Self-Help Work Sheets C12: Sequences & Series

Exercises

1. Prove (this means give an $\varepsilon-N$ proof) that

(a) $\lim_{n \to \infty} \frac{100}{n} = 0$

(b) $\lim_{n \to \infty} \frac{2n^2 + 1}{9n^2 + 5} = \frac{2}{9}$

(c) $\lim_{n \to \infty} \frac{1 - 2n}{3n + 5} = -\frac{2}{3}$

2. Determine whether each of the following sequences is increasing, decreasing or neither and whether Theorem 1 can be used to conclude convergence.

(a) $\left[ \frac{2n + 1}{3n + 2} \right]_{n=0}^{\infty}$

(b) $\left[ \frac{3^n}{1 + 3^n} \right]_{n=0}^{\infty}$

(c) $\{ \sin n\pi \}_{n=0}^{\infty}$

(d) $\{ (-1)^n \}_{n=0}^{\infty}$

3. Determine whether each sequence converges or diverges. If the sequence converges, find its limit. (Here you can use the limit as $x \to \infty$ trick, like $\lim_{n \to \infty} \frac{1}{n} = 0$, $\lim_{n \to \infty} \frac{1}{x^2} = 0$, etc.

(a) $\left[ \frac{n(n + 1)}{3n^2 + 7n} \right]_{n=0}^{\infty}$

(b) $\left[ \frac{\sin n}{n} \right]_{n=1}^{\infty}$

(c) $\left[ \frac{\sqrt{n + 1}}{\sqrt{3n + 1}} \right]_{n=0}^{\infty}$

(d) $\left[ \frac{2^n n!}{(2n + 1)!} \right]_{n=0}^{\infty}$

4. For each of the following use the integral test to test the convergence if it applies.
5. For each of the following, determine whether or not the series converges. Use the comparison test when it is needed.

(a) \( \sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \)

(b) \( \sum_{k=1}^{\infty} \frac{3k^2}{k^3 + 16} \)

(c) \( \sum_{k=1}^{\infty} \frac{1}{1 + \sqrt{k}} \)

(d) \( \sum_{n=1}^{\infty} \left( \frac{1000}{n} \right)^2 \)

6. Test each of the following for convergence or divergence using the ratio test.

(a) \( \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 3k + 1} \)

(b) \( \sum_{k=1}^{\infty} \frac{1}{k \cdot 5^k} \)

(c) \( \sum_{j=1}^{\infty} \frac{j + 1}{(j + 2) 7^j} \)

(d) \( \sum_{q=1}^{\infty} \frac{\sqrt{q}}{q + 2} \)
7. Using any appropriate method to decide whether each series converges (absolutely or conditionally) or diverges.

Here you should first check for absolute convergence once you know the $n$th terms go to zero. For series with positive terms, we have the integral, comparison, limit comparison and ratio tests. If you don’t get absolute convergence, then try for conditional convergence using the alternating series test.

(a) $\sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k + 10}$

(b) $\sum_{k=1}^{\infty} \frac{(-1)^k (1 + e^k)}{2^k}$

(c) $\sum_{k=1}^{\infty} \frac{k}{k + 1} \left( \frac{1}{9} \right)^k$

(d) $\sum_{k=1}^{\infty} \frac{(-1)^k}{\ln \left(1 + \frac{1}{k}\right)}$

(e) $\sum_{k=1}^{\infty} \frac{1 + (-1)^k}{k}$

(f) $\sum_{k=1}^{\infty} \frac{(-1)^k e^k}{(2k - 1)!}$

(g) $\sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots \cdots \cdots (2k - 1)}{3^k k!}$

(h) $\sum_{j=2}^{\infty} (-1)^j \ j e^{-j}$

(i) $\sum_{n=1}^{\infty} \coth n$

(j) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + (-1)^k}$

8. Find the Maclaurin series expansion for

(a) $f(x) = \sin 2x$
(b) \( f(x) = \sin x + \cos x \)
(c) \( f(x) = 10^x \)

9. Find the Taylor series expansion about \( x = 1 \) for

(a) \( f(x) = \sqrt{x} \)
(b) \( f(x) = e^{2x} \)

10. Find the radius of convergence, \( R \), for each of the following:

(a) \( \sum_{k=1}^{\infty} [1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2k - 1)] x^k \)
(b) \( \sum_{k=0}^{\infty} 7^k x^k \)
(c) \( \sum_{k=0}^{\infty} \frac{x^{k+1}}{\sqrt{k+1}} \)
(d) \( \sum_{k=1}^{\infty} \frac{k!}{k^k} x^k \)

**Proposed Answers and Solutions**

(Pay attention to the format given for the answers.)

1. (a) Let \( a_n = \frac{100}{n} \). Given any \( \varepsilon > 0 \)

\[ |a_n - 0| = \left| \frac{100}{n} - 0 \right| = \frac{100}{n} < \varepsilon \]

whenever \( n > \frac{100}{\varepsilon} \). Therefore, for \( n > N(\varepsilon) = \left\lceil \frac{100}{\varepsilon} \right\rceil \), \( |a_n - 0| < \varepsilon \); i.e.,

\[ \lim_{n \to \infty} \frac{100}{n} = 0. \]

(b) Let \( a_n = \frac{2n^2 + 1}{9n^2 + 5} \). Given any \( \varepsilon > 0 \),

\[ \frac{2n^2 + 1}{9n^2 + 5} - \frac{2}{9} = \left| \frac{9 (2n^2 + 1) - 2 (9n^2 + 5)}{9 (9n^2 + 5)} \right| = \left| \frac{-1}{9 (9n^2 + 5)} \right| = \frac{1}{9 (9n^2 + 5)} < \frac{1}{81n^2}. \]
But
\[ \frac{1}{81n^2} < \varepsilon \iff 81n^2 < \frac{1}{\varepsilon} \iff n > \frac{1}{9\sqrt{\varepsilon}}. \]

Therefore for any \( \varepsilon > 0 \), for all \( n > N(\varepsilon) = \left\lceil \frac{1}{9\sqrt{\varepsilon}} \right\rceil \) we have \( |a_n - \frac{2}{9}| < \varepsilon \).

Therefore \( \lim_{n \to \infty} a_n = \frac{2}{9} \).

(c) Let \( a_n = \frac{1-2n}{3n+5} \). Given any \( \varepsilon > 0 \),
\[
|a_n - \left( -\frac{2}{3} \right)| = \left| \frac{1-2n}{3n+5} + \frac{2}{3} \right| = \frac{13}{3n+5} < \frac{13}{3n}.
\]

But
\[ \frac{13}{3n} < \varepsilon \iff 3n > \frac{13}{\varepsilon} \iff n > \frac{13}{3\varepsilon} \]

Therefore, for all \( n > N(\varepsilon) = \left\lceil \frac{13}{3\varepsilon} \right\rceil \), \( |a_n - \left( -\frac{2}{3} \right)| < \varepsilon \); i.e., \( \lim_{n \to \infty} a_n = -\frac{2}{3} \).

2. (a) \( \left\{ \frac{2n+1}{3n+2} \right\}_{n=0}^{\infty} \): Let \( f(x) = \frac{2x+1}{3x+2} \). Then \( f'(x) = \frac{1}{(3x+2)^2} > 0 \) so \( f \) is increasing. Therefore \( \left\{ \frac{2n+1}{3n+2} \right\} \) is an increasing sequence. Also \( \frac{2n+1}{3n+2} < \frac{2}{3} \iff 6n+3 < 6n+6 \iff 3 < 6 \). The sequence is increasing and bounded above so Theorem 1 applies and says the sequence is convergent.

(b) \( \left\{ \frac{3^n}{1+3^n} \right\}_{n=0}^{\infty} \): Note that
\[
\frac{3^n}{1+3^n} < \frac{3^{n+1}}{1+3^{n+1}} \iff 3^n + 3^{n+1} < 3^{n+1} + 3^{2n+1} \iff 1 < 3
\]
is true. So the sequence is increasing. Also \( \frac{3^n}{1+3^n} < 1 \) for all \( n \) which means the sequence is bounded above. Therefore Theorem 1 applies and the sequence is convergent.
(c) $\{\sin n\pi\}_{n=0}^{\infty}$: $\sin n\pi = 0$ for $n = 1, 2, 3, \ldots$, so the sequence is a constant sequence which is neither increasing nor decreasing. Hence Theorem 1 does not apply. (The sequence is convergent.)

(d) $\left\{(-1)^{n^2}\right\}_{n=0}^{\infty}$: For $n$ odd, $n^2$ is odd and $(-1)^{n^2} = -1$. For $n$ even, $n^2$ is even and $(-1)^{n^2} = +1$. So the sequence is neither increasing nor decreasing and Theorem 1 does not apply. (Here, the sequence is divergent.)

3. (a) \[\left\{\frac{n(n+1)}{3n^2+7n}\right\}_{n=0}^{\infty}:\]

\[\frac{n(n+1)}{3n^2+7n} = \frac{n^2+n}{3n^2+7n} = \frac{1 + \frac{1}{n}}{3 + \frac{7}{n}}.\]

Here, \(\lim_{n \to \infty} \frac{1 + \frac{1}{n}}{3 + \frac{7}{n}} = \frac{1}{3}\)

(b) \[\left\{\frac{\sin n}{n}\right\}_{n=1}^{\infty}:\]

Note that $-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$. Since $\lim_{n \to \infty} -\frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0$,

\[\lim_{n \to \infty} \frac{\sin n}{n} = 0\] by the Squeeze Principle.

(c) \[\left\{\frac{\sqrt{n+1}}{\sqrt{3n+1}}\right\}_{n=0}^{\infty}:\]

\[\frac{\sqrt{n+1}}{\sqrt{3n+1}} = \sqrt{\frac{n+1}{3n+1}} = \sqrt{\frac{1 + \frac{1}{n}}{3 + \frac{1}{n}}}.\]

Then \(\lim_{n \to \infty} \frac{\sqrt{n+1}}{\sqrt{3n+1}} = \sqrt{\frac{1}{3}} = \sqrt{\frac{\sqrt{3}}{3}}\)

(d) \[\left\{\frac{2^n n!}{(2n+1)!}\right\}_{n=0}^{\infty}:\]

Note that

\[\frac{2^n n!}{(2n+1)!} = \frac{2^n}{(n+1)(n+2) \cdots (n+(n+1))} = \frac{2}{n+1} \cdot \frac{2}{n+2} \cdots \frac{2}{2n} \cdot \frac{1}{2n+1}.\]

there are \(n+1\) terms here
As \( n \to \infty \), \( \frac{2}{n + 1} \to 0 \). Since \( 0 < \frac{2}{n + k} < 1 \) for \( n > 1 \) and \( k = 1, 2, \ldots, n \),

\[
0 < \frac{2}{n + 1} \cdot \frac{2}{n + 2} \cdots \frac{2}{2n} \cdot \frac{1}{2n + 1} < \frac{2}{n + 1}
\]

for \( n > 1 \). Therefore by the squeeze principle \( \lim_{n \to \infty} \frac{2^n n!}{(2n + 1)!} = 0 \).

4. (a) \( \sum_{k=1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{\infty} k^4 \). Let \( f(x) = x^4 \). Then \( f''(x) = -\frac{4}{3}x^{-\frac{7}{3}} < 0 \) for \( x \geq 1 \) which means that \( f \) is decreasing. Thus the integral test applies. Consider

\[
\lim_{b \to \infty} \int_1^b x^{-\frac{4}{3}}dx = \lim_{b \to \infty} \left( -\frac{3}{4}\sqrt[4]{b} + 3 \right) = 3.
\]

The improper integral exists so the series converges and its sum is between 3 and 4.

(b) \( \sum_{k=1}^{\infty} \frac{3k^2}{k^3 + 16} \). Let \( f(x) = \frac{3x^2}{x^3 + 16} \). Then \( f''(x) = -\frac{3x(x^3 - 32)}{(x^3 + 16)^3} < 0 \) for \( x \geq 4 \), so the integral test applies to the series from \( k = 4 \) and after. Consider

\[
\lim_{b \to \infty} \int_4^b \frac{3x^2}{x^3 + 16}dx = \lim_{b \to \infty} \left[ \ln \left( b^3 + 16 \right) - \ln 80 \right].
\]

This limit does not exist. Therefore \( \sum_{k=4}^{\infty} \frac{3k^2}{k^3 + 16} \) diverges by the integral test, so

\[
\sum_{k=1}^{\infty} \frac{3k^2}{k^3 + 16}
\]

must diverge also.

(c) \( \sum_{k=1}^{\infty} \frac{1}{1 + \sqrt{k}} \). Let \( f(x) = \frac{1}{1 + \sqrt{x}} \). Then \( f''(x) = -\frac{1}{2\sqrt{x}(1 + \sqrt{x})^2} < 0 \) for \( x \geq 1 \). So \( f \) is decreasing for \( x \geq 1 \) and the integral test applies. Consider

\[
\lim_{b \to \infty} \int_1^b \frac{dx}{1 + \sqrt{x}} = \lim_{b \to \infty} \int_1^b \frac{2u}{1 + u}du
\]

substituting \( u = \sqrt{x} \)

\[
= \lim_{b \to \infty} \int_1^b \left( 2 - \frac{2}{u + 1} \right)du
\]

\[
= \lim_{b \to \infty} \left( 2\sqrt{b} - 2 \ln \left( \sqrt{b} + 1 \right) - 1 \right).
\]
This limit does not exist. Therefore the series diverges by the integral test.

(d) \[ \sum_{n=1}^{\infty} \left( \frac{1000}{n} \right)^2 = \sum_{n=1}^{\infty} \frac{10^6}{n^2}. \]
Let \( f(x) = \frac{10^6}{x^2} \). Then \( f'(x) = \frac{-2(10)^6}{x^3} < 0 \) for \( x \geq 1 \). Thus \( f \) is decreasing for \( x \geq 1 \) and the integral test applies. Consider

\[
\lim_{b \to \infty} b \int_{1}^{b} \frac{10^6}{x^2} \, dx = \lim_{b \to \infty} \left[ \frac{-10^6}{b} + 10^6 \right] = 10^6.
\]

Since the limit exists, the improper integral exists. Therefore the series converges by the integral test.

5. (a) \[ \sum_{k=1}^{\infty} \frac{k^2}{k^4 + 3k + 1} \]
Note that

\[
\frac{k^2}{k^4 + 3k + 1} < \frac{1}{k^2} \iff k^4 < k^4 + 3k + 1 \iff 0 < 3k + 1
\]
which is true for all \( k \geq 1 \). Therefore the series converges by comparison with \( \sum_{k=1}^{\infty} \frac{1}{k^2} \).

(b) \[ \sum_{k=1}^{\infty} \frac{1}{k5^k} \]
Since \( k \cdot 5^k \geq 1 \cdot 5^k \) for \( k > 1 \), we have \( \frac{1}{k \cdot 5^k} \leq \frac{1}{5^k} \) for \( k \geq 1 \). The series \( \sum_{k=1}^{\infty} \left( \frac{1}{5} \right)^k \) is convergent as a geometric series with ratio \( \frac{1}{5} < 1 \). Therefore \( \sum_{k=1}^{\infty} \frac{1}{k5^k} \) converges by comparison with \( \sum_{k=1}^{\infty} \left( \frac{1}{5} \right)^k \).

(c) \[ \sum_{j=1}^{\infty} \frac{(j+1)}{(j+2)7^j} \]
Since \( \frac{j+1}{j+2} = \frac{j+1}{(j+1)+1} < 1 \) for \( j \geq 1 \), we have \( \frac{j+1}{(j+2)7^j} < \frac{1}{7^j} \) for \( j \geq 1 \). Therefore the series converges by comparison with the geometric series \( \sum_{j=1}^{\infty} \left( \frac{1}{7} \right)^j \) with ratio \( \frac{1}{7} < 1 \).

(d) \[ \sum_{q=1}^{\infty} \frac{\sqrt{q}}{q+2} \]
Since \( \sqrt{q} \geq 1 \) for all \( q \geq 1 \), we have \( \frac{\sqrt{q}}{q+2} \geq \frac{1}{q+2} \) for all \( q \geq 1 \).
Since \( \frac{1}{x+2} \) is a decreasing function for \( x \geq 1 \), we apply the integral test to
\[ \sum_{q=1}^{\infty} \frac{1}{q + 2}. \] Note that
\[
\lim_{b \to \infty} \int_1^b \frac{dx}{x + 2} = \lim_{b \to \infty} (\ln (b + 2) - \ln 3)
\]
and the limit does not exist. Therefore \( \sum_{q=1}^{\infty} \frac{1}{q + 2} \) is divergent by the integral test and thus \( \sum_{q=1}^{\infty} \frac{\sqrt{q}}{q + 2} \) diverges by comparison.

6. (a) \( \sum_{k=1}^{\infty} \frac{2^k}{7^k (k + 1)} \): Here \( a_{n+1} = \frac{2^{n+1}}{7^{n+1} (n + 2)} \) and \( a_n = \frac{2^n}{7^n (n + 1)} \). Then
\[
\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{7^{n+1} (n + 2)} \cdot \frac{7^n (n + 1)}{2^n} = \frac{2}{7} \cdot \frac{1 + \frac{1}{n}}{1 + \frac{2}{n}}.
\]
Therefore \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{2}{7} (1) = \frac{2}{7} < 1 \) and the series converges by the ratio test.

(b) \( \sum_{k=1}^{\infty} \frac{5^k}{k!} \): Here \( a_{n+1} = \frac{5^{n+1}}{(n + 1)!} \) and \( a_n = \frac{5^n}{n!} \). Then
\[
\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n + 1)!} \cdot \frac{n!}{5^n} = \frac{5}{n + 1}.
\]
Therefore \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 < 1 \) and the series converges by the ratio test.

(c) \( \sum_{k=1}^{\infty} \frac{5^k}{k4^k} \): Here \( a_{n+1} = \frac{5^{n+1}}{(n + 1)4^{n+1}} \) and \( a_n = \frac{5^n}{n4^n} \). Then
\[
\frac{a_{n+1}}{a_n} = \frac{5^{n+1}}{(n + 1)4^{n+1}} \cdot \frac{n4^n}{5^n} = \frac{5}{4} \cdot \frac{1}{1 + \frac{1}{n}}.
\]
Therefore \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{5}{4} (1) = \frac{5}{4} > 1 \) and the series diverges by the ratio test.
(d) \( \sum_{k=1}^{\infty} \frac{k^3 + 1}{k!} \): In this series \( a_{n+1} = \frac{(n + 1)^3 + 1}{(n + 1)!} \) and \( a_n = \frac{n^3 + 1}{n!} \). Then

\[
a_{n+1} = \left( \frac{n^3 + 3n^2 + 3n + 2}{(n + 1)!} \right) \left( \frac{n!}{n^3 + 1} \right) = \left( \frac{1 + \frac{3}{n^2} + \frac{3}{n} + \frac{2}{n^3}}{1 + \frac{1}{n^2}} \right).
\]

Therefore \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 \cdot 1 = 0 < 1 \) and the series converges by the ratio test.

7. (a) \( \sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k + 10} \): First, to check for absolute convergence, look at

\[
\sum_{k=1}^{\infty} \left| \frac{(-1)^k \sqrt{k}}{k + 10} \right| = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k + 10}.
\]

For \( k \geq 1 \), \( \sqrt{k} \geq 1 \) so \( \frac{\sqrt{k}}{k + 10} \geq \frac{1}{k + 10} \). Since \( \sum_{k=1}^{\infty} \frac{1}{k + 10} \) diverges (by the integral test \( \lim_{b \to \infty} \int_1^b \frac{dx}{x + 10} = \lim_{b \to \infty} \ln (b + 10) - \ln 11 \)), therefore \( \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k + 10} \) diverges by the comparison test. Thus the series is not absolutely convergent.

Second, the given series is alternating with \( a_k = \frac{\sqrt{k}}{k + 10} \). For \( f(x) = \frac{\sqrt{x}}{x + 10} \), \( f'(x) = \frac{10 - x}{2\sqrt{x}(x + 10)^2} < 0 \) for \( x > 10 \). Therefore the \( a_k \) are decreasing for \( k > 10 \). Also \( \lim_{k \to \infty} \frac{\sqrt{k}}{k + 10} = \lim_{k \to \infty} \frac{1}{1 + \frac{10}{k}} = 0 \). Therefore we have convergence by the alternating series test. The series \( \sum_{k=1}^{\infty} \frac{(-1)^k \sqrt{k}}{k + 10} \) is conditionally convergent.

(b) \( \sum_{k=1}^{\infty} \frac{(-1)^k \left(1 + e^k\right)}{2^k} \): Note that

\[
\lim_{k \to \infty} \frac{1 + e^k}{2^k} = \lim_{k \to \infty} \left( \frac{1}{2^k} + \left( \frac{e^k}{2} \right) \right) \neq 0 \text{ since } \frac{e}{2} > 1.
\]

Therefore the series diverges.
(c) \[ \sum_{k=1}^{\infty} \frac{k}{k+1} \left( \frac{1}{9} \right)^{k} \]: All the terms are positive so we test directly. Note that \( a_{n+1} = \frac{n + 1}{n + 2} \left( \frac{1}{9} \right)^{n+1} \) and \( a_n = \frac{n}{n + 1} \left( \frac{1}{9} \right)^{n} \). Then
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{9} \left( \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{2}{n}} \right) = \frac{1}{9} < 1.
\]
Therefore the series is absolutely convergent by the ratio test.

(d) \[ \sum_{k=1}^{\infty} \frac{(-1)^k}{\ln(1 + \frac{1}{k})} \]: Since \( \lim_{k \to \infty} \ln \left( 1 + \frac{1}{k} \right) = \ln 1 = 0 \), we have that \( \lim_{k \to \infty} \frac{1}{\ln(1 + \frac{1}{k})} \to \infty \) not 0. Therefore the series diverges.

(e) \[ \sum_{k=1}^{\infty} \frac{1 + (-1)^k}{k} \] = \[ 0 + \frac{2}{1} + \frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \frac{2}{5} + \frac{2}{6} + \cdots \] For \( k \) odd, \( a_k = 0 \). For \( k \) even, \( k = 2n \) and \( a_k = a_{2n} = \frac{2}{2n} = \frac{1}{n} \). Therefore this series is identical to \( \sum_{n=1}^{\infty} \frac{1}{n} \) and is therefore divergent.

(f) \[ \sum_{k=1}^{\infty} \frac{(-1)^k e^k}{(2k - 1)!} \]: First, to check for absolute convergence, look at
\[
\sum_{k=1}^{\infty} \left| \frac{(-1)^k e^k}{(2k - 1)!} \right| \leq \sum_{k=1}^{\infty} \frac{e^k}{(2k - 1)!}
\]
Here \( a_{n+1} = \frac{e^{n+1}}{(2n + 1)!} \) and \( a_n = \frac{e^n}{(2n - 1)!} \). Thus
\[
\frac{a_{n+1}}{a_n} = \frac{e^{n+1}}{(2n + 1)!} \cdot \frac{(2n - 1)!}{e^n} = \frac{e}{2n(2n + 1)}.
\]
Therefore \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 0 < 1 \) and the series is absolutely convergent by the ratio test.

(g) \[ \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2k - 1)}{3^k k!} \]: All the terms are positive.
\[
\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1) (2n + 1)}{3^{n+1} (n + 1)!} \cdot \frac{3^n n!}{1 \cdot 3 \cdot 5 \cdots (2n - 1)} = \frac{1}{3} \left( \frac{2 + \frac{1}{n}}{1 + \frac{1}{n}} \right).
\]
Therefore \[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1}{3} (2) = \frac{2}{3} < 1 \] and the series is absolutely convergent by the ratio test.

(h) \( \sum_{j=1}^{\infty} (-1)^j j e^{-j} \): First, to check for absolute convergence, look at

\[ \sum_{j=1}^{\infty} \left| (-1)^j j e^{-j} \right| = \sum_{j=1}^{\infty} j e^{-j}. \]

Let \( f(x) = xe^{-x} \). Then \( f'(x) = e^{-x} (1-x) < 0 \) for \( x > 1 \). So \( f \) is decreasing and the integral test applies. Consider

\[ \lim_{b \to \infty} \frac{1}{2} \int_2^b xe^{-x} \, dx = \lim_{b \to \infty} \left[ \frac{-b}{e^b} - \frac{1}{e^b} + \frac{3}{e^2} \right] = \frac{3}{e^2}. \]

(By L’Hopital’s rule, \( \lim_{b \to \infty} \frac{b}{e^b} = \lim_{b \to \infty} \frac{1}{e^b} = 0 \).) The improper integral exists and the series is absolutely convergent by the integral test.

(i) \( \sum_{n=1}^{\infty} \coth n = \sum_{n=1}^{\infty} \frac{e^n + e^{-n}}{e^n - e^{-n}} = \sum_{n=1}^{\infty} \frac{e^{2n} + 1}{e^{2n} - 1} \): All the terms are positive. Note that

\[ \lim_{n \to \infty} \frac{e^{2n} + 1}{e^{2n} - 1} = \lim_{n \to \infty} \frac{1 + \frac{1}{e^{2n}}}{1 - \frac{1}{e^{2n}}} = 1 \neq 0. \]

Therefore the series diverges because the terms are not going to zero.

(j) \( \sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + (-1)^k} \): First, to check for absolute convergence, we look at

\[ \sum_{k=2}^{\infty} \left| \frac{(-1)^k}{k^2 + (-1)^k} \right| = \sum_{k=2}^{\infty} \frac{1}{k^2 + (-1)^k}. \]

Now \( k^2 + (-1)^k \geq k^2 - 1 \) for \( k = 2, 3, 4, \ldots \). Therefore we have \( \frac{1}{k^2 + (-1)^k} \leq \frac{1}{k^2 - 1} \) and we look at the series \( \sum_{k=2}^{\infty} \frac{1}{k^2 - 1} \) for a comparison. Let \( f(x) = \frac{1}{x^2 - 1} \). Then \( f'(x) = \frac{-2x}{(x^2 - 1)^2} < 0 \) for \( x > 1 \), thus the integral test applies.
Consider
\[
\lim_{b \to \infty} \int_1^b \frac{dx}{x^2 - 1} = \lim_{b \to \infty} \int_1^b \frac{1}{2} \left( \frac{1}{x - 1} - \frac{1}{x + 1} \right) dx
\]
\[
= \lim_{b \to \infty} \frac{1}{2} \left[ \ln \left( \frac{b - 1}{b + 1} \right) - \ln \left( \frac{1}{2} \right) \right]
\]
\[
= -\frac{1}{2} \ln \frac{1}{2} = \frac{1}{2} \ln 2.
\]

The improper integral exists. Therefore \( \sum_{k=2}^{\infty} \frac{1}{k^2 - 1} \) converges by the integral test. (Be careful here. We had to use the integral test because \( \frac{1}{k^2 - 1} \not< \frac{1}{k^2} \) and the ratio test gives 1.) Finally

\[
\sum_{k=2}^{\infty} \frac{(-1)^k}{k^2 + (-1)^k}
\]
converges absolutely by comparison with \( \sum_{k=2}^{\infty} \frac{1}{k^2 - 1} \).

8. (a) \( f (x) = \sin 2x \quad f (0) = 0 \quad a_0 = 0 \)
\( f' (x) = 2 \cos x \quad f' (0) = 2 \quad a_1 = 2 \)
\( f'' (x) = -2^2 \sin 2x \quad f'' (0) = 0 \quad a_2 = 0 \)
\( f''' (x) = -2^3 \cos 2x \quad f''' (0) = -2^3 \quad a_3 = \frac{-2^3}{3!} \)
\( f^{IV} (x) = 2^4 \sin 2x \quad f^{IV} (0) = 0 \quad a_4 = 0 \)
\[\vdots\]
\( f^{2n} (0) = 0 \quad a_{2n} = 0 \)
\( f^{2n+1} (0) = (-1)^{n+2} 2^{2n+1} \quad a_{2n+1} = \frac{(-1)^n (2)^{2n+1}}{(2n + 1)!} \)

Therefore

\[
f (x) = 2x - \frac{2^3}{3!} x^3 + \frac{2^5}{5!} x^5 - \frac{2^7}{7!} x^7 + \cdots + (-1)^k \frac{2^{2k+1}}{(2k + 1)!} x^{2k+1} + \cdots
\]

or \( f (x) = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k + 1)!} x^{2k+1} \) or \( f (x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} 2^{2k-1}}{(2k - 1)!} x^{2k-1} \).
(b) \( f(x) = \sin x + \cos x \) \quad f(0) = 1 \quad a_0 = 1
\[ f'(x) = \cos x - \sin x \] \quad f'(0) = 1 \quad a_1 = 1
\[ f''(x) = -\sin x - \cos x \] \quad f''(0) = -1 \quad a_2 = \frac{-1}{2!}
\[ f'''(x) = -\cos x + \sin x \] \quad f'''(0) = -1 \quad a_3 = \frac{-1}{3!}
\[ f^4(x) = \sin x + \cos x \] \quad f^4(0) = 1 \quad a_4 = \frac{1}{4!}
\[ f^5(x) = \cos x - \sin x \] \quad f^5(0) = 1 \quad a_5 = \frac{1}{5!}

Therefore

\[
f'(x) = 1 + x - \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 - \ldots
\]

This can also be obtained by adding

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots
\]

and

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots.
\]

(c) \( f(x) = 10^x = e^{x\ln 10} \) \quad f(0) = 1 \quad a_0 = 1
\[ f'(x) = 10^x (\ln 10) \] \quad f'(0) = \ln 10 \quad a_1 = \ln 10
\[ f''(x) = 10^x (\ln 10)^2 \] \quad f''(0) = (\ln 10)^2 \quad a_2 = \frac{(\ln 10)^2}{2!}
\[ f^3(x) = 10^x (\ln 10)^3 \] \quad f^3(0) = (\ln 10)^3 \quad a_3 = \frac{(\ln 10)^3}{3!}
\[ f^4(x) = 10^x (\ln 10)^4 \] \quad f^4(0) = (\ln 10)^4 \quad a_4 = \frac{(\ln 10)^4}{4!}
\[ f^5(x) = 10^x (\ln 10)^n \] \quad f^n(0) = (\ln 10)^n \quad a_n = \frac{(\ln 10)^n}{n!}

Therefore

\[
f(x) = 1 + (\ln 10)x + \frac{(\ln 10)^2}{2!}x^2 + \frac{(\ln 10)^3}{3!}x^3 + \ldots + \frac{(\ln 10)^n}{n!}x^n + \ldots.
\]
9. (a) \[ f(x) = x^\frac{1}{2} \]

\[ f'(x) = \frac{1}{2} x^{-\frac{1}{2}} \]

\[ f''(x) = -\frac{1}{2^2} x^{-\frac{3}{2}} \]

\[ f'''(x) = \frac{1 \cdot 3}{2^3} x^{-\frac{5}{2}} \]

\[ f''''(x) = -\frac{1 \cdot 3 \cdot 5}{2^4} x^{-\frac{7}{2}} \]

\[ \vdots \]

\[ f^n(x) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} x^{\frac{-2n-1}{2}} \]

\[ f^n(1) = (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n} \]

where, for \( n > 1 \),

\[ a_n = \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} \]

Therefore

\[ f(x) = 1 + \frac{1}{2} (x - 1) - \frac{1}{2^2 2!} (x - 1)^2 + \frac{1 \cdot 3}{2^3 3!} (x - 1)^3 + \cdots + \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} (x - 1)^n + \cdots \]

(b) \[ f(x) = e^{2x} \]

\[ f'(x) = 2e^{2x} \]

\[ f''(x) = 2^2 e^{2x} \]

\[ f'''(x) = 2^3 e^{2x} \]

\[ \vdots \]

\[ f^n(x) = 2^n e^{2x} \]

\[ f^n(1) = 2^n e^2 \]

\[ a_0 = e^2 \]

\[ a_1 = 2e^2 \]

\[ a_2 = \frac{2^2 e^2}{3!} \]

\[ a_3 = \frac{2^3 e^2}{3!} \]

\[ \vdots \]

\[ a_n = \frac{2^n e^2}{n!} \]

Therefore
10. (a) \( \sum_{k=1}^{\infty} [1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1)] x^k \): Here

\[
a_k = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1)
\]

and

\[
\frac{a_{k+1}}{a_k} = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1) \cdot (2k+1)}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2k-1)} = 2k + 1.
\]

Therefore \( \left| \frac{a_{k+1}}{a_k} \right| \to \infty \), so \( R = 0 \) and the series diverges everywhere except at \( x = 0 \).

(b) \( \sum_{k=0}^{\infty} 7^k x^k \): Here \( a_k = 7^k \) and \( \frac{a_{k+1}}{a_k} = \frac{7^{k+1}}{7^k} = 7 \). Since \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = 7 = L \), the radius of convergence is \( R = \frac{1}{7} \).

(c) \( \sum_{k=0}^{\infty} \frac{x^{k+1}}{\sqrt{k+1}} \): Here \( a_k = \frac{1}{\sqrt{k+1}} \). Since \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \frac{\sqrt{k+2}}{\sqrt{k+1}} = 1 = L \), the radius of convergence is \( R = 1 \).

(d) \( \sum_{k=1}^{\infty} \frac{k!}{k^k} x^k \): Here \( \frac{a_{k+1}}{a_k} = \frac{(k+1)!}{(k+1)^{k+1}} \cdot \frac{k^k}{k!} = \left( \frac{k}{k+1} \right)^{k} = \frac{1}{(1+\frac{1}{k})^k} \). Since

\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \frac{1}{e} = L ,
\]

the radius of convergence is \( R = \frac{1}{L} = e \).