I. For each of the following functions, find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

(10 points)
(a) $f(x,y) = [\sin(x^2) + \cos(y^2)]^5$

(10 points)
(b) $f(x) = \ln(\tan \frac{y}{x})$

II. (10 points) Find all fourth roots of $-16$. State the answers in the form $z = a + bi$.

III. (15 points) The radius and the height of a right circular cone are measured at $a$ and $b$ respectively, with a possible error of measurement of at most 2% in each. Use differentials to estimate the maximum percentage error in the measured volume of the cone, $V = \frac{1}{3} \pi a^2 b$.

V. Suppose that the density $\sigma$ of a ball of radius 4 is proportional to the distance from its center, so that $\sigma(x,y,z) = \alpha \sqrt{x^2 + y^2 + z^2}$, for some constant $\alpha$.

(a) (10 points) Compute the mass of the top half of the ball (that is, $z \geq 0$).

(b) (10 points) Compute the center of mass of the region in part (a) above.
VI. (15 points) Evaluate the integral $\int_{0}^{3} \int_{\frac{x}{3}}^{1} x^2 \, dx \, dy$.

(Note: $\arcsin y = \sin^{-1} y$.)

VII. For each of the following, determine convergence or divergence. Justify your answers.

(a) (10 points) $\sum_{n=1}^{\infty} \frac{n^2 + 2}{n^5 - 3}$

(b) (10 points) $\sum_{n=1}^{\infty} \frac{\sin^3 n}{n^{4/3}}$

VIII. For each of the following, determine convergence or divergence. Justify your answers. If a series converges, give its sum.

(a) (10 points) $\sum_{n=1}^{\infty} \frac{(n + 4)!}{8^n}$

(b) (10 points) $\sum_{n=1}^{\infty} \frac{(-1)^n}{(3 + \cos n\pi)^n}$

IX. (15 points) Find all real values of $x$ for which the following converges. Give your reasons.

$\sum_{n=2}^{\infty} \frac{2^n (x + 1)^n}{n (2n)}$
X. (a) (5 points) Write \( f(x) = \frac{1}{1 + x^2} \) as a geometric series.

(b) (5 points) Use your answer in (a) to find a power series for \( g(x) = \arctan x \).

(c) (5 points) Find the interval of convergence for the series in part (b).

(d) (10 points) Since \( \arctan \left( \sqrt{3} \right) = \frac{\pi}{3} \), can the series in part (b) be used with \( x = \sqrt{3} \) to find a series that converges to \( \frac{\pi}{3} \)? If so, do it. If not, use a different value of \( x \) and find a series that converges to \( \frac{\pi}{3} \).

XI. (20 points) Find by use of calculus the point on the plane \( 3x + 4y - z = 26 \) which is nearest the origin.

**MAT21C Final - from F'93**

**Problem 1**: Determine if the following series converge or diverge. State which tests you are using.

a) \( \sum_{n=1}^{\infty} \left( \frac{n+3}{n-1} \right)^n \frac{1}{n^{3/2}} \) (5 points)

b) \( \sum_{n=1}^{\infty} \left( \frac{n+1}{n} \right)^n \frac{\sqrt{n}}{3^n} \) (5 points)
Problem 2:

a) Find the radius of convergence of the power series $\sum_{n=2}^{\infty} \frac{(x-4)^n}{3^n \ln n}$ (5 points)

b) Find the interval of convergence of the series in a). (5 points)

c) Test for convergence at the upper endpoint of the interval of convergence. (5 points)

d) Test for convergence at the lower endpoint of the interval of convergence. (5 points)

Problem 3: Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n(\ln n)^p}$ converges absolutely, conditionally, or diverges for

a) $p = 1$ (5 points)

b) $p > 1$ (5 points)

c) $0 < p < 1$ (5 points)

Problem 4: Use Maclaurin series only to evaluate

$$\lim_{x \to 0} \frac{\cos x - e^{-x^2/2}}{x^4}$$

HINT: Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$. (10 points)
Problem 5: Find all 5 fifth roots of \( z = -32 \). Express your answers in polar form. (10 points)

Problem 6: Let \( f(x, y) = \ln(x^2 + y^2) \). Calculate \( f_{xx} \) and \( f_{yy} \) and show that this function satisfies the differential equation

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \quad (10 \text{ points})
\]

Problem 7: Let \( z = f(x, y) \), where \( x = u^2 - v^2 \) and \( y = v^2 - u^2 \). Show that

\[
u \frac{\partial z}{\partial v} + u \frac{\partial z}{\partial u} = 0. \quad (10 \text{ points})
\]

Problem 8: Find the maximum and minimum of \( f(x, y) = 2x^2 - xy - x \) on the region bounded by the triangle with vertices \((0,0)\), \((1,0)\) and \((1,1)\). Show all your work.

(20 points)

Problem 9: Suppose that \( z = f(x, y) \) and that \((a, b)\) is a critical point of \( f \). Suppose that \( f_{xx}(a, b) = 3 \) and \( f_{yy}(a, b) = 12 \). For what values of \( k = f_{xy}(a, b) \) do we know that \( f \) has a minimum? Maximum? Saddlepoint? For what values of \( k \) can we not tell from this information what type of critical point we have. (10 points)

Problem 10: By evaluating different paths of approach, prove that

\[
\lim_{{(x, y) \to (0,0)}} \frac{2x^3y}{x^6 + y^2} \text{ does not exist.} \quad (10 \text{ points})
\]
Problem 11: Replace the double integral

\[ \int_{0}^{1} \int_{x}^{2x} \sqrt{1 + y^2} \, dy \, dx \]

with an equivalent one in which the order of integration is reversed. **DO NOT EVALUATE THE INTEGRAL** (10 points)

Problem 12: Let \( R \) be the region bounded by the trapezoid whose vertices are \((1,0), (5,0), (5,1), \) & \((2,1)\). Set up but do **not** evaluate the double integrals necessary to evaluate each of the following, using the density function \( \sigma(x, y) = \sqrt{xy} \).

a) Mass of \( R \). (5 points)

b) Center of Mass \((\bar{x}, \bar{y})\). (10 points)

Problem 13: Evaluate the double integral

\[ \int_{0}^{1} \int_{0}^{y} \frac{1}{(x^2 + 1)(y^2 + 1)} \, dx \, dy \] (10 points)

Problem 14: Evaluate

\[ \int_{0}^{2} \int_{0}^{\sqrt{1-(x-1)^2}} \frac{2}{\sqrt{4-x^2-y^2}} \, dy \, dx \]

**Hint** Use polar coordinates (10 points)

Problem 15: Setup the integral for the volume of the region bounded below by the curve \( z = \sqrt{x^2 + y^2} \) and above by the sphere \( x^2 + y^2 + z^2 = 4 \),

a) In spherical coordinates (5 points)

b) In cylindrical coordinates (5 points)
**Problem 16**: Use cylindrical coordinates to set up the integrals for the mass of the solid under the plane \( z - x - y = 4 \) and above the region in the \( xy \) plane inside the circle \( r = 2 \) and outside the circle \( r = 2 \sin \theta \). Use the density function \( \delta(x, y, z) = x^2 + y^2 + z^2 \). (10 points)

**Problem 17**: Convert the integral

\[
\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x^2+y^2+z^2)^{3/2}} \, dx \, dy \, dz
\]

to spherical coordinates and solve. (10 points)

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MAT21C Final Exam — from W'95

#1 (15 points)
Find the equation for the surface of revolution obtained by rotating

the hyperbola \( \frac{z^2}{4} - y^2 = 1 \) about the \( z \)-axis. Identify the resulting quadric surface by name.

#2 (20 points)
(a) Let \( f \) be a function that depends upon three variables \( x, y, \) and \( z \). Let \( x, y, \) and \( z \) each depend upon two other variables \( u \) and \( v \). Finally, suppose that \( u \) and \( v \) are functions of another variable \( t \).

Draw an appropriate tree diagram and find an expression for

\[
\frac{df}{dt}
\]

calculating using the chain rule for partial derivatives.

(b) Use the above to calculate \( \frac{df}{dt} \) (the derivative of \( f \) with respect to \( t \) evaluated at \( t=1 \)) Where

\[
\begin{align*}
\dot{f}(x, y, z) &= xy + z \cos(y) \\
x(u, v) &= uv \\
y(u, v) &= u - v \\
z(u, v) &= 3u + 2v \\
u(t) &= t^2 \\
v(t) &= 3t - 2.
\end{align*}
\]
#3 (20 points)
(a) Find and classify all critical points for the function
\[ f(x, y) = x^6 - 6x + y^3 - 3y^2 + 5. \]
(b) Find the maximum and minimum values of this function over the rectangle with vertices at (0,0), (0,2), (1,0), and (1,2). Be sure to specify the points where these values occur.

#4 (15 points)
Let \( R \) be the region in the first quadrant bounded by the cardioid \( r = 1 + \cos(\theta) \). Set up, but Do Not Evaluate a double integral in polar coordinates to find the first moment of \( R \) about the \( x \)-axis where the density function is given by \( \sigma(x, y) = xy \).

#5 (20 points)
(a) Use spherical coordinates and a triple integral to prove that the volume of a sphere of radius \( k \) is \[ \frac{4\pi k^3}{3}. \]
(b) Integrate the function \( f(x, y, z) = x^3 - y^5 + z + 3 \) over the region bounded by a sphere of radius 2 centered at the origin.

#6 (20 points)
Use a triple integral to find the volume of the region that is above the \( xy \) plane, and between the paraboloid \( z = 9 - x^2 - y^2 \) and the sphere of radius 3.

#7 (20 points)
Calculate the 4th degree Taylor polynomial for \( f(x) = \sqrt{x} \) at \( x = 9 \).
#8 (5 points each)
Determine if the following series converge absolutely, converge conditionally, or diverge:

(a) \[ \sum_{n=1}^{\infty} \frac{(1)(3)(5)\ldots(2n-1))}{n!} \]

(b) \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{2n^2 + 4n + 1} \]

(c) \[ \sum_{n=1}^{\infty} \frac{1}{5^{n+c(n-1)^2}} \]

#9 (15 points)
(a) Use a 3rd degree Taylor polynomial for \( e^x \) expanded about \( x=0 \) to approximate the value of \( e^{i\theta} \). (You do not need to simplify the result).

(b) Use the fact that \( e^{i\theta} < 2 \) and the Lagrange remainder formula to give a bound on the error (remainder) in the above approximation.

#10 (20 points)
Find the center and radius of convergence of the following power series:
\[ \sum_{n=1}^{\infty} e^{2n} \left( \frac{n}{n+1} \right)^n (x+1)^n \]

#11 (15 points)
Use the Maclaurin series for \( \sin(x) \) and \( \cos(x) \) to find the first two non-zero terms of the Maclaurin series for \( \tan(x) \). Does this series have an infinite radius of convergence? Why or why not? (You do NOT need to use the Lagrange form of the remainder to answer this)

#12 (15 points)
Find 4 distinct 4th roots of \( z = 8 + 8\sqrt{3}i \). Express your answers in the form \( w = re^{i\theta} \).
1. (10 points) Let \( f(x, y) = \frac{\cos x}{y} + \ln(x^2y^3 + e^x) \). Compute \( f_x \). Do not simplify your answer.

2. (12 points) Show that the following limit does not exist as \((x, y) \to (0, 0)\) by considering different paths of approach.
\[
\lim_{(x,y) \to (0,0)} \frac{x^2y + y^2}{x^4 + y^2}.
\]

3. (12 points) Evaluate the following triple integral using spherical coordinates:
\[
\int \int \int_{R} e^{(x^2 + y^2 + z^2)^{3/2}} \, dV
\]
where \( R \) is the solid ball of radius 2 centered at \((0, 0, 0)\).

4. (12 points) Evaluate the double integral or an equivalent one.
\[
\int_{0}^{4} \int_{0}^{2} \frac{9y^2}{\sqrt{1 + x^7}} \, dx \, dy.
\]

5. a) (10 points) Find the critical points of the function
\[
f(x, y) = x^4 - x^2y + \frac{3}{4}y^2 - 2y + 5.
\]
b) (8 points) Use part a) to determine all relative maximum, relative minimum, and saddle points.

6. Let \( S \) be the solid region bounded below by the \( xy \)-plane, on the side by the cylinder \( x^2 + y^2 = 8 \), and above by the paraboloid \( z = 100 - x^2 - y^2 \).

a) (12 points) Set up BUT DO NOT EVALUATE a triple integral in rectangular coordinates which represents the volume of the solid \( S \).

b) (12 points) Set up BUT DO NOT EVALUATE a triple integral in cylindrical coordinates which represents the volume of the solid \( S \).
7. (9 points) Express the complex number $z = (1 + i)^{50}$ in form $z = a + bi$, where $a$ and $b$ are appropriately chosen real numbers.

8. (12 points) Determine if the series

$$
\sum_{n=1}^{\infty} \frac{(-1)^n \sin(e^{\sqrt{n}})}{\sqrt{n^3 + 10}}
$$

converges absolutely, conditionally or diverges. State clearly the reasons for your conclusions.

9. Determine if the following series converge or diverge. State clearly what tests you are using, justifying the use of any particular test you wish to apply.

a) (10 points) $\sum_{n=5}^{\infty} \frac{2}{n(\ln 4n)^4}$

b) (10 points) $\sum_{n=2}^{\infty} \frac{(\ln n)^5}{n^{5/4}}$

10. (12 points) Find the interval and radius of convergence of the following power series. (Do not check convergence at the endpoints of the interval):

$$
\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (x + 1)^{n-1}
$$

11. (15 points) Compute the entire Taylor series for $f(x) = \frac{1}{2 - x}$ centered at $c = 1$. Simplify your answer as much as possible. (To receive full credit you must indicate the form of the nth term.)

12. (12 points) Use a Taylor polynomial of degree 4 for $f(x) = \frac{e^{-4x^2}}{x^2} - \frac{1}{x^2}$ to approximate the value of the definite integral

$$
\int_{0}^{1/2} \frac{e^{-4x^2}}{x^2} - 1 \, dx .
$$
13. (12 points) Let \( w = f(u, v) \) be a function whose derivatives of all orders exist.

Suppose that

\[
\frac{\partial^2 f}{\partial u^2}(3, 0) = -3
\]
\[
\frac{\partial^2 f}{\partial u \partial v}(3, 0) = 3
\]
\[
\frac{\partial^2 f}{\partial v^2}(3, 0) = -1
\]

If \( u = y + e^{2x} \) and \( v = xy \), what is the value of \( \frac{\partial^2 w}{\partial y^2} \) evaluated at the point \((x, y) = (0, 2)\)?

14. Let \( \{a_n\} \) be a sequence of non-zero real numbers such that \( \lim_{n \to \infty} a_n = 3 \).

a) (10 points) What can you conclude (if anything) about the convergence or divergence of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{a_n} \)? EXPLAIN.

b) (10 points) What can you conclude (if anything) about the convergence or divergence of the series \( \sum_{n=1}^{\infty} (a_n - 3) \)? EXPLAIN.
1. (15 points) Let \( f(x, y) = 1 + \sqrt{x^2 + y^2} \).

   a) Sketch the graph of \( f \), and show at least three different curves obtained by intersecting the graph with planes.

   b) Explain how the graph of \( f \) can be generated by a curve via rotation. Give an equation for such a curve.

2. (15 points) Let \( R \) be the annular region in the plane enclosed between the circles of radius 1 and 2 centered at the origin. Suppose the density per unit area of the annulus is \( |xy| \). Find the mass of the annulus.

3. (25 points) Let \( S \) be the solid right circular cone of radius \( a \) and height \( h \), with axis along the \( z \)-axis, as in the picture at the right. For each point \( P \) in \( S \) let \( f(P) \) be the square of the height of \( P \) above the \((x, y)\)-plane.

   a) Using cylindrical coordinates, calculate \( \iiint_S f(P) \, dV \).

   b) Set up BUT DO NOT EVALUATE a triple integral in spherical coordinates for \( \iiint_S f(P) \, dV \).

4. (20 points) Define a function \( f(x, y) \) by

\[
f(x, y) = \begin{cases} 
\sin\left(\frac{x^2 + xy^2}{x^2 + y^2}\right) & \text{if } (x, y) \neq (0, 0) \\
1 & \text{if } (x, y) = (0, 0) 
\end{cases}
\]

   a) Compute \( f_x(x, y) \) when \((x, y) \neq (0, 0)\). (Do not simplify your answer.)

   b) Does \( f_x(0, 0) \) exist? If it doesn't exist, explain why. If it does exist, compute it.
5. (15 points) Let \( z = g(x^2 - y^2, y^2 - x^2 + 3) \), where \( g \) is a differentiable function of two variables. Show that
\[
y \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0.
\]

6. (40 points) In each of the following four parts, determine if the given series converges absolutely, converges conditionally, or diverges, and justify your answers.

   a) \[ \sum_{n=0}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdots (4n+1)}{5 \cdot 8 \cdot 11 \cdots (3n+5)}. \]

   b) \[ \sum_{n=1}^{\infty} \frac{5 + \sin\left(\frac{1}{n}\right)}{n^2}. \]

   c) \[ \sum_{n=1}^{\infty} (-1)^n \left( \sqrt{n} + 1 - \sqrt{n} \right). \]

   d) \[ \sum_{n=1}^{\infty} 2\left(\frac{-2}{n^3}\right). \]

7. (15 points) Determine the interval of convergence of the power series
\[
\sum_{n=1}^{\infty} (-1)^n \frac{(x - 2)^n}{n4^n}.
\]

8. (15 points) Let \( f(t) = \frac{1}{2}(e^t + e^{-t}) \). Use a Maclaurin series representation for \( f(t) \) to obtain a power series for \( \int_0^x f(t) \, dt \). Express your answer in summation notation.

9. (20 points) Find the minimum cost of a closed rectangular box with volume 48 cubic feet, where the front and back of the box cost one dollar per square foot, the top and bottom of the box cost two dollars per square foot, and the two ends of the box cost three dollars per square foot.
10. (20 points) The following is an open ended question. The grade will be based on the quality of writing and clarity of discussion as well as mathematical content and thoroughness.

Let \( \{a_n\}_{n=1}^{\infty} \) be a sequence and suppose there are real constants \( m \) and \( M \) with \( m > 0, M > 0, \) and \( m < a_n < M \) for each \( n \).

a) What can be said about the convergence or divergence of \( \{a_n\}_{n=1}^{\infty} \)?

b) What can be said about the convergence or divergence of \( \sum_{n=1}^{\infty} a_n \)?

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**SPRING 1990**

I. **Determine convergence or divergence for the following series. Be sure to justify your answer.**

a) (10 points) \( \sum_{n=1}^{\infty} \frac{n^6}{6^{n+1}} \)

b) (10 points) \( \sum_{n=2}^{\infty} \frac{1}{n(n\ln n)} \)

II. **Using an appropriate test, determine if each of the following series is absolutely convergent, conditionally convergent, or divergent.**

a) (10 points) \( \sum_{n=1}^{\infty} \frac{(-1)^n (3n^2 + 1)}{5n^2 + 3} \)

b) (10 points) \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(n + 5)^2} \)

III. (20 points) Determine all values of \( x \) for which the following series converges.

\[
\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{n^2 + 4}
\]

IV. (20 points) Using polar coordinates, evaluate \( \int_{0}^{\pi/2} \int_{0}^{x} \frac{1}{x^2 + y^2 + 1} \, dx \, dy \).

V. (15 points) Find the critical points of the function \( f(x, y) = 2y^4 + x^2 - 2xy \). Then determine whether each critical point corresponds to a local maximum, a local minimum, or a saddle point.
VI. (15 points) Find a function \( f(x,y) \) such that \( \frac{\partial f}{\partial x}(x,y) = \cos y - \cos x \) and \( \frac{\partial f}{\partial y}(x,y) = \cos y - x \sin y \).

VII. a) (15 points) Set up, but do not evaluate, a repeated integral in cylindrical coordinates for the mass of the solid bounded above by the plane \( z = 4 \) and below by the paraboloid \( z = x^2 + y^2 - 5 \), if the density at any point \( P \) is the distance from \( P \) to the \( z \)-axis.

VII. b) (15 points) Set up, but do not evaluate, a repeated integral in spherical coordinates for the volume of the region inside the sphere \( x^2 + y^2 + z^2 = 10z \) and above the cone \( z = 4 \sqrt{x^2 + y^2} \).

VIII. a) (15 points) Approximate \( \int_0^{0.3} \cos x^5 \, dx \) using the first 3 nonzero terms of a Maclaurin series. Leave answer without carrying out the arithmetic in your last step.

b) (5 points) Find an upper bound for the error in your approximation. Leave answer without carrying out the arithmetic.

IX. (20 points) Consider the region in the first quadrant bounded by the graphs of \( y = 0, y = \sqrt{x}, \) and \( x = -y^2 + 1 \). Use polar coordinates to describe the region in the form \( \theta_1 \leq \theta \leq \theta_2 \) and \( r_1(\theta) \leq r \leq r_2(\theta) \).

X. (20 points) Let \( \mathbf{v} = f(x - y, y - z, z - x) \). Show that \( \frac{\partial \mathbf{v}}{\partial x} + \frac{\partial \mathbf{v}}{\partial y} + \frac{\partial \mathbf{v}}{\partial z} = 0 \).

1. (40 points) For each of the following series, determine convergence or divergence, stating the tests you use, and carefully justifying your answers. (Do NOT test for absolute convergence.)

   a) \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \)

   b) \( \sum_{n=1}^{\infty} \frac{1/n - 3}{n!} \)

   c) \( \sum_{n=1}^{\infty} \frac{\sin \sqrt{n}}{\sqrt{n^3 + 1}} \)

   d) \( \sum_{n=2}^{\infty} \left[ \frac{-1}{\ln n} \right]^n \)
2. (15 points) Find the radius of convergence for the power series
\[ \sum_{n=1}^{\infty} \frac{(2x + 1)^n}{n^2} \].

3. (10 points) Determine the convergence or divergence of the following sequence. If the sequence converges, give its limit:
\[ \left[ \frac{n^2}{2n - 1} - \frac{n^2}{2n + 1} \right] \cdot \]

4. (15 points) Find the five fifth roots of \(-4 - 4i\), leaving your answers in \(re^{i\theta}\) form.

5. (10 points) Let
\[ u(x, y) = \ln(x^2 + y^2) \]
and
\[ v(x, y) = 2 \arctan \left( \frac{y}{x} \right) \].

Show that \(u\) and \(v\) satisfy the equation:
\[ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \].

6. (20 points)
   a) Use the Maclaurin series for \(\cos x\) to state the first three nonzero terms of the Maclaurin series for \(\frac{1 - \cos x}{x^2}\).
   b) Use your answer in part (a) to estimate
\[ \int_{0}^{1} \frac{1 - \cos x}{x^2} \, dx \].
   c) Give a numerical bound on the error in the answer in part (b), explaining your reason for doing so.

7. (20 points) A jeweler wishes to make a rectangular pill box such that the top and bottom will be copper plated and the sides silver plated. If silver plating costs 6 dollars per \(\text{cm}^2\) and copper plating costs 2 dollars per \(\text{cm}^2\), what are the dimensions of the least costly pill box having \(\frac{8}{3}\) \(\text{cm}^3\) as its volume? Justify your answer.
8. (20 points) Use polar coordinates to evaluate the following integral:
\[ \int_{-3}^{3} \int_{0}^{\sqrt{9 - y^2}} \sin(x^2 + y^2) \, dx \, dy. \]

9. (20 points) Let \( R \) be the three dimensional region which is bounded by the parabolic cylinder \( x = y^2 \) and the planes \( z = 0 \) and \( x + z = 1 \) (see graph). Find the volume of \( R \).

10. (15 points) Rewrite the integral
\[ \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{r} r^2 \, dz \, dr \, d\theta \]
as an integral in spherical coordinates. (Do not evaluate.)

11. (15 points) Let \( \{a_n\} \) be the sequence for which \( a_1 = \sqrt{1} \) and \( a_{n+1} = \sqrt{1 + a_n} \) for \( n \geq 1 \).
   a) Find \( a_2, a_3, \) and \( a_4 \).
   b) It can be shown that the sequence \( \{a_n\} \) defined above converges. Assuming this to be true, find its limit \( L \). (HINT: \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} a_{n+1} = L \).)

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Final Exam - March 19, 1982
Mathematics 21C

I. (20 points) Find the point on \( z = xy + 1 \) that is nearest to the origin.
II. (20 points) Set up the integral that would give the volume of the solid \( S \) that remains when the hemisphere \( x^2 + y^2 + (z-8)^2 \leq 16, \ z \leq 8 \), is removed from the cylinder \( x^2 + y^2 \leq 16, \ 0 \leq z \leq 8 \). USE CYLINDRICAL COORDINATES AND AND DO NOT EVALUATE THE INTEGRAL.

III. (20 points) Find the volume of the solid \( S \) whose base is bounded by the curve \( xy = \pi \) and the lines \( x = 0, \ x = \pi, \ y = 0 \) and whose height at a point \((x,y)\) is given by \( z = x^2 \sin xy \).

IV. (20 points) Consider the solid \( S \) obtained by removing the upper branch of the cone bounded by \( x^2 + y^2 = 3z^2 \) from the hemisphere \( x^2 + y^2 + z^2 \leq 25, \ z \geq 0 \). If the density at a point \( P = (x,y,z) \) is inversely proportional to the square of the distance from the origin to \( P \), USE SPHERICAL COORDINATES to find the moment of inertia of \( S \) about the z-axis.
V. (32 points) Determine whether each of the following converges (absolutely or conditionally) or diverges. **JUSTIFY** the positions taken.

(a) \[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \]

(b) \[ \sum_{n=1}^{\infty} (-1)^n \cos \left( \frac{n}{n^2} \right) \]

(c) \[ \sum_{k=2}^{\infty} \frac{(-1)^k}{k \sqrt{k/n^2}} \]

(d) \[ \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k} (\sqrt{k} + 1)} \]

VI. (28 points) **TRUE** or **FALSE**. **Justify** the position taken.

(a) \[ \sqrt{2} = -1, \quad (2+3i)^2 = 3 + \frac{2(2+10i)}{1-i} \]

(b) \[ \text{If a function } y = f(x) \text{ has continuous derivatives, } \]
\[ f(0) = f'(0) = f''(0) = 0 \text{ and } 4 \leq f'''(x) \leq 12 \text{ for all } x, \]
\[ -1 \leq x \leq 1, \text{ then } \frac{3}{2} x^3 \leq f(x) \leq 2x^3 \text{ for all } x \text{ in } [-1,1]. \]

(c) \[ \text{Suppose } z = f(x,y) \text{ has continuous first and second partials.} \]
\[ \text{If } f_x(a,b) = f_y(a,b) = 0, \ f_{xx}(a,b) < 0 \text{ and } [f_{xy}(a,b)]^2 < f_{xx}(a,b) f_{yy}(a,b), \text{ then } f \text{ has a local maximum at } (a,b). \]

(d) \[ \text{Given } \sum_{k=1}^{\infty} a_k, \ a_k \text{ real. If } a_k \leq \frac{1}{k^2} \text{ for all } k, \text{ then the } \]
\[ \text{series is convergent by the comparison test.} \]
VII. (20 points) Water for a city water system is stored in a spherical water tank whose base is 60 feet above the ground and whose top is 100 feet above the ground. If water is pumped from the bottom of a well 120 feet deep, find the work done to fill the tank. (Water weighs 62.4 lbs/cu. ft and the volume of a sphere with radius $r$ is given by $\frac{4}{3} \pi r^3$.)

VIII. (20 points) Find the Taylor series expansion about $x = 2$ for

$$f(x) = (x+7)^{-2/3}.$$ 

IX. (20 points) Given $z = f(u, v)$ where $u = x + 2y^2$ and $v = ye^{x^2y}$, assume that $f$ has continuous first and second partials. Find $z_x$ and $z_{xy}$ in terms of functions of $x$ and $y$ and the partials of $f$. 

\[ z_x = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad z_{xy} = \frac{\partial^2 z}{\partial u \partial x} \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial y} \frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v \partial x} \frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial y} \frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v \partial y} \frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v \partial y} \frac{\partial v}{\partial y}. \]
NOTE: A key for this one is posted in the glass cabinets that are next to 017 Wellman Hall.

I. (12 points) For each of the following, show your work in the space that is provided and write your final answers in the boxes that are provided. Correct answers will count ONLY if some work is shown. Incorrect answers could receive partial credit based on the work that is shown.

(a) For \( f(x, y) = (x^2 + 2y^3), x = g(u) = e^{3u}, y = h(u, v) = 2u^3 + 5v^2, \) \( u = p(t) = \sin(2t^4), \) and \( v = q(t) = t^2, \) find \( \frac{\partial f}{\partial t}. \)

(b) Find the first four terms of the Maclaurin series for \( e^{-x^2} \ln(1 + x). \)

II(A). (8 points) Define each of the following.

(i) For a real-valued function of two real variables \( z = f(x, y) \) that is defined in a disk centered at the point \((a, b)\) except possibly at the point \((a, b)\), \( \lim_{(x,y)\to(a,b)} f(x, y) = L \)

(ii) A power series about \( x = c \)

B. (12 points) For each of the following, simply state the Maclaurin series for the given function and indicate where it is valid.

<table>
<thead>
<tr>
<th>Maclaurin Series Expansion</th>
<th>Where valid</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( \sin x = )</td>
<td></td>
</tr>
<tr>
<td>(b) ( \frac{1}{1 - x} =)</td>
<td></td>
</tr>
<tr>
<td>(c) ( \cos x = )</td>
<td></td>
</tr>
</tbody>
</table>
C. (6 points) In the box to the left of the given equation, provide the name of the surface for which the equation is given in the right column.

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>( \frac{x^2}{4} - \frac{y^2}{16} - \frac{z^2}{9} = 1 )</td>
</tr>
<tr>
<td>(b)</td>
<td>( z - \frac{x^2}{4} - \frac{y^2}{9} = 1 )</td>
</tr>
<tr>
<td>(c)</td>
<td>( \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} = 1 )</td>
</tr>
</tbody>
</table>

III. (a) (5 points) For a sequence of real numbers \( \{a_n\}_{n=1}^{\infty} \), define \( \lim_{n \to \infty} a_n = L \).

(b) (7 points) Use the Theorem on Bounded Monotone Sequences to justify carefully that \( \lim_{n \to \infty} \frac{(12)^n}{n!} \) exists.

(c) (8 points) Use the definition for convergence of a series to show carefully that

\[
\sum_{k=1}^{\infty} \left( \frac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k^2 + k}} \right) = 1.
\]

IV. (A) (10 points) For \( f(x, y) = x^3 + y^3 - 3x - 3y \), classify the extrema and identify all saddle points.

(B) (10 points) Find the maximum volume of a rectangular box that has three faces in the coordinate planes and a vertex in the first octant on the plane \( x + y + z = 6 \).

V. (16 points) Evaluate each of the following. Switch the order of integration or the coordinate system, if necessary. Do only obvious simplifications.

(a) \( \int_{0}^{2} \int_{0}^{\sqrt{x}} \cos(y^3 + 4) \, dy \, dz \, dx \)

(b) \( \int_{-5}^{5} \int_{0}^{\sqrt{25-x^2}} \int_{0}^{\sqrt{25-x^2-y^2}} \sqrt{x^2+y^2+z^2} \, dz \, dy \, dx \)

VI. (36 points) For each of the following SET-UP the integral that would yield what is requested. DO NOT EVALUATE THE INTEGRALS; however, the given integral should be in a form for which all that remains to be carried out is the process of integration; all obvious simplifications should be carried out. Whenever one is needed, the density at a point \( P = (x, y, z) \) in the solid \( S \) is \( \delta(P) = 3xy \).
(a) Let $S$ be the solid that is bounded between the elliptic cylinder $x^2 + 9y^2 = 9$ and the planes $z = 0$ and $z = x + 3$. SET-UP an iterated integral in rectangular coordinates that would yield the volume of $S$.

(b) Let $S$ be the solid contained in the first octant that is bounded above by the paraboloid $z = 4 - x^2 - y^2$ below by the plane $z = 0$ and laterally by the cylinder $x^2 + y^2 = 2x$ and the plane $y = 0$. SET-UP an iterated integral in cylindrical coordinates that would yield the mass of $S$.

(c) Let $S$ be the solid located in the upper hemisphere that is bounded above by the ellipsoid $\frac{x^2}{9} + \frac{y^2}{9} + \frac{z^2}{36} = 1$, below by the sphere $x^2 + y^2 + z^2 = 1$ and laterally by the right circular cone $3z^2 = 4(x^2 + y^2)$. SET-UP an iterated integral in spherical coordinates that would yield the moment of inertia $I_z$ of $S$ about the z-axis.
VII. (16 points) Suppose that \( R \) is the closed square having vertices \((0,0), (0,2), (2,2),\) and \((2,0)\) in the xy-plane. Find the global maximum and global minimum of \[ f(x, y) = x^2 - 3y^2 - 2x + 6y \] over the region \( R \). Carefully justify your conclusion.

VIII. (21 points) Determine whether each of the following converges (absolutely or conditionally) or diverges. Carefully justify your conclusions.

(a) \[ \sum_{k=0}^{\infty} (-1)^k \frac{2^k + k}{5^k (k + 1)!} \]

(b) \[ \sum_{n=0}^{\infty} \left[ \frac{(n + 2)}{(n + 1)} \right]^n \frac{(n + 3)^3}{3^n} \]

(c) \[ \sum_{k=1}^{\infty} (-1)^k k^2 \sin^2 \left( \frac{1}{k} \right) \]

IX. (A) (7 points) Showing your work carefully, find the sets of convergence and divergence of \[ \sum_{n=1}^{\infty} \frac{(-1)^n n}{3^n} (x + 3)^{2n+1} \] and illustrate your conclusions on the number line that is provided.

(B) (18 points) Find the Taylor series expansion for \( f(x) = (3x + 7)^{-1/4} \) about \( x = 3 \).

X. Note that \( \mathbb{C} = \{ x + iy : x, y \in \mathbb{R} \text{ and } i^2 = -1 \} \)

(A) (5 points) Simplify \( (i^3) + 2i (1 + 3i) -(3 + 2i) \) until it is in proper form for membership in \( \mathbb{C} \).

(B) (12 points) Find all of the cube roots of \(-8 + 8i\); graph and label your solutions on the Argand coordinate system provided below.