if $\int f(x)\,d\alpha(x) \le 0$. Then

$$\left|\int_a^b f(x)\,d\alpha(x)\right| = \gamma \int_a^b f(x)\,d\alpha(x) \quad \text{and} \quad \gamma f(x) \le |f(x)| \text{ for } x \in [a, b].$$

It follows from Algebraic Properties of the Riemann-Stieltjes Integrals (a) and (c) that

$$\left|\int_a^b f(x)\,d\alpha(x)\right| = \gamma \int_a^b f(x)\,d\alpha(x) = \int_a^b \gamma f(x)\,d\alpha(x) \leq \int_a^b |f(x)|\,d\alpha(x).$$

■

One glaring absence from our discussion has been specific examples of finding the integral for integrable functions using the definition. Think for a moment or so about what the definition requires us to find: First, we need to determine the set of all upper Riemann-Stieltjes sums and the set of all lower Riemann-Stieltjes sums; this is where the subdivisions of the interval over which we are integrating range over all possibilities. We have no uniformity, no simple interpretation for the suprema and infima we need, and no systematic way of knowing when we "have checked enough" subdivisions or sums. On the other hand, whenever we have general conditions that insure integrability, the uniqueness of the least upper and greatest lower bounds allows us to find the value of the integral from considering wisely selected special subsets of the set of all subdivisions of an interval.

The following result offers a sufficient condition under which a Riemann-Stieltjes integral is obtained as a point evaluation. It makes use of the characteristic function. Recall that, for a set S and $A \subset S$, the **characteristic function** $\chi_A : S \to \{0, 1\}$ is defined by

$$\chi_A(x) = \begin{cases} 1 & , \text{if} \quad x \in A \\ 0 & , \text{if} \quad x \in S - A \end{cases}$$

In the following, $\chi_{(0,\infty)}$ denotes the characteristic function with $S = \mathbb{R}$ and $A = (0, \infty)$; i.e.,

$$\chi_{(0,\infty)}(x) = \begin{cases} 1 & , \text{if} \quad x > 0 \\ 0 & , \text{if} \quad x \leq 0 \end{cases}.$$

Lemma 7.1.40 *Suppose that f is bounded on $[a, b]$ and continuous at $s \in (a, b)$. If $\alpha(x) = \chi_{(0,\infty)}(x - s)$, then*

$$\int_a^b f(x)\,d\alpha(x) = f(s).$$

Proof. For each $\mathcal{P} = \{x_0 = a, x_1, ..., x_{n-1}, x_n = b\}$ be an arbitrary partition for $[a, b]$, there exists a $j \in \{1, 2, ..., n\}$ such that $s \in \left[x_{j-1}, x_j\right)$. From the definition of α, we have that $\alpha(x_k) = 0$ for each $k \in \{1, 2, ..., j-1\}$ and $\alpha(x_k) = 1$ for each $k \in \{j, ..., n\}$. Hence,

$$\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1}) = \begin{cases} 1 & \text{, if} \quad k = j \\ \\ 0 & \text{, if} \quad k \in \{1, 2, ..., j-1\} \cup \{j+1, ..., n\} \end{cases},$$

from which we conclude that

$$U(\mathcal{P}, f, \alpha) = \sup_{x_{j-1} \le x \le x_j} f(x) \text{ and } L(\mathcal{P}, f, \alpha) = \inf_{x_{j-1} \le x \le x_j} f(x).$$

Since f is continuous at s and $\left(x_j - x_{j-1}\right) \le \operatorname{mesh}\mathcal{P}$, $\sup_{x_{j-1} \le x \le x_j} f(x) \to s$ and $\inf_{x_{j-1} \le x \le x_j} f(x) \to s$ as $\operatorname{mesh}\mathcal{P} \to 0$. Therefore, $\int_a^b f(x)\, d\alpha(x) = f(s)$. ■

If the function f is continuous on an interval $[a, b]$, then Lemma 7.1.40 can be extended to a sequence of points in the interval.

Theorem 7.1.41 *Suppose the sequence of nonnegative real numbers* $\{c_n\}_{n=1}^{\infty}$ *is such that* $\sum_{n=1}^{\infty} c_n$ *is convergent,* $\{s_n\}_{n=1}^{\infty}$ *is a sequence of distinct points in* (a, b)*, and* f *is a function that is continuous on* $[a, b]$*. If* $\alpha(x) = \sum_{n=1}^{\infty} c_n \chi_{(0,\infty)}(x - s_n)$*, then*

$$\int_a^b f(x)\, d\alpha(x) = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof. For $u, v \in (a, b)$ such that $u < v$, let $S_u = \{n \in \mathbb{J} : a < s_n \le u\}$ and $T_v = \{n \in \mathbb{J} : a < s_n \le v\}$. Then

$$\alpha(u) = \sum_{n \in S_u} c_n \le \sum_{n \in T_v} c_n = \alpha(v)$$

from which we conclude that α is monotonically increasing. Furthermore, $\alpha(a) = 0$ and $\alpha(b) = \sum_{n=1}^{\infty} c_n$.

Let $\varepsilon > 0$ be given. Since $\sum_{n=1}^{\infty} c_n$ is convergent, there exists a positive integer K such that

$$\sum_{n=K+1}^{\infty} c_n < \frac{\varepsilon}{M}$$

where $M = \sup_{x \in [a,b]} |f(x)|$. Let $\alpha_1(x) = \sum_{n=1}^{K} c_n \chi_{(0,\infty)}(x - s_n)$ and $\alpha_2(x) = \sum_{n=K+1}^{\infty} c_n \chi_{(0,\infty)}(x - s_n)$. It follows from Lemma 7.1.40 that

$$\int_a^b f(x)\, d\alpha_1(x) = \sum_{j=1}^{K} c_n f(s_n);$$

while $\alpha_2(b) - \alpha_2(a) < \dfrac{\varepsilon}{M}$ yields that

$$\left| \int_a^b f(x)\, d\alpha_2(x) \right| < \varepsilon.$$

Because $\alpha = \alpha_1 + \alpha_2$, we conclude that

$$\left| \int_a^b f(x)\, d\alpha(x) - \sum_{n=1}^{K} c_n f(s_n) \right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\int_a^b f(x)\, d\alpha(x) = \sum_{n=1}^{\infty} c_n f(s_n)$. ■

Theorem 7.1.42 *Suppose that α is a monotonically increasing function such that $\alpha' \in \Re$ on $[a,b]$ and f is a real function that is bounded on $[a,b]$. Then $f \in \Re(\alpha)$ if and only if $f\alpha' \in \Re$. Furthermore,*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\, \alpha'(x)\, dx.$$

Excursion 7.1.43 *Fill in what is missing in order to complete the following proof of Theorem 7.1.42.*

Proof. Suppose that $\varepsilon > 0$ is given. Since $\alpha' \in \Re$ on $[a, b]$, by the Integrability Criterion, there exists a partition $\mathcal{P} = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ such that

$$U\left(\mathcal{P}, \alpha'\right) - \underset{(1)}{\underline{\hspace{3cm}}} < \frac{\varepsilon}{M} \tag{7.12}$$

where $M = \sup |f(x)|$. Furthermore, from the Mean-Value Theorem, for each $j \in \{1, 2, ..., n\}$ there exists a $t_j \in \left[x_{j-1}, x_j\right]$ such that

$$\Delta\alpha_j = \underset{(2)}{\underline{\hspace{6cm}}} = \alpha'\left(t_j\right) \Delta x_j. \tag{7.13}$$

By Theorem 7.1.26(b) and (7.12), for any $s_j \in \left[x_{j-1}, x_j\right]$, $j \in \{1, 2, ..., n\}$

$$\sum_{j=1}^{n} \left|\alpha'\left(s_j\right) - \alpha'\left(t_j\right)\right| \Delta x_j < \varepsilon. \tag{7.14}$$

With this set-up, we have that

$$\sum_{j=1}^{n} f\left(s_j\right) \Delta\alpha_j = \sum_{j=1}^{n} \underset{(3)}{\underline{\hspace{3cm}}}$$

and

$$\begin{aligned} & \left|\sum_{j=1}^{n} f\left(s_j\right) \Delta\alpha_j - \sum_{j=1}^{n} f\left(s_j\right) \alpha'\left(s_j\right) \Delta x_j\right| \\ = & \left|\underset{(4)}{\underline{\hspace{6cm}}} - \sum_{j=1}^{n} f\left(s_j\right) \alpha'\left(s_j\right) \Delta x_j\right| \\ = & \left|\sum_{j=1}^{n} f\left(s_j\right) \left[\alpha'\left(t_j\right) - \alpha'\left(s_j\right)\right] \Delta x_j\right| \\ \leq & M \left|\sum_{j=1}^{n} \left[\alpha'\left(t_j\right) - \alpha'\left(s_j\right)\right] \Delta x_j\right| < \varepsilon. \end{aligned}$$

That is,

$$\left|\sum_{j=1}^{n} f\left(s_j\right) \Delta\alpha_j - \sum_{j=1}^{n} f\left(s_j\right) \alpha'\left(s_j\right) \Delta x_j\right| < \varepsilon \tag{7.15}$$

for any choice of points $s_j \in [x_{j-1}, x_j]$, $j = 1, 2, ..., n$. Then

$$\sum_{j=1}^{n} f(s_j)\, \Delta\alpha_j \leq U(\mathcal{P}, f\alpha') + \varepsilon$$

and

$$U(\mathcal{P}, f, \alpha) \leq U(\mathcal{P}, f\alpha') + \varepsilon.$$

Equation (7.15) also allows us to conclude that

$$U(\mathcal{P}, f\alpha') \leq U(\mathcal{P}, f, \alpha) + \varepsilon.$$

Hence,

$$\left|U(\mathcal{P}, f, \alpha) - U(\mathcal{P}, f\alpha')\right| \leq \varepsilon. \tag{7.16}$$

Since $\mathcal{P}$ was arbitrary, if follows that (7.16) holds for all $\mathcal{P} \in \wp[a, b]$, the set of all partitions of $[a, b]$. Therefore,

$$\left|\overline{\int_a^b} f(x)\, d\alpha(x) - \underset{(5)}{\underline{\hspace{6cm}}}\right| \leq \varepsilon.$$

Because $\varepsilon > 0$ was arbitrary, we conclude that, for any function f that is bounded on $[a, b]$,

$$\overline{\int_a^b} f(x)\, d\alpha(x) = \overline{\int_a^b} f(x)\, \alpha'(x)\, dx.$$

Equation (7.15) can be used to draw the same conclusion concerning the comparable lower Riemann and Riemann-Stieltjes integrals in order to claim that

$$\underline{\int_a^b} f(x)\, d\alpha(x) = \underline{\int_a^b} f(x)\, \alpha'(x)\, dx.$$

The combined equalities leads to the desired conclusion. ■

Acceptable responses are: (1) $L(\mathcal{P}, \alpha')$, (2) $\alpha(x_j) - \alpha(x_{j-1})$ (3) $f(s_j)\, \alpha'(t_j)\, \Delta x_j$ (4) $\sum_{j=1}^{n} f(s_j)\, \alpha'(t_j)\, \Delta x_j$, (5) $\overline{\int_a^b} f(x)\, \alpha'(x)\, dx$.

Recall that our original motivation for introducing the concept of the Riemann integral was adapting formulas such as $A = l \cdot w$, $d = r \cdot t$ and $m = d \cdot l$ to more general situations; the Riemann integral allow us to replace one of the "constant dimensions" with functions that are at least bounded where being considered. The Riemann-Stieltjes integral allows us to replace both of the "constant dimensions" with functions. Remark 6.18 on page 132 of our text describes a specific example that illustrates to flexibility that has been obtained.

The last result of this section gives us conditions under which we can transfer from one Riemann-Stieltjes integral set-up to another one.

Theorem 7.1.44 (Change of Variables) *Suppose that ϕ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$, α is monotonically increasing on $[a, b]$, and $f \in \Re(\alpha)$ on $[a, b]$. For $y \in [A, B]$, let $\beta(y) = \alpha(\phi(y))$ and $g(y) = f(\phi(y))$. Then $g \in \Re(\beta)$ and*

$$\int_A^B g(y)\, d\beta(y) = \int_a^b f(x)\, d\alpha(x).$$

Proof. Because ϕ is strictly increasing and continuous, each partition $\mathcal{P} = \{x_0, x_1, ..., x_n\} \in \wp[a, b]$ if and only if $\mathcal{Q} = \{y_0, y_1, ..., y_n\} \in \wp[A, B]$ where $x_j = \phi(y_j)$ for each $j \in \{0, 1, ..., n\}$. Since $f([x_{j-1}, x_j]) = g([y_{j-1}, y_j])$ for each j, it follows that

$$U(\mathcal{Q}, g, \beta) = U(\mathcal{P}, f, \alpha) \qquad \textit{and} \qquad L(\mathcal{Q}, g, \beta) = L(\mathcal{P}, f, \alpha).$$

The result follows immediately from the Integrability Criterion. ■

7.2 Riemann Integrals and Differentiation

When we restrict ourselves to Riemann integrals, we have some nice results that allow us to make use of our knowledge of derivatives to compute integrals. The first result is both of general interest and a useful tool for proving some to the properties that we seek.

Theorem 7.2.1 (Mean-Value Theorem for Integrals) *Suppose that f is continuous on $I = [a, b]$. Then there exists a number ξ in I such that*

$$\int_a^b f(x)\, dx = f(\xi)(b - a).$$

Proof. This result follows directly from the bounds on integrals given by Theorem 7.1.13 and the Intermediate Value Theorem. Since f is continuous on $[a, b]$, it is integrable there and, by Theorem 7.1.13,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

where $m = \inf_{x\in I} f(x) = \min_{x\in I} f(x) = f(x_0)$ for some $x_0 \in I$ and $M = \sup_{x\in I} f(x) = \max_{x\in I} f(x) = f(x_1)$ for some $x_1 \in I$. Now, $A = \dfrac{\int_a^b f(x)\,dx}{(b-a)}$ is a real number such that $m \leq A \leq M$. By the Intermediate Value Theorem, $f(x_0) \leq A \leq f(x_1)$ implies that there exists a $\xi \in I$ such that $f(\xi) = A$. ■

The following two theorems are two of the most celebrated results from integral calculus. They draw a clear and important connection between integral calculus and differential calculus. The first one makes use of the fact that integrability on an interval allows us to define a new function in terms of the integral. Namely, if f is Riemann integrable on the interval $[a, b]$, then, by the Theorem on Restrictions of Integrable Functions, we know that it is integrable on every subinterval of $[a, b]$. In particular, for each $x \in [a, b]$, we can consider

$$f : x \longrightarrow \int_a^x f(t)\,dt.$$

This function is sometimes referred to as the accumulation of f–probably as a natural consequence of relating the process of integration to finding the area between the graph of a positive function and the real axis. The variable t is used as the dummy variable because x is the argument of the function. The accumulation function is precisely the object that will allow us to relate the process of integration back to differentiation at a point.

Theorem 7.2.2 (The First Fundamental Theorem of Calculus) *Suppose that* $f \in \mathfrak{R}$ *on* $I = [a, b]$. *Then the function* F *given by*

$$F(x) = \int_a^x f(t)\,dt$$

is uniformly continuous on $[a, b]$. *If* f *is continuous on* I, *then* F *is differentiable in* (a, b) *and, for each* $x \in (a, b)$, $F'(x) = f(x)$.

Proof. Suppose that $u, v \in [a, b]$. Without loss of generality we can assume that $u < v$. Then, from the Algebraic Properties of Riemann-Stieltjes Integrals (d) and (e),

$$|F(v) - F(u)| = \left| \int_u^v f(t)\,dt \right| \leq M\,|u - v|$$

where $M = \sup_{t \in I} |f(t)|$. Thus,

$$(\forall \varepsilon > 0)\,(\forall u)\,(\forall v)\left(u, v \in I \wedge |u - v| < \delta = \frac{\varepsilon}{M} \Rightarrow |F(v) - F(u)| < \varepsilon\right); \text{i.e.,}$$

F is uniformly continuous on I.

For the second part, suppose f is continuous on $[a, b]$ and $x \in (a, b)$. Then there exists δ_1 such that $\{x + h : |h| < \delta_1\} \subset (a, b)$. Since f is continuous it is integrable on every subinterval of I, for $|h| < \delta_1$, we have that each of $\int_a^{x+h} f(t)\,dt$, $\int_a^x f(t)\,dt$, and $\int_x^{x+h} f(t)\,dt$ exists and

$$\int_a^{x+h} f(t)\,dt = \int_a^x f(t)\,dt + \int_x^{x+h} f(t)\,dt.$$

Consequently, for any h, with $|h| < \delta_1$, we have that

$$F(x + h) - F(x) = \int_x^{x+h} f(t)\,dt.$$

By the Mean-Value Theorem for Integrals, there exists ξ_h with $|x - \xi_h| < \delta_1$ such that

$$\int_x^{x+h} f(t)\,dt = f(\xi_h) \cdot h.$$

Hence, for $|h| < \delta_1$,

$$\frac{F(x + h) - F(x)}{h} = f(\xi_h)$$

where $|x - \xi_h| < \delta_1$. Now, suppose that $\epsilon > 0$ is given. Since f is continuous at x, there exists a $\delta_2 > 0$ such that $|f(w) - f(x)| < \epsilon$ whenever $|w - x| < \delta_2$. Choose $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|h| < \delta$, we have

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| = |f(\xi_h) - f(x)| < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, we conclude that

$$\lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = f(x)\text{; i.e.,}$$

$F'(x) = f(x)$. Since $x \in (a,b)$ was arbitrary, we conclude that F is differentiable on the open interval (a,b). ■

Theorem 7.2.3 (The Second Fundamental Theorem of Calculus) *If $f \in \Re$ on $I = [a,b]$ and there exists a function F that is differentiable on $[a,b]$ with $F' = f$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Excursion 7.2.4 *Fill in what is missing in order to complete the following proof.*

Proof. Suppose that $\varepsilon > 0$ is given. For $f \in \Re$, by the $\underset{(1)}{\underline{\hspace{3cm}}}$, we can choose a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$. By the Mean-Value Theorem, for each $j \in \{1,2,\ldots,n\}$ there is a point $t_j \in [x_{j-1}, x_j]$ such that

$$F(x_j) - F(x_{j-1}) = F'(t_j)\,\Delta x_j = \underset{(2)}{\underline{\hspace{3cm}}}.$$

Hence,

$$\sum_{j=1}^{n} f(t_j)\,\Delta x_j = \underset{(3)}{\underline{\hspace{6cm}}}$$

On the other hand, from Theorem 7.1.26(c), $\left|\sum_{j=1}^{n} f(t_j)\,\Delta x_j - \int_a^b f(x)\,dx\right| < \varepsilon$.

Therefore,

$$\left|\underset{(3)}{\underline{\hspace{6cm}}} - \int_a^b f(x)\,dx\right| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $\int_a^b f(t)\,dt = F(b) - F(a)$. ■
Acceptable responses are: (1) Integrability Criterion, (2) $f(t_j)\,\Delta x_j$, (3) $F(b) - F(a)$.

Remark 7.2.5 *The statement of the First Fundamental Theorem of Calculus differs from the one that you had in elementary Calculus. If instead of taking f to be integrable in $I = [a, b]$, we take f to be integrable on an open interval containing I, we can claim that $G(x) = \int_a^x f(t)\,dt$ is differentiable on $[a, b]$ with $G'(x) = f(x)$ on $[a, b]$. This enables us to offer a slightly different proof for the Second Fundamental Theorem of Calculus. Namely, it follows that if F is any antiderivative for f then $F - G = c$ for some constant c and we have that*

$$F(b) - F(a) = [G(b) + c] - [G(a) + c] = [G(b) - G(a)] = \int_a^b f(t)\,dt.$$

Remark 7.2.6 *The Fundamental Theorems of Calculus give us a circumstance under which finding the integral of a function is equivalent to finding a primitive or antiderivative of a function. When f is a continuous function, we conclude that it has a primitive and denote the set of all primitives by $\int f(x)\,dx$; to find the definite integral $\int_a^b f(x)\,dx$, we find any primitive of f, F, and we conclude that*

$$\int_a^b f(x)\,dx = F(x)\big|_a^b = F(b) - F(a).$$

7.2.1 Some Methods of Integration

We illustrate with two methods, namely substitution and integration-by-parts. The theoretical foundation for the method of substitution is given by Theorem 7.1.42 and the Change of Variables Theorem.

Theorem 7.2.7 *Suppose that the function f is continuous on a segment I, the functions u and $\dfrac{du}{dx}$ are continuous on a segment J, and the range of u is contained in I. If $a, b \in J$, then*

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du.$$

Proof. By the First Fundamental Theorem of Calculus, for each $c \in I$, the function

$$F(u) = \int_c^u f(t)\,dt$$

is differentiable with $F'(u) = f(u)$ for $u \in I$ with $c \leq u$. By the Chain Rule, if $G(x) = F(u(x))$, then $G'(x) = F'(u(x))\,u'(x)$. Hence, $G'(x) = f(u(x))\,u'(x)$. Also, G' is continuous from the continuity of f, u, and u'. It follows from the Second Fundamental Theorem of Calculus and the definition of G that

$$\int_a^b f(u(x))\,u'(x)\,dx = \int_a^b G'(x)\,dx = G(b) - G(a) = F(u(b)) - F(u(a)).$$

From the definition of F, we conclude that

$$F(u(b)) - F(u(a)) = \int_c^{u(b)} f(t)\,dt - \int_c^{u(a)} f(t)\,dt = \int_{u(a)}^{u(b)} f(t)\,dt.$$

The theorem follows from the transitivity of equals. ■

Example 7.2.8 *Use of the Substitution Method of Integration to find*

$$\int_0^{\pi/4} \cos^2\left(3t + \frac{\pi}{4}\right)\sin\left(3t + \frac{\pi}{4}\right)dt$$

(a) Take $u = \cos\left(3t + \frac{\pi}{4}\right)$

(b) Take $u = \cos^3\left(3t + \frac{\pi}{4}\right)$.

The other common method of integration with which we should all be familiar is known as Integration by Parts. The IBP identity is given by

$$\int u\,dv = uv - \int v\,du$$

for u and v differentiable and follows from observing that $d(uv) = udv + vdu$, which is the product rule in differential notation. This enables us to tackle integrands that "are products of functions not related by differentiation" and some special integrands, such as the inverse trig functions.

Example 7.2.9 *Examples of the use of the Integration-by-Parts method of integration.*

1. *Find* $\int x^3 \cdot \sqrt{1 - x^2} dx$.

2. *Find* $\int \arctan(x)\, dx$

3. *Find* $\int e^{2x} \sin(3x)\, dx$

7.2.2 The Natural Logarithm Function

The Fundamental Theorems enable us to find integrals by looking for antiderivatives. The formula $(x^n)' = nx^{n-1}$ for n an integer leads us to conclude that $\int x^k dx = \frac{x^{k+1}}{k+1} + C$ for any constant C as long as $k+1 \neq 0$. So we can't use a simple formula to determine

$$\int_a^b \frac{1}{x} dx,$$

though we know that it exists for any finite closed interval that does not contain 0 because x^{-1} is continuous in any such interval. This motivates us to introduce a notation for a simple form of this integral.

Definition 7.2.10 *The* ***natural logarithm function****, denoted by* ln *, is defined by the formula*

$$\ln x = \int_1^x \frac{1}{t} dt, \text{ for every } x > 0.$$

As fairly immediate consequences of the definition, we have the following Properties of the Natural Logarithm Function. Let $f(x) = \ln x$ for $x > 0$ and suppose that a and b are positive real numbers. Then, the following properties hold:

1. $\ln(ab) = \ln(a) + \ln(b)$,
2. $\ln(a/b) = \ln(a) - \ln(b)$,
3. $\ln(1) = 0$,

4. $\ln(a^r) = r \cdot \ln(a)$ for every rational number r,
5. $f'(x) = \dfrac{1}{x}$,
6. f is increasing and continuous on $I = \{x : 0 < x < +\infty\}$,
7. $\dfrac{1}{2} \leq \ln(2) \leq 1$,
8. $\ln x \longrightarrow +\infty$ as $x \to +\infty$,
9. $\ln x \longrightarrow -\infty$ as $x \to 0^+$, and
10. the range of f is all of $\mathbb{R}$.

Remark 7.2.11 *Once we have property (6), the Inverse Function Theorem guarantees the existence of an inverse function for* $\ln x$. *This leads us back to the function* e^x.

7.3 Integration of Vector-Valued Functions

Building on the way that limits, continuity, and differentiability from single-valued functions translated to vector-valued functions, we define Riemann-Stieltjes integrability of vector-valued functions by assignment of that property to the coordinates.

Definition 7.3.1 *Given a vector-valued (n-valued) function* $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ *from* $[a, b]$ *into* $\mathbb{R}^n$ *where the real-valued functions* $f_1, f_2, \ldots, f_n$ *are bounded on the interval* $I = [a, b]$ *and a function* α *that is defined and monotonically increasing on* I, $\mathbf{f}$ *is* ***Riemann-Stieltjes integrable with respect to*** α *on* I, *written* $\mathbf{f} \in \mathfrak{R}(\alpha)$, *if and only if* $(\forall j)\left(j \in \{1, 2, \ldots, n\} \Rightarrow f_j \in \mathfrak{R}(\alpha)\right)$. *In this case,*

$$\int_a^b \mathbf{f}(x)\, d\alpha(x) \underset{def}{=} \left(\int_a^b f_1(x)\, d\alpha(x), \int_a^b f_2(x)\, d\alpha(x), \ldots, \int_a^b f_n(x)\, d\alpha(x)\right).$$

Because of the nature of the definition, any results for Riemann-Stieltjes integrals that involved “simple algebraic evaluations” can be translated to the vector-valued case.

Theorem 7.3.2 *Suppose that the vector-valued functions* $\mathbf{f}$ *and* $\mathbf{g}$ *are Riemann-Stieltjes integrable with respect to* α *on the interval* $I = [a, b]$.

(a) *If* k *is a real constant, then* $k\mathbf{f} \in \Re(\alpha)$ *on* I *and*

$$\int_a^b k\mathbf{f}(x)\, d\alpha(x) = k \int_a^b \mathbf{f}(x)\, d\alpha(x).$$

(b) *If* $\mathbf{h} = \mathbf{f} + \mathbf{g}$, *then* $\mathbf{h} \in \Re(\alpha)$ *and*

$$\int_a^b \mathbf{h}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_a^b \mathbf{g}(x)\, d\alpha(x).$$

(c) *If the function* $f \in \Re(\alpha)$ *also on* $I^* = \{x : b \le x \le c\}$, *then* f *is Riemann-Stieltjes integrable with respect to* α *on* $I \cup I^*$ *and*

$$\int_a^c \mathbf{f}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_b^c \mathbf{f}(x)\, d\alpha(x).$$

(d) *If* $\mathbf{f} \in \Re(\alpha^*)$ *on* I, *then* $\mathbf{f} \in \Re(\alpha + \alpha^*)$ *and*

$$\int_a^b \mathbf{f} d\left(\alpha + \alpha^*\right) = \int_a^b \mathbf{f}(x)\, d\alpha(x) + \int_a^b \mathbf{f}(x)\, d\alpha^*(x).$$

(e) *If* c *is any positive real constant, then* $\mathbf{f} \in \Re(c\alpha)$ *and*

$$\int_a^b \mathbf{f} d(c\alpha) = c \int_a^b \mathbf{f}(x)\, d\alpha(x).$$

Theorem 7.3.3 *Suppose that* α *is a monotonically increasing function such that* $\alpha' \in \Re$ *on* $[a, b]$ *and* $\mathbf{f}$ *is a vector-valued function that is bounded on* $[a, b]$. *Then* $\mathbf{f} \in \Re(\alpha)$ *if and only if* $\mathbf{f}\alpha' \in \Re$. *Furthermore,*

$$\int_a^b \mathbf{f}(x)\, d\alpha(x) = \int_a^b \mathbf{f}(x)\, \alpha'(x)\, dx.$$

Theorem 7.3.4 *Suppose that* $\mathbf{f} = (f_1, f_2, \ldots, f_n) \in \Re$ *on* $I = [a, b]$.

(a) *Then the vector-valued function* $\mathbf{F}$ *given by*

$$\mathbf{F}(x) = \left(\int_a^x f_1(t)\,dt, \int_a^x f_2(t)\,dt, ..., \int_a^x f_n(t)\,dt \right) \text{ for } x \in I$$

is continuous on $[a, b]$*. Furthermore, if* $\mathbf{f}$ *is continuous on* I*, then* $\mathbf{F}$ *is differentiable in* (a, b) *and, for each* $x \in (a, b)$, $\mathbf{F}'(x) = \mathbf{f}(x) = (f_1(x), f_2(x), ..., f_n(x))$.

(b) *If there exists a vector-valued function* $\mathbf{G}$ *on* I *that is differentiable there with* $\mathbf{G}' = \mathbf{f}$*, then*

$$\int_a^b \mathbf{f}(t)\,dt = \mathbf{G}(b) - \mathbf{G}(a).$$

On the other hand, any of the results for Riemann-Stieltjes integrals of real-valued functions that involved inequalities require independent consideration for formulations that might apply to the vector-valued situation; while we will not pursue the possibilities here, sometimes other geometric conditions can lead to analogous results. The one place where we do have an almost immediate carry over is with Theorem 7.1.39 because the inequality involved the absolute value which generalizes naturally to an inequality in terms of the Euclidean metric. The generalization–natural as it is–still requires proof.

Theorem 7.3.5 *Suppose that* $\mathbf{f} : [a, b] \to \mathbb{R}^n$ *and* $\mathbf{f} \in \mathfrak{R}(\alpha)$ *on* $[a, b]$ *for some* α *that is defined and monotonically increasing on* $[a, b]$*. Then* $|\mathbf{f}| \in \mathfrak{R}(\alpha)$ *and*

$$\left| \int_a^b \mathbf{f}(x)\,d\alpha(x) \right| \leq \int_a^b |\mathbf{f}(x)|\,d\alpha(x). \tag{7.17}$$

Excursion 7.3.6 *Fill in what is missing in order to complete the following proof.*

Proof. Suppose that $\mathbf{f} = (f_1, f_2, ..., f_n) \in \mathfrak{R}(\alpha)$ on $I = [a, b]$. Then

$$|\mathbf{f}(x)| = \sqrt{f_1^2(x) + f_2^2(x) + ... + f_n^2(x)} \geq 0 \text{ for } x \in I \tag{7.18}$$

and, because $\mathbf{f}$ is $\underset{(1)}{\underline{\hspace{3cm}}}$ on I, there exists $M > 0$ such that $|\mathbf{f}(I)| = \{|\mathbf{f}(x)| : x \in I\} \subset [0, M]$. Since $(\forall j)\left(j \in \{1, 2, ..., n\} \Rightarrow f_j \in \mathfrak{R}(\alpha)\right)$ and the function $\phi(t) = t^2$ is continuous on $\mathbb{R}$, by $\underset{(2)}{\underline{\qquad\qquad}}$, $(\forall j)\left(j \in \{1, 2, ..., n\} \Rightarrow f_j^2 \in \Re(\alpha)\right)$. From Algebraic Property (b) of the Riemann-Stieltjes integral, $f_1^2(x) + f_2^2(x) + ... + f_n^2(x) \in \Re(\alpha)$. Taking $\phi^*(t) = \sqrt{t}$ for $t \geq 0$ in Theorem 7.1.28 yields that $|\mathbf{f}|$ $\underset{(3)}{\underline{\qquad\qquad}}$.

Since (7.17) is certainly satisfied if $\mathbf{f}$, we assume that $\mathbf{f} \neq \mathbf{0}$. For each $j \in \{1, 2, ..., n\}$, let $\omega_j = \int_a^b f_j(x)\, d\alpha(x)$ and set $\mathbf{w} = \int_a^b \mathbf{f}(x)\, d\alpha(x)$. Then

$$|\mathbf{w}|^2 = \sum_{j=1}^{n} \omega_j^2 = \sum_{j=1}^{n} \omega_j \int_a^b f_j(t)\, d\alpha(t) = \int_a^b \left(\sum_{j=1}^{n} \omega_j f_j(t)\right) d\alpha(t).$$

From Schwarz's inequality,

$$\sum_{j=1}^{n} \omega_j f_j(t) \leq \underset{(4)}{\underline{\quad}} |\mathbf{f}(t)| \text{ for } t \in [a, b]. \tag{7.19}$$

Now $\sum_{j=1}^{n} \omega_j f_j(t)$ and $|\mathbf{w}|\,|\mathbf{f}(t)|$ are real-valued functions on $[a, b]$ that are in $\Re(\alpha)$. From (7.19) and Algebraic Property (c) of Riemann-Stieltjes integrals, it follows that

$$|\mathbf{w}|^2 = \int_a^b \left(\sum_{j=1}^{n} \omega_j f_j(t)\right) d\alpha(t) \leq \int_a^b |\mathbf{w}|\,|\mathbf{f}(t)|\, d\alpha(t) = |\mathbf{w}| \int_a^b |\mathbf{f}(t)|\, d\alpha(t).$$

Because $\mathbf{w} \neq 0$, $|\mathbf{w}|^2 \leq |\mathbf{w}| \int_a^b |\mathbf{f}(t)|\, d\alpha(t)$ implies that $|\mathbf{w}| \leq \int_a^b |\mathbf{f}(t)|\, d\alpha(t)$ which is equivalent to equation 7.17). ■

*******Acceptable responses are: (1) bounded, (2) Theorem 7.1.28, (3) $\in \Re(\alpha)$, (4) $|\mathbf{w}|$.*******

7.3.1 Rectifiable Curves

As an application of Riemann-Stieltjes integration on vector-valued functions we can prove a result that you assumed when you took elementary vector calculus. Recall the following definition.

Definition 7.3.7 *A continuous function γ from an interval $[a, b]$ into $\mathbb{R}^n$ is called a **curve** in $\mathbb{R}^n$ or a curve on $[a, b]$ in $\mathbb{R}^n$; if γ is one-to-one, then γ is called an **arc**, and if $\gamma(a) = (\beta)$, then γ is a **closed curve**.*

Remark 7.3.8 *In the definition of curve, we want to think of the curve as the actual mapping because the associated set of points in $\mathbb{R}^n$ is not uniquely determined by a particular mapping. As a simple example, $\gamma_1(t) = (t, t)$ and $\gamma_2(t) = \left(t, t^2\right)$ are two different mappings that give the same associated subset of $\mathbb{R}^2$.*

Given a curve γ on $[a, b]$, for any partition of $[a, b]$,

$$\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\}$$

let

$$\Lambda(\mathcal{P}, \gamma) = \sum_{j=1}^{m} \left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right|.$$

Then $\Lambda(\mathcal{P}, \gamma)$ is the length of a polygonal path having vertices $\gamma(x_0), \gamma(x_1), ..., \gamma(x_m)$ which, if conditions are right, gives an approximation for the length of the curve γ. For $\wp[a, b]$ the set of all partitions of $[a, b]$, it is reasonable to define the length of a curve γ as

$$\Lambda(\gamma) = \sup_{\mathcal{P} \in \wp[a,b]} \Lambda(\mathcal{P}, \gamma);$$

if $\Lambda(\gamma) < \infty$, then γ is said to be **rectifiable**.

In various applications of mathematics integrating over curves becomes important. For this reason, we would like to have conditions under which we can determine when a given curve is rectifiable. We close this chapter with a theorem that tells us a condition under which Riemann integration can be used to determine the length of a rectifiable curve.

Theorem 7.3.9 *Suppose that γ is a curve on $[a, b]$ in $\mathbb{R}^n$. If γ' is continuous on $[a, b]$, then γ is rectifiable and*

$$\Lambda(\gamma) = \int_a^b \left|\gamma'(t)\right| dt.$$

Proof. Suppose that γ is a curve on $[a, b]$ in $\mathbb{R}^n$ such that γ' is continuous. From the Fundamental Theorem of Calculus and Theorem 7.1.39, for $\left[x_{j-1}, x_j\right] \subset [a, b]$,

$$\left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right| = \left|\int_{x_{j-1}}^{x_j} \gamma'(t)\, dt\right| \leq \int_{x_{j-1}}^{x_j} \left|\gamma'(t)\right| dt.$$

Hence, for any partition of $[a,b]$, $\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\}$,

$$\Lambda(\mathcal{P}, \gamma) \le \int_a^b \left|\gamma'(t)\right| dt$$

from which it follows that

$$\Lambda(\gamma) \le \int_a^b \left|\gamma'(t)\right| dt \tag{7.20}$$

Let $\varepsilon > 0$ be given. By the Uniform Continuity Theorem, γ' is uniformly continuous on $[a,b]$. Hence, there exists a $\delta > 0$ such that

$$|s-t| < \delta \Rightarrow \left|\gamma'(s) - \gamma'(t)\right| < \frac{\varepsilon}{2(b-a)}. \tag{7.21}$$

Choose $\mathcal{P} = \{x_0 = a, x_1, ..., x_{m-1}, x_m = b\} \in \wp[a,b]$ be such that mesh $\mathcal{P} < \delta$. It follows from (7.21) and the (other) triangular inequality that

$$t \in \left[x_{j-1}, x_j\right] \Rightarrow \left|\gamma'(t)\right| \le \left|\gamma'\left(x_j\right)\right| + \frac{\varepsilon}{2(b-a)}.$$

Thus,

$$\begin{aligned}
\int_{x_{j-1}}^{x_j} \left|\gamma'(t)\right| dt &\le \left|\gamma'\left(x_j\right)\right| \Delta x_j + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&= \left|\int_{x_{j-1}}^{x_j} \left[\gamma'(t) + \gamma\left(x_j\right) - \gamma'(t)\right] dt\right| + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&\le \left|\int_{x_{j-1}}^{x_j} \gamma'(t)\, dt\right| + \left|\int_{x_{j-1}}^{x_j} \left[\gamma\left(x_j\right) - \gamma'(t)\right] dt\right| + \frac{\varepsilon}{2(b-a)} \Delta x_j \\
&\le \left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right| + 2\left(\frac{\varepsilon}{2(b-a)} \Delta x_j\right) \\
&= \left|\gamma\left(x_j\right) - \gamma\left(x_{j-1}\right)\right| + \frac{\varepsilon}{(b-a)} \Delta x_j.
\end{aligned}$$

Summing the inequalities for $j = 1, 2, ...m$ yields that

$$\int_a^b \left|\gamma'(t)\right| dt \le \Lambda(\mathcal{P}, \gamma) + \varepsilon. \tag{7.22}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\int_a^b \left|\gamma'(t)\right| dt \le \Lambda(\mathcal{P}, \gamma)$.

Combining the inequalities (7.20) and (7.21) leads to the desired conclusion. ■

7.4 Problem Set G

1. Let $f(x) = x^2$, $g(x) = \lfloor 2x \rfloor$, and, for $n \in \mathbb{J}$, $\mathcal{P}_n$ denote the partition of $[0,2]$ that subdivides the interval into n subintervals of equal length. Find each of the following.

   (a) $U(\mathcal{P}_3, f)$

   (b) $U(\mathcal{P}_5, g)$

   (c) $L(\mathcal{P}_4, f)$

   (d) $L(\mathcal{P}_6, g)$

2. For $f(x) = 2x^2 + 1$, $\alpha(t) = t + \lfloor 3t \rfloor$ and Δ the subdivision of $[0,1]$ consisting of 4 subintervals of equal length, find $U(\Delta, f, \alpha)$ and $L(\Delta, f, \alpha)$.

3. For $f(x) = 3x$ in $\left[-\frac{1}{2}, 1\right]$, $\alpha(t) = t$, and $\mathcal{P} = \left\{-\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 1\right\}$, find $U(\mathcal{P}, f, \alpha)$ and $L(\mathcal{P}, f, \alpha)$.

4. Suppose that the function f in bounded on the interval $[a,b]$ and $g = kf$ for a fixed negative real number k. Prove that $\sup\limits_{x \in I} g(x) = k \inf\limits_{x \in I} f(x)$.

5. Suppose that the function f in bounded on the interval $I = [a,b]$ and $g = kf$ for a fixed negative integer k. Show that
$$\underline{\int_a^b} g(x)\,dx = k \overline{\int_a^b} f(x)\,dx.$$

6. Suppose that the functions f, f_1, and f_2 are bounded and defined on the closed interval $I = [a,b]$ and α is a function that is defined and monotonically increasing in I. Prove each of the following:

   (a) If $h = f_1 + f_2$, then $\underline{\int_a^b} h(x)\,d\alpha(x) \geq \underline{\int_a^b} f_1(x)\,d\alpha(x) + \underline{\int_a^b} f_2(x)\,d\alpha(x)$

   (b) If $h = f_1 + f_2$, then $\overline{\int_a^b} h(x)\,d\alpha(x) \leq \overline{\int_a^b} f_1(x)\,d\alpha(x) + \overline{\int_a^b} f_2(x)\,d\alpha(x)$

   (c) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then $\underline{\int_a^b} f_1(x)\,d\alpha(x) \leq \underline{\int_a^b} f_2(x)\,d\alpha(x)$

   (d) If $f_1(x) \leq f_2(x)$ for all $x \in I$, then $\overline{\int_a^b} f_1(x)\,d\alpha(x) \leq \overline{\int_a^b} f_2(x)\,d\alpha(x)$.

(e) If $a < b < c$ and f is bounded on $I^* = \{x : a \leq x \leq c\}$ and α is monotonically increasing on I^*, then

$$\overline{\int_a^c} f(x)\,d\alpha(x) = \overline{\int_a^b} f(x)\,d\alpha(x) + \overline{\int_b^c} f(x)\,d\alpha(x).$$

7. Suppose that f is a bounded function on $I = [a, b]$. Let $M = \sup_{x\in I} f(x)$, $m = \inf_{x\in I} f(x)$, $M^* = \sup_{x\in I} |f(x)|$, and $m^* = \inf_{x\in I} |f(x)|$.

   (a) Show that $M^* - m^* \leq M - m$.

   (b) If f and g are nonnegative bounded functions on I, $N = \sup_{x\in I} g(x)$, and $n = \inf_{x\in I} g(x)$, show that

   $$\sup_{x\in I}(fg)(x) - \inf_{x\in I}(fg)(x) \leq MN - mn.$$

8. Suppose that f is bounded and Riemann integrable on $I = [a, b]$.

   (a) Prove that $|f|$ is Riemann integrable on I.

   (b) Show that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

9. Suppose that f and g are nonnegative, bounded and Riemann integrable on $I = [a, b]$. Prove that fg is Riemann integrable on I.

10. Let $A = \left\{\frac{j}{2^n} : n, j \in \mathbb{J} \wedge j < 2^n \wedge 2 \nmid j\right\}$ and

$$f(x) = \begin{cases} \dfrac{1}{2^n} & \text{, if } \quad x \in A \\ 0 & \text{, if } \quad x \in [0, 1] - A \end{cases}.$$

    Is f Riemann integrable on $[0, 1]$. Carefully state and prove your conclusion.

11. Let $f(x) = x^2$ and $\alpha(t) = \lfloor 3t \rfloor$ where $\lfloor .. \rfloor$ denotes the greatest integer function.

(a) For the partition $\mathcal{P} = \left\{ x_j = \frac{2j}{3} : j = 0, 1, 2, 3 \right\}$ with associated subdivision $\Delta = \{I_1, I_2, I_3\}$, find $U(\mathcal{P}, f, \alpha)$.

(b) If $\mathcal{P}^* = \left\{ u_j = \frac{j}{3} : j = 0, 1, 2 \right\} \cup \left\{ u_{j+2} = \frac{2}{3} + \frac{2j}{9} : j = 1, 2, 3, 4, 5, 6 \right\}$ and Δ^* denotes the associated subdivision of $[0, 2]$, then $\mathcal{P}^*$ is a refinement of $\mathcal{P}$. For each $k \in \{1, 2, 3\}$, let $\Delta(k)$ be the subdivision of I_k consisting of the elements of Δ^* that are contained in I_k. Find $L(\Delta(2), f, \alpha)$.

12. For $a < b$, let $\mathcal{C}([a, b])$ denote the set of real-valued functions that are continuous on the interval $I = [a, b]$. For $f, g \in \mathcal{C}([a, b])$, set

$$d(f, g) \underset{def}{=} \int_a^b |f(x) - g(x)|\, dx.$$

Prove that $(\mathcal{C}([a, b]), d)$ is a metric space.

13. If f is monotonically increasing on an interval $I = [a, b]$, prove that f is Riemann integrable. Hint: Appeal to the Integrability Criterion.

14. For nonzero real constants $c_1, c_2, ..., c_n$, let $f(x) = \sum_{j=1}^{n} c_j \lfloor x \rfloor \chi_{[j, j+1)}(x)$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function and χ denotes the characteristic function on $\mathbb{R}$. Is f Riemann integrable on $\mathbb{R}$? Carefully justify the position taken; if yes, find the value of the integral.

15. Prove that if a function f is "R" integrable (see Remark 7.1.27) on the interval $I = [a, b]$, then f is Riemann integrable on I.

16. Suppose that f and g are functions that are positive and continuous on an interval $I = [a, b]$. Prove that there is a number $\zeta \in I$ such that

$$\int_a^b f(x)\, g(x)\, dx = f(\zeta) \int_a^b g(x)\, dx.$$

17. For $a < b$, let $I = [a, b]$. If the function f is continuous on $I - \{c\}$, for a fixed $c \in (a, b)$, and bounded on I, prove that f is Riemann integrable on I.

18. Suppose that f in integrable on $I = [a, b]$ and

$$(\exists m)(\exists M)(m > 0 \wedge M > 0 \wedge (\forall x)(x \in I \Rightarrow m \leq f(x) \leq M)).$$

Prove $\dfrac{1}{f(x)} \in \Re$ on I.

19. For $f(x) = x^2 + 2x$, verify the Mean-Value Theorem for integrals in the interval $[1, 4]$.

20. Find $\displaystyle\int_1^{\sin x^3} e^{(t^3+1)} dt$.

21. For $x > 0$, let $G(x) = \displaystyle\int_{\sqrt{x}}^{e^{x^2}} \sin^9 3t\, dt$. Make use of the First Fundamental Theorem of Calculus and the Chain Rule to find $G'(x)$. Show your work carefully.

22. Suppose that $f \in \Re$ and $g \in \Re$ on $I = [a, b]$. Then each of f^2, g^2, and fg are Riemann integrable on I. Prove the Cauchy-Schwarz inequality:

$$\left(\int_a^b f(x)\, g(x)\, dx\right)^2 \leq \left(\int_a^b f^2(x)\, dx\right)\left(\int_a^b g^2(x)\, dx\right).$$

[Note that, for $\alpha = \int_a^b f^2(x)\, dx$, $\beta = \int_a^b f(x)\, g(x)\, dx$, and $\gamma = \int_a^b g^2(x)\, dx$, $\alpha^2 x + 2\beta x + \gamma$ is nonnegative for all real numbers x.]

23. For $f(x) = \ln x = \int_1^x \dfrac{dt}{t}$ for $x > 0$ and a and b positive real numbers, prove each of the following.

(a) $\ln(ab) = \ln(a) + \ln(b)$,

(b) $\ln(a/b) = \ln(a) - \ln(b)$,

(c) $\ln(a^r) = r \cdot \ln(a)$ for every rational number r,

(d) $f'(x) = \dfrac{1}{x}$,

(e) f is increasing and continuous on $I = \{x : 0 < x < +\infty\}$, and

(f) $\dfrac{1}{2} \leq \ln(2) \leq 1$,