Self-Help Work Sheets C11: Triple Integration

These problems are intended to give you more practice on some of the skills the chapter on Triple Integration has sought to develop. They do not cover everything so a careful review of the Chapter and your class notes is also in order.

Problems for Fun and Practice

1. For each of the following solids give a description in rectangular coordinates in the order specified:

   (a) S is bounded above by $\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{16} = 1$ and below by $z = 2$. Description: in terms of range on z, then y, then x.

   (b) S is bounded above by $\frac{x^2}{16} + \frac{y^2}{36} + \frac{z^2}{16} = 1$ and below by $\frac{y}{6} + \frac{z}{4} = 1$. Description: in terms of range on z, then x, then y.

2. Each of the following iterated integrals cannot be easily done in the order given. Convince yourself that this is true and then convert each one to an equivalent iterated integral that can be done and evaluate it.

   (a) $$\int_0^2 \int_0^1 \int_y^1 \sinh\left(z^2\right) dz\,dy\,dx$$

   (b) $$\int_0^2 \int_0^4 \int_z^2 yze^{x^3}\,dx\,dy\,dz$$

3. Convert each of the following to an equivalent triple integ4al in cylindrical coordinates and evaluate.

   (a) $$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^6 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2}}$$

   (b) $$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{x^2+y^2}{2}} \frac{z\,dz\,dy\,dx}{\sqrt{x^2+y^2}}$$

4. Convert each of the following to an equivalent triple integral in spherical coordinates and evaluate.

(a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{1+x^2+y^2+z^2}$$

(b) $$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xzdzdydx$$

5. Convert to cylindrical coordinates and evaluate the integral

(a) $\iiint_S \sqrt{x^2+y^2}dV$ where S is the solid in the first octant bounded by the coordinate plane, the plane $z=4$, and the cylinder $x^2+y^2=25$.

(b) $\iiint_S \left(x^2+y^2\right)^{\frac{3}{2}} dV$ where S is the solid bounded above by the paraboloid $z = \frac{1}{2}\left(x^2+y^2\right)$, below by the xy-plane, and laterally by the cylinder $x^2+y^2=4$.

6. Convert to spherical coordinates and evaluate the integral.

(a) $\iiint_S \left(x^2+y^2+z^2\right)^{\frac{3}{2}} dV$ where S is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=25$, the cone $z=\sqrt{x^2+y^2}$, and the cone $z=2\sqrt{x^2+y^2}$.

(b) $\iiint_S \sin\sqrt{x^2+y^2+z^2}dV$ where S is the solid bounded above by the sphere $x^2+y^2+z^2=49$ and below by the cone $z=\sqrt{x^2+y^2}$.

7. Find the coordinates of the center of gravity of the solid S with indicated mass density $\delta = \delta(x,y,z)$. (Choose whichever system of coordinates would be best.)

(a) S: $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1; \delta = 3 - x - y - z$.

(b) S: $x^2 + y^2 \le a^2$, $\frac{b}{a}\sqrt{x^2 + y^2} \le z \le b$ for constants $b > 0$, $a > 0$ and $\delta = x^2 + y^2 + z^2$.

8. Find the centroid for each of the following solids S:

   (a) S: $x^2 + y^2 \le 1, x \ge 0, y \ge 0, 0 \le z \le xy$
   (b) S: $9 \le x^2 + y^2 + z^2, z \ge 0$

9. Find the designated moment for each solid S with density $\delta = \delta(x, y, z)$.

   (a) M_{xy} – the moment relative to the xy-plane where
   S: $0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1$
   $\delta = 3xy$
   (b) M_{xz} – the moment relative to the xz-plane where
   S: $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le xy$
   $\delta = 2(x + y)$
   (c) M_{yz} – the moment relative to the yz-plane where
   S: bounded in the first octant by $x + z = 1$, $x = y$, and the coordinate planes
   $\delta = 5y$

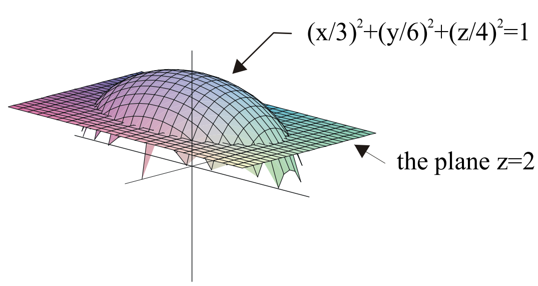
10. Find the moment of inertia about the z-axis for the solid S bounded by $0 \le x \le 1$, $0 \le y \le 1 - x$, $0 \le z \le 5$ with density $\delta = \delta(x, y, z) = 3$.

11. Find the moment of inertia about the y-axis of the solid S in the first octant bounded by $x^2 + z^2 = 1$, $y = x$, $y = 0$, $z = 0$ with density $\delta = 2z$.

12. Find the moment of inertia about the central axes of a homogeneous right circular cylindrical shell with total mass m, inner radius a, outer radius b and height h.

Proposed Solutions/Answers

1. (a)

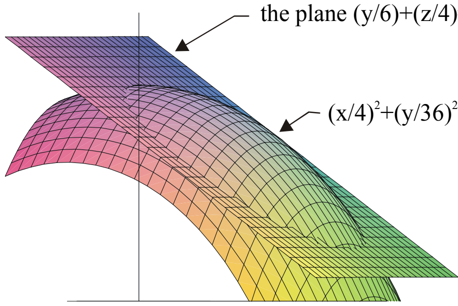


To find the range of x and y we substitute the value of z into the equation $\frac{x^2}{9} + \frac{y^2}{36} + \frac{z^2}{16} = 1$ that will give the largest range for x and y. From inspecting the figure we see that this value is $z = 2$.

For $z = 2$, we have $\dfrac{x^2}{9} + \dfrac{y^2}{36} + \dfrac{(2)^2}{16} = 1$ or $\dfrac{x^2}{9} + \dfrac{y^2}{36} = \dfrac{3}{4}$. From this we obtain:

$$2 \le z \le 4\sqrt{1 - \frac{x^2}{9} - \frac{y^2}{36}}$$
$$-6\sqrt{\frac{3}{4} - \frac{x^2}{9}} \le y \le 6\sqrt{\frac{3}{4} - \frac{x^2}{9}}$$
$$-3 \le x \le 3$$

(b)

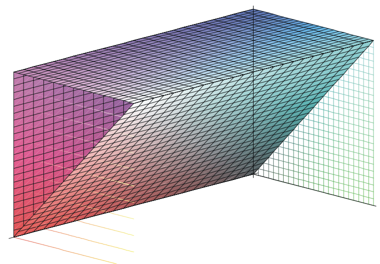


To find the scope of x and y we notice from the figure that the largest range of x and y in the solid occur on the plane $\dfrac{y}{6} + \dfrac{z}{4} = 1$. Thus we solve this for z and substitute this into the equation $\dfrac{x^2}{16} + \dfrac{y^2}{36} + \dfrac{z^2}{16} = 1$. Solving $\dfrac{y}{6} + \dfrac{z}{4} = 1$ for z we get $z = 4\left(1 - \dfrac{y}{6}\right)$. Substitution this value of z into $\dfrac{x^2}{16} + \dfrac{y^2}{36} + \dfrac{z^2}{16} = 1$, we get $\dfrac{x^2}{16} + \dfrac{y^2}{36} + \dfrac{16\left(1 - \frac{y}{6}\right)^2}{16} = 1$ or $\dfrac{x^2}{16} + \dfrac{y^2}{18} + \dfrac{y}{3} = 1$ or $\dfrac{x^2}{16} + \dfrac{(y-3)^2}{18} = \dfrac{1}{2}$. From this we see that y will be maximized when $x = 0$. Thus we obtain the desired description.

$$4\left(1 - \frac{y}{6}\right) \le z \le 4\sqrt{1 - \frac{x^2}{16} - \frac{y^2}{36}}$$
$$-2\sqrt{2}\sqrt{1 - \frac{(y-3)^2}{9}} \le x \le 2\sqrt{2}\sqrt{1 - \frac{(y-3)^2}{9}}$$
$$0 \le y \le 6.$$

2. (a)

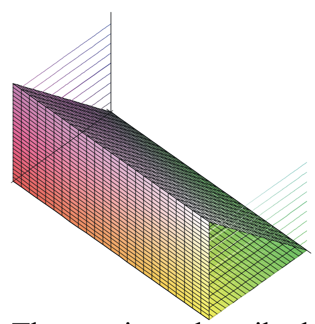
$$\int_0^2\int_0^1\int_y^1 \sinh\left(z^2\right) dzdydx$$
$$= \int_0^2\int_0^1\int_0^z \sinh\left(z^2\right) dydzdx$$
$$= \int_0^2\int_0^1 z\sinh\left(z^2\right) dzdx$$
$$= \int_0^2 \frac{1}{2}\left(\cosh(1) - \cosh(0)\right) dx$$
$$= \cosh(1) - \cosh(0)$$
$$= \boxed{\frac{e^1 + e^{-1}}{2} - 1}$$



The region indicated by the integral is bounded by $z = y$, $y = 0$, $z = 1$, $x = 0$, and $x = 2$ which is indicated by the figure above. The difficulty with integrating the original triple integral is that to easily integrate $\sinh\left(z^2\right)$, we need a zdz rather than just dz. Note that if we switch the dz and dy, we might get a z where we need it.

(b)

$$\int_0^2\int_0^4\int_z^2 yze^{x^3} dxdydz$$
$$= \int_0^2\int_0^4\int_0^x yze^{x^3} dzdydx$$
$$= \int_0^2\int_0^4 y\frac{x^2}{2}e^{x^3} dydx$$
$$= \int_0^2 \left(\frac{y^2}{2}\right]_0^4 \frac{x^2}{2}e^{x^3} dx$$
$$= 4\int_0^2 x^2e^{x^3} dx$$
$$= \frac{4}{3}e^{x^3}\Big]_0^2 = \boxed{\frac{4}{3}\left(e^8 - 1\right)}$$

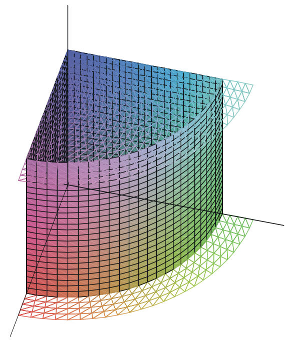


The region described by the integral is bounded by $y = 0$, $y = 4$, $z = 0$, $z = x$, and $x = 2$. A picture of the region is indicated above. In the original integral, if we try to integrate $e^{x^3}dx$ we have a problems. We can easily integrate $x^2e^{x^3}$, so this suggests switching dx and dz.

3. (a)

$$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^6 \frac{dzdydx}{\sqrt{x^2+y^2}}$$

$$= \int_0^{\pi/2} \int_0^5 \int_0^6 \frac{rdzdrd\theta}{r} = 15\pi$$

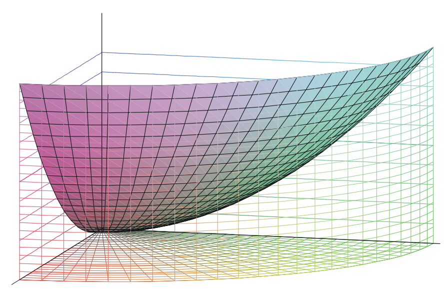


The region described by the integral is the set of all point (x, y, z) satisfying the three inequalities $0 \le z \le 6$, $0 \le y \le \sqrt{25 - x^2}$, and $0 \le x \le 5$. It is $\frac{1}{4}$ of a cylinder as shown in the figure above.

(b)

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{x^2+y^2}{2}} \frac{zdzdydx}{\sqrt{x^2+y^2}}$$

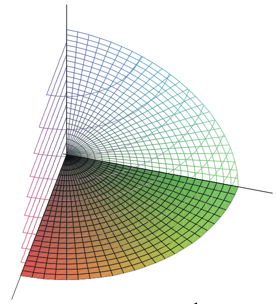
$$= \int_0^{\pi/2} \int_0^2 \int_0^{r^2/2} \frac{zrdzdrd\theta}{r} = \boxed{\frac{2\pi}{5}}$$



The region or solid is in the first octant between the paraboloid and the cylinder as indicated in the picture above. It is the set of all points (x, y, z) satisfying the three inequalities $0 \le z \le \dfrac{x^2 + y^2}{2}$, $0 \le y \le \sqrt{4 - x^2}$, and $0 \le x \le 2$.

4. (a)

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dzdydx}{1+x^2+y^2+z^2}$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{\rho^2 \sin\phi d\rho d\phi d\theta}{1+\rho^2}$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \sin\phi \left[1 - \frac{1}{1+\rho^2}\right] d\rho d\phi d\theta$$
$$= \left(1 - \frac{\pi}{4}\right) \int_0^{\pi/2} (\cos\phi]_0^{\pi/2}\, d\theta$$
$$= \left(1 - \frac{\pi}{4}\right) \frac{\pi}{2} = \boxed{\frac{\pi}{2} - \frac{\pi^2}{8}}$$



The solid is $\frac{1}{8}$ of the sphere with radius 1 centered at the origin as indicated in the figure.

(b)

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xzdzdydx$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 (\rho\sin\phi\cos\theta)(\rho\cos\phi)\,\rho^2 \sin\phi d\rho d\phi d\theta$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 \rho^4 \cos\theta \sin^2\phi \cos\phi d\rho d\phi d\theta$$
$$= \frac{3^5}{5}\int_0^{\pi/2} \cos\theta \left(\frac{\sin^3\phi}{3}\right]_{\phi=0}^{\phi=\pi/2} d\theta = \boxed{\frac{3^4}{5}}$$

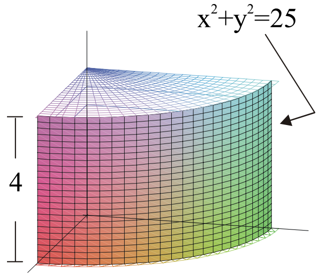
The solid is $\frac{1}{8}$ of the sphere with radius 3 centered at the origin as depicted in the figure in part a.
$x = \rho\sin\phi\cos\theta$
$z = \rho\cos\phi$

5. (a)

$$\iiint_S \sqrt{x^2+y^2}\,dV$$

$$= \int_0^{2\pi}\int_0^{5}\int_0^{4} r\cdot r\,dz\,dr\,d\theta$$

$$= \boxed{\frac{4}{3}(5)^3\,2\pi}$$



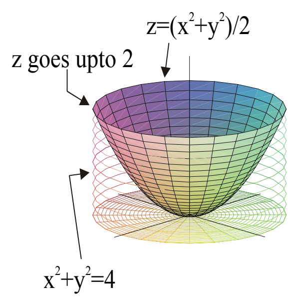
(b)

$$\iiint_S \left(x^2+y^2\right)^{\frac{3}{2}}\,dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_0^{r^2/2} r^3\cdot r\,dz\,dr\,d\theta$$

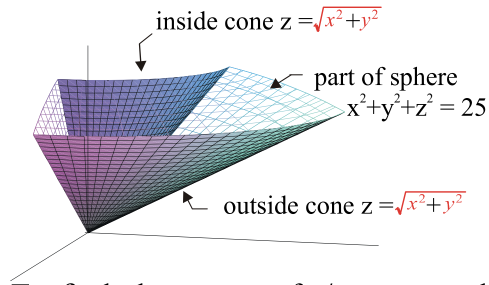
$$= \int_0^{2\pi}\int_0^{2} \frac{r^6}{2}\,dr\,d\theta$$

$$= \int_0^{2\pi} \left.\frac{r^7}{14}\right]_0^2 d\theta = \boxed{\left(\frac{2^6}{7}\right)(2\pi)}$$



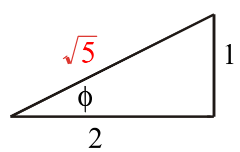
6. (a)

$$\iiint_S \left(x^2+y^2+z^2\right)^{\frac{3}{2}} dV$$
$$= \int_0^{\pi/2} \int_{\tan^{-1}\left(\frac{1}{2}\right)}^{\pi/4} \int_0^5 \rho^3 \cdot \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$
$$= \left(\frac{5^6}{6}\right) \int_0^{\pi/2} \cos\phi \Big]_{\tan^{-1}\left(\frac{1}{2}\right)}^{\pi/4} d\theta$$
$$= \boxed{\left(\frac{5^6}{6}\right)\left(\frac{4-\sqrt{10}}{2\sqrt{5}}\right)\left(\frac{\pi}{2}\right)}$$



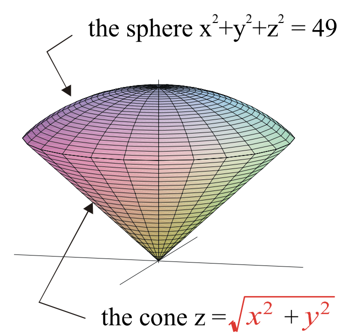
To find the range of ϕ we use the fact that $\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$. For the inside $z = 2\sqrt{x^2+y^2}$, so $\tan\phi = \frac{\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}} = \frac{1}{2}$. For the outside $z = \sqrt{x^2+y^2}$, so $\tan\phi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$.

Aside: If $\tan\phi = \frac{1}{2}$, then $\cos\phi = \frac{2}{\sqrt{5}}$.



(b)

$$\iiint_S \sin\sqrt{x^2+y^2+z^2}\, dV$$
$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^7 (\sin\rho)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= \boxed{2\pi\left(\frac{\sqrt{2}}{2}-1\right)(-47\cos 7 + 14\sin 7 - 2)}$$



Using $\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$ we get $\tan\phi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$ or $\phi = \frac{\pi}{4}$.

7. (a) First find the mass (which we denote by m):

$$m = \int_0^1\int_0^1\int_0^1 (3 - x - y - z)\, dxdydz = \boxed{\frac{3}{2}}$$

Then

$$M_{xy} = \int_S z\delta dV = \int_0^1\int_0^1\int_0^1 \left(3z - xz - yz - z^2\right) dxdydz = \boxed{\frac{2}{3}}$$

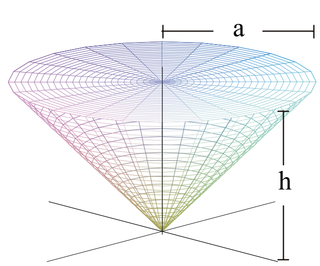
$$M_{xz} = \int_S y\delta dV = \int_0^1\int_0^1\int_0^1 \left(3y - xy - y^2 - zy\right) dxdydz = \boxed{\frac{2}{3}}$$

$$M_{yz} = \int_S x\delta dV = \int_0^1\int_0^1\int_0^1 \left(3x - x^2 - yx - zx\right) dxdydz = \boxed{\frac{2}{3}}$$

The coordinates of the center of gravity are

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \boxed{\left(\frac{4}{9}, \frac{4}{9}, \frac{4}{9}\right)}.$$

(b)

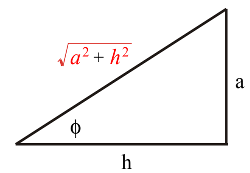


The solid is the cone with height h and cover radius a. To compute the integral in spherical coordinates we need ϕ.

$$\tan\phi = \frac{a}{h} \text{ so } \phi = \tan^{-1}\left(\frac{a}{h}\right).$$

also

$$z = \rho\cos\phi = h \text{ gives } \rho = \frac{h}{\cos\phi}.$$



$$\cos\phi = \frac{h}{\sqrt{a^2+h^2}}$$

$$\begin{aligned}
\text{mass} &= \iiint_S \delta dV \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{\frac{h}{\cos\phi}} \rho^2\cdot\rho^2 \sin\phi d\rho d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \frac{h^5}{5}\frac{\sin\phi}{\cos^5\phi} d\phi d\theta \\
&= \int_0^{2\pi} \frac{h^5}{5}\left(\frac{1}{\cos^4\phi}\right]_0^{\tan^{-1}\left(\frac{a}{h}\right)} d\theta \\
&= \boxed{\frac{2\pi h^5}{5}\left(\frac{\left(a^2+h^2\right)^2}{h^4}-1\right)}
\end{aligned}$$

Also

$$\int_S x\delta dV = \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{\frac{h}{\cos\phi}} \rho\sin\phi\cos\theta\cdot\rho^2\cdot\rho^2\sin\phi d\rho d\phi d\theta = 0$$

$$\int_S y\delta dV = \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{\frac{h}{\cos\phi}} \rho\sin\phi\sin\theta\rho^2\cdot\rho^2\sin\phi d\rho d\phi d\theta = 0$$

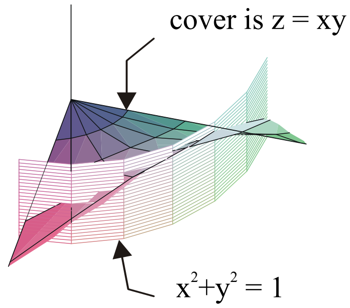
$$\int_S z\delta dV = \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{\frac{h}{\cos\phi}} \rho\cos\phi\rho^2\cdot\rho^2\sin\phi d\rho d\phi d\theta = \frac{\pi h^6}{3}\left(\frac{\left(a^2+h^2\right)^2}{h^4}-1\right)$$

Therefore $(\bar{x},\bar{y},\bar{z}) = \boxed{\left(0, 0, \frac{5}{6}h\right)}$.

8. The centroid of a solid S has coordinates $(\bar{x},\bar{y},\bar{z})$ where

$$\bar{x} = \frac{\int_S xdV}{\text{volume of } S}, \bar{y} = \frac{\int_S ydV}{\text{volume of } S}, \bar{z} = \frac{\int_S zdV}{\text{volume of } S}.$$

(a)



$$\text{volume of } S = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{xy} dz\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} xy\,dx\,dy$$

$$= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{y\left(1-y^2\right)}{2} dy = \left(\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 = \boxed{\frac{1}{8}}$$

$$\int_S x\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{xy} x\,dz\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} x^2 y\,dx\,dy = \int_0^1 \frac{\left(1-y^2\right)^{\frac{3}{2}} y}{3} dy$$

$$= -\frac{1}{6}\left(1-y^2\right)^{\frac{5}{2}} \frac{2}{5}\Bigg]_0^1 = \boxed{\frac{1}{15}}$$

$$\int_S y\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{xy} y\,dz\,dx\,dy = \int_0^1 \frac{x^2 y^2}{2}\Bigg]_0^{\sqrt{1-y^2}} dy = \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \boxed{\frac{1}{15}}$$

$$\int_S z\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{xy} z\,dz\,dx\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} x^2 y^2\,dx\,dy = \frac{1}{6}\int_0^1 y^2\left(1-y^2\right)^{\frac{3}{2}} dy$$

$$= \frac{1}{6}\int_0^{\pi/2} \sin^2\theta\cos^3\theta\cos\theta d\theta$$

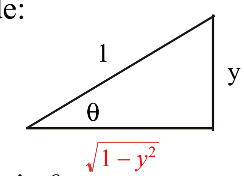
$$= \frac{1}{6}\int_0^{\pi/2} \sin^2\theta\cos^4\theta d\theta$$

$$= \frac{1}{6}\int_0^{\pi/2} \frac{(1-\cos 2\theta)(1+\cos 2\theta)^2}{8} d\theta$$

$$= \frac{1}{6}\int_0^{\pi/2} \frac{\left(1-\cos^2 2\theta+\cos 2\theta-\cos^3 2\theta\right)}{8} d\theta$$

$$= \frac{1}{6}\left(\frac{\pi}{32}\right)$$

Aside:

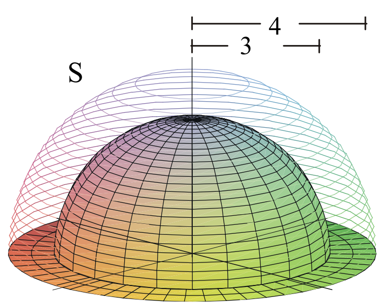


$y = \sin\theta$

$\sqrt{1-y^2} = \cos\theta$

Therefore the centroid is $\boxed{\left(\frac{8}{15}, \frac{8}{15}, \frac{\pi}{24}\right)}$.

(b)



The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Using this the volume of S is

$$\frac{1}{2}\left(\frac{4}{3}\pi(4)^3 - \frac{4}{3}\pi(3)^3\right) = \frac{2}{3}\pi(64-27)$$

$$= \boxed{\frac{74\pi}{3}}$$

$$\int_S x\,dV = \int_3^4\int_0^{\pi/2}\int_0^{2\pi} \rho\sin\phi\cos\theta\rho^2\sin\phi d\theta d\phi d\rho = \boxed{0}$$

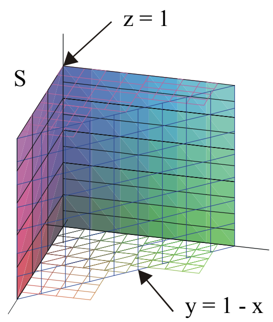
$$\int_S y\,dV = \int_3^4\int_0^{\pi/2}\int_0^{2\pi} \rho\sin\phi\sin\theta\rho^2\sin\phi d\theta d\phi d\rho = \boxed{0}$$

$$\int_S z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_3^4 \rho\cos\phi\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{4^4-3^4}{4}\int_0^{2\pi}\left(\frac{-\cos^2\phi}{2}\right]_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{\pi\left(4^4-3^4\right)}{4} = \boxed{\frac{175\pi}{4}}$$

The centroid is $\left(0, 0, \frac{525}{296}\right)$.

9. (a)



$\delta = 3xy$

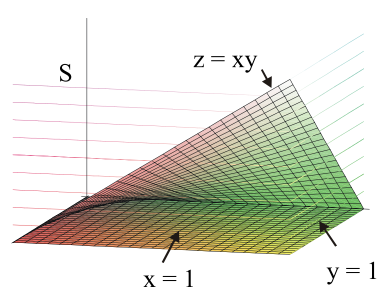
$$M_{xy} = \int_S z\delta\,dV = \int_0^1\int_0^{1-x}\int_0^1 3xyz\,dz\,dy\,dx$$

$$= \int_0^1\int_0^{1-x}\frac{3}{2}xy\,dy\,dx = \frac{3}{4}\int_0^1\left(1-x^2\right)dx = \boxed{\frac{1}{4}}$$

(b) $\delta = 2(x+y)$

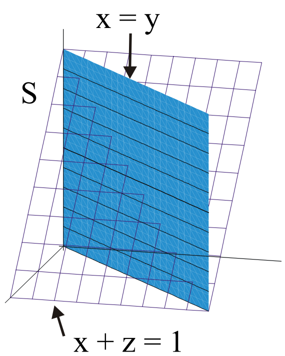
$$M_{xz} = \int_S y\delta\,dV$$

$$= \int_0^1\int_0^1\int_0^{xy} 2y(x+y)\,dz\,dx\,dy$$



$$= \int_0^1\int_0^1 2xy^2(x+y)\,dx\,dy = \int_0^1\left(\frac{2}{3}y^2+y^3\right)dy = \frac{2}{9}+\frac{1}{4} = \boxed{\frac{17}{36}}$$

(c)



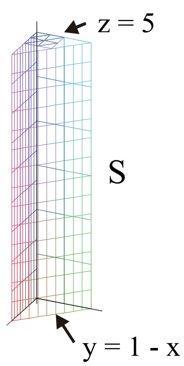
$$\delta = 5y$$

$$M_{yz} = \int_S x\delta dV = \int_0^1\int_0^x\int_0^{1-x} 5xydzdydx$$

$$= \int_0^1\int_0^x 5xy(1-x)\,dydx = \int_0^1 \frac{5}{2}x^3(1-x)\,dx$$

$$= \frac{5}{2}\int_0^1\left(x^3 - x^4\right)dx = \frac{5}{2}\left(\frac{1}{4} - \frac{1}{5}\right) = \boxed{\frac{1}{8}}$$

10.



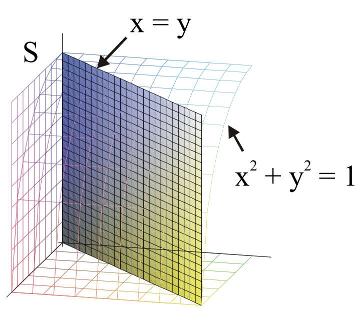
$$\delta = 3$$

$$I_z = \int_S \left(x^2 + y^2\right)\delta dV = 3\int_0^1\int_0^{1-x}\int_0^5 \left(x^2 + y^2\right)dzdydx$$

$$= 15\int_0^1\int_0^{1-x}\left(x^2 + y^2\right)dydx = 15\int_0^1\left(x^2 - x^3 + \frac{(1-x)^3}{3}\right)dx$$

$$= 15\left(\frac{x^3}{3} - \frac{x^4}{4} - \frac{(1-x)^4}{12}\right]_0^1 = 15\left(\frac{2}{12}\right) = \boxed{\frac{5}{2}}$$

11.



$$\delta = 2z$$

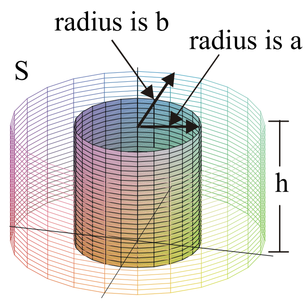
$$I_y = \int_S 2z\left(x^2 + z^2\right)\delta dV$$

$$= \int_0^1\int_0^x\int_0^{\sqrt{1-x^2}}\left(2zx^2 + 2z^2\right)dzdydx$$

$$= \int_0^1\int_0^x\left(x^2\left(1-x^2\right) + \frac{1}{2}\left(1-x^2\right)^2\right)dydx$$

$$= \int_0^1\left(x^3\left(1-x^2\right) + \frac{x}{2}\left(1-x^2\right)^2\right)dx = \int_0^1\left(\frac{x}{2} - \frac{x^5}{2}\right)dx = \boxed{\frac{1}{6}}$$

12.



The mass is m. Since the solid is homogeneous $\delta = \frac{\text{mass of } S}{\text{volume of } S}$. To find the volume, we can use the formula for the volume of a cylinder of radius r and height h which is $\pi r^2 h$.

volume of $S = \pi b^2 h - \pi a^2 h = \pi \left(b^2 - a^2\right) h$.

Thus $\delta = \frac{m}{\pi \left(b^2 - a^2\right) h}$.

$$I_z = \int_S \left(x^2 + y^2\right) \delta dV = \frac{m}{\pi \left(b^2 - a^2\right) h} \int_0^{2\pi} \int_0^h \int_a^b r^2 \cdot r dr dz d\theta$$

$$= \frac{m}{\pi \left(b^2 - a^2\right) h} \frac{b^4 - a^4}{4} (h) (2\pi) = \frac{m \left(b^4 - a^4\right)}{2 \left(b^2 - a^2\right)} = \boxed{\frac{m \left(b^2 + a^2\right)}{2}}$$