Chapter 1

Mathematical Processes

Expanding our Mathematical View

To prepare for the study of college level course work and, later, careers in the mathematical sciences or engineering, you need to continue the move towards a new level of understanding and experience with mathematics that your earlier exposure to Calculus began. Your long term goal is to be prepared to handle problems that come in the form of situations or challenges:

- design a tool so that it will work better;
- redesign a part of some equipment so that it won’t malfunction so quickly;
- determine how efficiently something is working and suggest improvements in set-up or design;
- study the pattern of growth or disappearance of a species and design a program for repopulation, continued growth, or better control;
- determine the stresses or life expectancy of a particular object;
- given several options, study their feasibility, desirability and/or likelihood; etc.

The challenge doesn’t come as a well stated word problem with terminology that suggests an approach towards a solution. You will need to access your knowledge, skills and the tools available in your discipline to translate the given situation to a model that can be used to seek solutions or partial solutions. When you believe that you have an answer, you will need to explain the rationale and justify your conclusions. There will be “no answer in the back of the book;” consequently, your (or your team’s) explanation will need to be well presented and documented.
Your short term goal is to make the first level of transition from routine or basic problems to more open–ended problems that pull together old and new techniques and/or concepts. Our study of Calculus will move us towards the transition. It is important that we have a more realistic understanding of what constitutes doing mathematics. Hopefully, your initiation into this new perspective was sparked by earlier parts of the Calculus sequence.

The most common mathematical situations or problems that people experience prior to the study of college level mathematics are directly related to solving very specific problems that involve computations, simplifications, practice of some techniques or approaches, and/or application of basic skills that have been taught and re–taught. (The outstanding exception to that is usually encountered with the study of Plane Geometry.) It is quite possible to leave pre–college experiences with mathematics having the wrong impression of how one goes about doing mathematics and, even, with the wrong conception of how one either learns mathematics or approaches open–ended mathematical situations. Calculus is an area of mathematics that builds on some of the computational aspects of mathematics while developing a rich theory that allows for a diverse array of applications. In earlier parts of this Calculus sequence, you saw some of the theoretical development of differential and integral calculus, the foundation for which is the very deep concept of the limit of a function, while working to become proficient at finding derivatives and integrals. The applications studied included, but were not limited to, analysis of the behavior (graphing) of functions, optimization problems, and determination of the area or the mass of given objects.

It will be helpful for us to have some loose categories into which we can classify mathematics problems. In Chapter 1 of *Cameos of Differential Calculus* (@math.ucdavis.edu/~emsilvia), the categories of Routine, Eclectic, and Investigations were introduced and discussed. The examples given there made use of pre-calculus mathematics. In the following section, we focus on examples that make use of differential and integral calculus.

**Some Problem Categories**

**Routine** problems are straightforward and require very little interpretation or analysis. They stimulate practice in order to help one learn basic skills, terminology, and techniques. Routine or basic problems are direct applications of concepts that are being learned at the time of assignment and can be done or started almost immediately after they are read. The expectation is that providing a good deal of practice or repeated use of a skill will enable the students to recall that skill when it is needed in other contexts. We will also use this term for elementary word problems for which translation to equations is very straightforward. Some examples are:

- Factor \(6x^2 - 19x - 7\).
- Graph the function \(f(x) = 7x^3 + 9\).
EXPANDING OUR MATHEMATICAL VIEW

- Find the first derivative of \( f(x) = \frac{3x^4 - 7}{x^2 + 5} \)
- Find the area of the region that is bounded by \( x = 1, x = 4, y = 0, y = 2x^3 + 3x - 1 \).
- After justifying its applicability, verify the conclusions of the Mean-Value Theorem for the function \( f(x) = x^4 - 5x + 7 \) over the interval \([0, 2]\).

**Eclectic** mathematical experiences or problems call for the use of some combination of elementary logic, skills and techniques that have been learned in other situations, application of tools that have been studied in order to solve certain types of challenge problems, etc. For these it is rare that someone can simply read the problems and begin to write down answers. On the other hand, they do offer an idea or description of what is to be found; consequently, after some (non–routine) translation of what has been given, they can be solved by use of some of the skills learned from routine problems and/or a calculator. The translation process may involve a fair amount of cleverness or creativity. Interpretation may be more complicated because problems given “out of chapter or text section context” don’t have the chapter or section topics as hints for how to tackle the problems. Not all of the following examples of eclectic problems require calculus.

- Determine whether or not the function \( f(x) = x \cos x \) has a horizontal tangent in the interval \( \left( 0, \frac{\pi}{2} \right) \).
- Find all parabolas that have \((-2, 1)\) as a vertex, a line parallel to an axis as a directrix, and \( y = 6x + 4 \) as a tangent.
- Show that the equation \( \cos x + x \sin x - x^2 = 0 \) has exactly two (real-valued) solutions.
- Under what conditions will the area of a right triangle equal the square of the length of its hypotenuse?
- A boat is 4 miles from the nearest point on a straight shore line which is 6 miles from a shoreside restaurant. A woman plans to row to a point on shore, and then walk to the restaurant. If she can walk 3 mph, at what speed must she be able to row in order for the quickest way to the restaurant to be by rowing directly?
- Consider the plane region that is bounded by the curves \( y = x \) and \( y = x^2 \). One solid region can be obtained by rotating the plane region about the \( x \)-axis, assuming that everything swept out becomes part of the solid region. Another solid region can be obtained by rotating the plane region about the \( y \)-axis. A third solid region can be obtained by rotating the plane region about the line \( y = x \). On separate representations of the Cartesian co-ordinate system, illustrate the three solids. Find the volumes of the three solids.
Finally, mathematical Investigations offer open-ended situations that are to be studied. The situations are given without prompts that suggest how to start the problem. Such challenges seek to put you nearer to the set-up that you might encounter as a practicing mathematical scientist or engineer. On the other hand, our investigations will only expect the student to call upon things learned in Calculus or courses that are prerequisite to Calculus. A mathematical investigation is a process in itself. There isn’t necessarily an answer at the end. Sometimes the process leads to a list of things that occur under various conditions, sometimes it leads to a very rich theory, sometimes it leads no place. No matter what, a major part of the value is gained from the processes used and the experiences that are either called upon or gained. Some examples are:

- **In Search of a Function**

  Suppose that a function $f$: (i) is defined for all real $x$, (ii) is continuous and nonzero at 0, (iii) is differentiable at $x = 0$, and (iv) satisfies the functional equation $f(x + y) = f(x) f(y)$. What else can be said about $f$? Use carefully prepared and well presented mathematical arguments to justify any conclusions that are claimed.

- **An Analysis of The Product of Limits**

  Use examples to give a complete and well-written discussion of the outcome of the limit of the product of two functions as $x$ approaches a finite value $a$, when one or both of the limits of the given functions does not exist.

- **An Area Shortcut?**

  When asked to find the area bounded between $y = -(x - 1)^2 + 9$ and $y = 0$, a student in integral calculus sketched a triangle having vertices $(-2, 0)$, $(1, 9)$, and $(4, 0)$ and offered the answer $\frac{4}{3} \left( \frac{1}{2} (4 - (-2)) (9) \right) = 36$ without any further explanation. When the graded exam was returned to the student, he discovered that he had received only 2 points out of a possible 20. Since the answer was correct, he decided to discuss the matter with the instructor. During the discussion that focused on the need to clearly justify the work that was done, the student reports that he had observed that the answer to several problems of similar type on the homework had always agreed with his shortcut formula $\frac{4}{3} A$ where $A$ is the area of the triangle that is formed by the vertex of the parabola and the two points where the parabola intersects the x-axis. Though the instructor is impressed with the insight demonstrated, she informs the student that “A few examples does not a theorem make.” On the other hand, the instructor offers the student the option of earning the 18 points missed by producing a well-written proof for the claim. What are the chances that the student will be able earn the 18 points?
A Problem Solving Process

Essential to likelihood of success with mathematical challenges is developing the ability to make use of some form of a problem solving process. The structure described here is loose and open-ended because people think differently and bring different experiences or basic knowledge to a problem solving situation. Drawing a picture might help one person to organize information while making a table or chart is what someone else would need. We describe three phases that are generally applicable. However, the actual ways in which they are practiced will vary considerably with the nature of the problem that is under consideration.

Phase I: Getting to Know the Problem. At minimum, this includes reading the problem and making sure that the terminology is understood. It may also involve asking other people about the problem, trying a special case or several examples, drawing pictures, making tables or charts, using a calculator to generate data, and/or looking things up.

Phase II: Scratch work. This is where a solution is actively sought and, hopefully, found. The pursuit may involve working out the details of a simpler version of the problem, establishing unknowns in order to form equations, acting out the problem, and/or working backwards. There is a certain amount of “seat of the pants” work involved, particularly when the problems are not the simple types of word problems that were studied in high school algebra. The work from this phase is for the person or people doing the work and is not to be turned in for grading.

N.B. One thing that you definitely cannot do in solving a problem is add objects or features that were not given. For example, if a problem is given with two jugs, you cannot work with three jugs. You are allowed, of course, to derive things from what is given by mathematical argument.

Phase III: Presenting a Solution. Solutions should be well written and well organized. The reader should not have to guess what the writer is trying to say. The exposition should be accomplished with complete and correctly punctuated sentences. In the last section of this chapter, we will discuss some of the basic rules for writing up solutions. At a minimum, justifications should consist of complete sentences. Sentences should begin with capital letters and end with periods (question marks, exclamation points).

At this point, think about the phases while you briefly consider some problems. Again, examples of problems that do not use calculus can be found in the first chapter of Cameos of Differential Calculus.
Some Problems

Before going on, take some time to work on, at least, Phase I and Phase II for problems (A) – (C). Solutions are offered after the statements of the four problems: You will get more out of the solutions if you look at them after you have thought about the problems.

(A) In Search of Zeros.

Determine the exact number of x-intercepts for function \( f(x) = \cos x + x \sin x - x^2 \).

(B) An Integral Translation Challenge.

Use your knowledge of integral calculus to find
\[
\lim_{n \to \infty} \sum_{j=1}^{n} \frac{2n}{(n+j)^2}.
\]

(C) A Fundamental Question.

Carefully justifying your work, find
\[
\frac{d}{dx} \left[ \int_{1}^{x^4 \ln(x+2)} \sqrt{2 + t + \cos^2 2t} \, dt \right].
\]
Some Solutions

(A) In Search of Zeros.

Phase I. A careful reading of the problem coupled with remembering that an x-intercept of a function \( f \) occurs whenever \( f(x) = 0 \) quickly leads us to the observation that we are being asked to find exactly the number of real values \( x \) for which \( \cos x + x \sin x - x^2 = 0 \). At this point, you might plug in some values of \( x \) or, if one is handy, generate a graph the function on a scientific calculator. The graph—which looks somewhat like a parabola in an interval about the origin—can not be used to simply claim the result; it does, however, strongly suggest an approach to solving the problem.

Phase II. (What is given here is not in the format of typical scratch work. This presentation seeks to offer a train of thought. As a rule, scratch work would most likely consist of short phrases, the names of results, and some formulas. You do want to be neat enough to be able to extract what you are going to present and/or to recover the justifications for observations that were made, when you prepare to present your argument.) For \( f(x) = \cos x + x \sin x - x^2 \), we note that \( f(0) = 1 \) while \( f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \left(1 - \frac{\pi}{2}\right) < 0 \). Since \( f \) is a continuous function, we immediately get to conclude that \( f \) has a zero in the interval \( \left(-\frac{\pi}{2}, 0\right) \) and a zero in the interval \( (0, \frac{\pi}{2}) \) by the Intermediate Value Theorem. This tells us that we can claim at least two intercepts, which isn’t enough. Noticing that \( f\left(\frac{\pi}{2}\right) = f\left(-\frac{\pi}{2}\right) \) might prompt us to see if \( f(x) = f(-x) \) in general; in fact,

\[
f(-x) = \cos(-x) + (-x)\sin(-x) - (-x)^2 = \cos x + (x)(-\sin x) - x^2 = f(x).
\]

So now we know that \( f \) is an even function which, because \( f(0) \neq 0 \), means that it has an even number of zeros, half of which are in \((-\infty, 0)\) while the other half is in \((0, \infty)\). We still haven’t closed in on an exact number. Since the given function is differentiable and, (if you did it) the graph looks very well behaved, checking the derivative of \( f \) seems like the next best thing to try. Here the news turns out to be really good. Note that

\[
f'(x) = -\sin x + x \cos x + \sin x - 2x = -x (2 - \cos x).
\]

Because \( |\cos x| \leq 1 \), we know that \( 2 - \cos x \neq 0 \) for all real values of \( x \). This means that \( f''(x) = 0 \) if and only if \( x = 0 \). In the language of differential
calculus, we have the only critical point for $f$ is $(0, 1)$. Furthermore, because $(2 - \cos x) > 0$ for all values of $x$, we have that $f' > 0$ in $(-\infty, 0)$ and $f' < 0$ in $(0, \infty)$; i.e., $f$ is increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$. Therefore, if $f$ has a zero in $(-\infty, 0)$, then it has only one zero there. Combined with our other observations, we now know that $f$ has exactly two x-intercepts.

Phase III.

Since $f(-x) = \cos(-x) + (-x)\sin(-x) = (-x)^2 = f(x)$, $f$ is an even function. From $(2 - \cos x) > 0$ for all values of $x$, it follows that $f'(x) = -x(2 - \cos x) > 0$ in $(-\infty, 0)$. Hence, $f$ is strictly increasing in $(-\infty, 0)$, from which we conclude that each value achieved by $f$ in $(-\infty, 0)$ is achieved only once. Because $f$ is continuous on $\mathbb{R}$, $f\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} \left(1 - \frac{\pi}{2}\right) < 0$ and $f(0) = 1 > 0$, it follows from the Intermediate Value Theorem that there exists a real number $w$ in $\left(-\frac{\pi}{2}, 0\right)$ such that $f(w) = 0$. Thus, $f$ has exactly one real root in $(-\infty, 0)$ and, since $f$ is even, exactly one real root in $(0, \infty)$. Therefore, $f(x) = \cos x + x\sin x - x^2$ has exactly two x-intercepts.

N.B. Not all of the observations made during the scratch work phase went into the proof that was offered. If seeking a well written and concise argument, such omissions are to be expected. It takes a bit of fortitude to avoid the trap of trying to report on all of the work that was done in search of a solution. A common error made by students is submission of narrations of what they did, rather than clearly stated arguments that focus only on what was needed for a correct response.

(B) An Integral Translation Challenge.

Phase I. The scratch work for this one gets us all the way back to looking at the definition of the definite integral over an interval. This leads us towards looking to see if the given summation can be written as

$$\lim_{n \to \infty} \sum_{j=1}^{n} f(x_j) (x_j - x_{j-1})$$

for some function $f$ over some interval $[a, b]$ where \{\(a = x_0, x_1, \ldots, x_{n-1}, x_n = b\)\} are partitions of $[a, b]$. The definition tells us that is the limit exists over all partitions of the interval then it is the definite integral $\int_{a}^{b} f(t) \, dt$. This will only help us if we can find a relevant function and know that the definite integral exists. Then we will be able to consider the limit over specially chosen
sums that match what was given. Now we are ready to play with the given sum.

Phase II. We want to rewrite the summand so that we can see an expression involving \( j \) and \( n \) that could be renamed \( x_j \); the \( x_j \) will have to lead to an \((x_j - x_{j-1})\) that goes to zero as \( n \) goes to infinity. Note that

\[
\sum_{j=1}^{n} \frac{2n}{(n+j)^2} = \sum_{j=1}^{n} \frac{2n}{n^2 \left(1 + \frac{j}{n}\right)^2} = \sum_{j=1}^{n} \frac{2}{\left(1 + \frac{j}{n}\right)^2} \left(\frac{1}{n}\right).
\]

Trying \( x_j = \frac{j}{n} \) gives \( x_j - x_{j-1} = \frac{j}{n} - \frac{j-1}{n} = \frac{1}{n} \). This looks really good! If we take \( f(x) = \frac{2}{(1 + x)^2} \), then the integrand can be rewritten as \( f(x_j) (x_j - x_{j-1}) \) where \( x_0 = 0 \) and \( x_n = 1 \). So the given sum can be considered as an estimating sum for the function \( f \) over the interval \([0, 1]\). The limit over all the estimating sums as the mesh goes to zero, if it exists, is the value of the integral. Since \( f \) is continuous on \([0, 1]\) and \( F(x) = -2 (1 + x)^{-1} \) is an antiderivative for \( f \), from the Fundamental Theorem of Calculus, we have that \( \int_{0}^{1} f(t) \, dt = F(1) - F(0) = -\frac{2}{2} - \left(-\frac{2}{1}\right) = 1 \). We’ve done enough work to justify that the limit that we seek exists and is equal to 1.

Phase III.

Let \( f(x) = \frac{2}{(1 + x)^2} \). From the Fundamental Theorem of Calculus, since \( f \) is continuous in the closed interval \([0, 1]\) and \( F(x) = -2 (1 + x)^{-1} \) is an antiderivative for \( f \), we have that \( f \) is integrable on \([0, 1]\) and \( \int_{0}^{1} f(t) \, dt = F(1) - F(0) = -\frac{2}{2} - \left(-\frac{2}{1}\right) = 1 \). In view of the definition of the integral,

\[
\int_{0}^{1} f(t) \, dt = \lim_{\text{mesh} \mathcal{P}_n([0,1]) \to 0} \sum_{j=1}^{n} f(\xi_j) (x_j - x_{j-1})
\]

where \( \mathcal{P}_n ([0, 1]) \) denotes the set of all partitions \( \{x_0 = 0, x_1, ..., x_{n-1}, x_n = 1\} \) of the interval \([0, 1]\) and \( \xi_j \in [x_{j-1}, x_j] \). Now consider all partitions \( \{x_0 = 0, x_1, ..., x_{n-1}, x_n = 1\} \) of \([0, 1]\) that have \( n \) sections of equal length.
Then, for \(0 \leq j \leq n\), we have \(x_j = \frac{j}{n}\) and \((x_j - x_{j-1}) = \frac{1}{n}\). Choosing \(\xi_j\) as \(x_j\) gives the following Riemann sum

\[
\sum_{j=1}^{n} f(\xi_j) (x_j - x_{j-1}) = \sum_{j=1}^{n} \frac{2}{\left(1 + \frac{j}{n}\right)^2} \left(\frac{1}{n}\right)
\]

which simplifies to \(\sum_{j=1}^{n} \frac{2n}{(n + j)^2}\). Since the limit is unique, we conclude that

\[
1 = \int_{0}^{1} f(t) \, dt = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{2n}{(n + j)^2}.
\]

\[\blacksquare\]

N.B. Take the time to notice how different the order of presentation of the argument was in comparison to the order of discovery of what was going to be used.

\(\textbf{(C) A Fundamental Question..}\)

**Phase I.** Our initial reading of this problem, hopefully, brought the Fundamental Theorem of Calculus to mind. Looking it up, for good measure, we are reminded that the First Fundamental Theorem of Calculus says that if \(f\) is a continuous function on an open interval containing the interval \([a, b]\), then the function defined for \(x \in [a, b]\) by \(G(x) = \int_{a}^{x} f(t) \, dt\) is differentiable on \([a, b]\) and \(G'(x) = f(x)\) there. The integral part of the given problem has \(f(t) = \sqrt{2 + t + \cos^2 2t}\) as an integrand which is continuous for \(t \geq -2\). In the other hand, the upper limit is not just the variable \(x\). This is enough for us to move one to the next phase.

**Phase II.** Finding the answer would be a routine application of the First Fundamental Theorem of Calculus if we had been asked to differentiate \(H(x) \equiv \left[\int_{1}^{x} \sqrt{2 + t + \cos^2 2t} \, dt\right].\) Once we see that we are being asked to differentiate \(H(x^3 \ln(x + 2))\), the idea of using the chain rule quickly follows. If \(g(x) = x^3 \ln(x + 2)\) then \((H \circ g)(x) = \left[\int_{1}^{x^3 \ln(x + 2)} \sqrt{2 + t + \cos^2 2t} \, dt\right].\) The chain rule tells us that if \(H\) and \(g\) are differentiable over the appropriate domains, then \((H \circ g)'(x) = H'(g(x)) \cdot g'(x)\). The 1st FTC gives \(H'\) while \(g'\) follows from our derivative formulas.
Phase III.

For \( x \geq 1 \), let \( H(x) \overset{\text{def}}{=} \int_1^x \sqrt{2 + t + \cos^2 2t} \, dt \) and \( g(x) = x^3 \ln (x + 2) \).

Because \( \sqrt{2 + t + \cos^2 2t} \) is continuous at least for \( t \geq 1 \), by the First Fundamental Theorem of Calculus, it follows that \( H \) is differentiable in \([1, \infty)\) and \( H'(x) = \sqrt{2 + x + \cos^2 2x} \). For \( x \geq 1 \), \( g(x) \geq \ln 3 > 1 \) and \( g \) is differentiable. Hence, \( G = H \circ g \) is well-defined and differentiable. From the Chain Rule, for \( x \geq 1 \), \( (H \circ g)'(x) = H'(g(x)) \, g'(x) \). Substitution and simplification yields that

\[
\begin{align*}
\frac{d}{dx} \left[ \int_1^{x^3 \ln (x + 2)} \sqrt{2 + t + \cos^2 2t} \, dt \right] &= (H \circ g)'(x) \\
= \left( \sqrt{2 + (x^3 \ln (x + 2)) + \cos^2 2(x^3 \ln (x + 2))} \right) \left( \frac{x^3}{x + 2} + 3x^2 \ln (x + 2) \right).
\end{align*}
\]

In this workbook, you will find all of the types of problems that we have discussed. The basic skill and literacy builders can prepare you to make use of the concepts that are being learned. You need to know how to look things up and what things are called in order to explain work you have done and in order to make meaningful use of graphing calculators or computers. To successfully complete eclectic problems and investigations, you will need to develop a better idea of what constitutes appropriate justifications or rationales.

While “free rein” is the rule for Phase II of our problem solving process, Phase III allows only for a limited number of options. In the next section, we will focus attention on the basic approaches to proving (or justifying) mathematical claims.

**Problem Set MPA**

**Building Literacy**

1. Make up three apparently different routine problems and solve them.

2. Make up three eclectic problems, each calling for the use of at least three ideas, concepts, and/or techniques, and solve them.

3. Classify each of the following problems according to the categories

   - **R**: routine
   - **E**: eclectic
   - **I**: investigation.
(a) Find a function \( y = f(x) \) such that \( \frac{dy}{dx} = \frac{x^4 + xe^{4x} + 3}{x \sin(2y)} \).

(b) Suppose that the base of a certain solid is the region in the xy-plane that is bounded by \( y = \sqrt{x} \) and \( y = x^2 \). If each section of the solid that is cut by a plane perpendicular to the y-axis is a rectangle whose height is double its base, find the volume of the solid.

(c) Find the average value of \( f(x) = x^2 \) over the interval \([0, 2]\).

(d) Find the dimension for a box having surface area 324 square centimeters that maximized the volume.

(e) If possible, construct a continuous function that has exactly three relative extrema, one global extremum and two points of inflection. If you cannot construct such a function, then justify why it can not be done; if you offer such a function, carefully show that it satisfies the prescribed list of properties.

(f) For \( k \) a real number and \( n \) a positive integer, explain why \( \int \frac{x^{3n+2}}{\sqrt{x^3 + k}} \, dx \) is elementary.

**Beyond the Basics**

1. Let \( f(x) = x^4 - Ax^2 + x \) where \( A \) is a constant. Are there values of \( A \) for which \( f \) has inflection points at both \( x = 0 \) and \( x = 1 \)? Carefully justify your response.

2. Explain why \( x^3 - 3x + b = 0 \) can have at most one solution in the interval \([-1, 1]\) for any given real number \( b \).

3. Prove that the “Area Shortcut” works for all parabolas of the form \( y = -(x - a)^2 + b^2 \) for any real positive numbers \( a \) and \( b \).

4. Let \( k \) denote a real number, \( m \) denote a positive integer such that \( m \geq 3 \).

   (a) What general form must the function \( g \) have in order for \( \int \frac{g(x)}{\sqrt{x^3 + k}} \, dx \) to be elementary?

   (b) What general form must the function \( h \) have in order for \( \int \frac{h(x)}{\sqrt{x^m + k}} \, dx \) to be elementary?

**Methods of Proof**

Solutions to eclectic problems and investigations require justifications. In particular, when asked to show or prove something, you will need to use what is given to reason towards
what you want to conclude. The process of reasoning that argues from something given—making use of a combination of accepted truths (axioms), basic algebraic manipulations, other proved results; etc.—to a specific conclusion is called **deductive reasoning** or the process of deduction. The proofs of theorems and examples in this text are illustrations of deductive reasoning.

The area of mathematics that supplies the structure that justifies systems of mathematical reasoning is Logic. In this section, we will restrict ourselves to a discussion of the few ideas from elementary Logic that could be immediately helpful.

The building blocks for mathematical claims or theorems are propositions. A **proposition** is a declarative sentence that has a definite truth value of true (T) or false (F). Some examples of propositions are

- The function $f(x) = \frac{x^2}{x^2 - 4}$ is continuous at $x = 2$.
- A square is a rectangle.
- For $x$ real, $\cos^2 x + \sin^2 x = 1$.

while each of the following is not a proposition

- The function $f$ is differentiable.
- A rectangle is a square.
- $\int_0^1 (f(x))^2 \, dx = \left(\int_0^1 f(x) \, dx\right)^2$.

We want a symbolic form for various combinations of propositions so that we can make general claims for how to justify statements involving them. We will use the term **propositional form** for a proposition that is given in symbolic form with capital letters representing propositions. Because the truth value of a propositional form depends on the truth values of the propositions that are substituted for the capital letters, we use truth tables to indicate all of the possibilities.

We want to be able to have several conditions either being given or being derived from our work. Given two propositions $P$ and $Q$, there are two **compound propositions** that we can form: The **conjunction** is the proposition “$P$ and $Q$”, while the **disjunction** is the proposition “$P$ or $Q$”. In the following truth tables, the symbol “∧” is read as “and”, while the symbol “∨” is read as “or”.

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<tr>
<th>$P$</th>
<th>$Q$</th>
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</table>

Thus, we see that the compound propositional form “$P \land Q$” is true only when both of the substituted propositions are true, while the compound propositional form “$P \lor Q$” is false only when both of the substituted propositions are false.
For some approaches to proving statements, it is useful to be able to claim the opposite of either what is given or what is to be proved. Given a proposition $P$, the negation of $P$ is denoted by $\neg P$ and is read as “not $P$”. Of course, the truth value of $\neg P$ is opposite to that of $P$. This is shown in the following truth table.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\neg P$</th>
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<tr>
<td>T</td>
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</table>

Simply negating a proposition by putting a “not” or “it is not the case that” in front of it may not yield a proposition that is well written or easy to understand. The term denial is used for any statement that is logically equivalent to the negation; this allows for rewording of a negation in order to offer clarity and good exposition.

Before going on, use the following list of propositions to form the propositions requested and decide on a truth value.

- $P : 2 + 5 = 10$
- $Q : 3 \cdot 8 = 24$
- $R : \lim_{x \to 2} (x^2 + 7) < 12$
- $M : \text{The product of two continuous functions is a continuous function.}$
- $T : \text{Polynomials are differentiable.}$

- $\neg R$
- $P \land (\neg Q)$
- $(\neg R) \lor (\neg M)$
- $T \land Q$

For mathematical claims and/or theorems, the usual set up is that we are given a hypothesis (some information, conditions, or facts), and are asked to show or told that something else must be true as a consequence of what has been given. The symbolic form for the type of propositions that fit this situation is given in the following definition.

Given two propositions $P$ and $Q$, the proposition “$P$ implies $Q$” is called a conditional statement. When written in this manner, the $P$ is the antecedent or hypothesis, while the $Q$ is the consequent or conclusion. The expression “$P \implies Q$” is read as “$P$ implies $Q$”. The truth table for $P \implies Q$ justifies the process that we just described as a direct proof. Namely, we have the following truth table.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \implies Q$</th>
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<tbody>
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</table>
If the hypothesis is false, the conditional statement is always taken to be true. Consequently, we need only concern ourselves with justifying that a true hypothesis leads to a true conclusion.

The approach to proving that you have seen many times, though it may not have been formally explained, draws conclusions directly from the given information: For a **direct proof**, we start with the hypothesis; then we manipulate, substitute, and make use of other things that we know until we deduce what we were asked to show. Buried with this process is the **transitive property of implies**; i.e.,

For propositions $P$, $Q$, and $R$, if $[(P \Rightarrow Q) \land (Q \Rightarrow R)]$, then $P \Rightarrow R$.

We can show this with a truth table. Suppose that $P$, $Q$, and $R$ are propositions and let $M$ denote the compound proposition $[(P \Rightarrow Q) \land (Q \Rightarrow R)]$. Fill-in the missing truth values to complete the following truth table.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P \Rightarrow Q$</th>
<th>$Q \Rightarrow R$</th>
<th>$M$</th>
<th>$P \Rightarrow R$</th>
<th>$M \Rightarrow (P \Rightarrow R)$</th>
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</table>

If you did the fill-ins correctly, the truth table indicates that, for all truth value assignments for the propositions $P$, $Q$, and $R$, we have that $[(P \Rightarrow Q) \land (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$ is always true. This proves the transitive property.

A propositional form that is true for all truth value assignments to its parts is called a **tautology**. The truth table that you just completed tells us that

$$[(P \Rightarrow Q) \land (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$$

is an example of a tautology.

The transitive property provides the justification that allows us to go from step to step in the deductive reasoning process. In practice, many of the intermediate statements are left off. For illustrative purposes only, we’ll do a simple algebra problem, but we’ll show the sequence of implications that normally are hidden.

**Example.** Show that $5x^2 + 1 = 126$ implies that $x = 5$ or $x = -5$. 
Proof.

If \( 5x^2 + 1 = 126 \), then \( 5x^2 = 125 \).

If \( 5x^2 = 125 \), then \( x^2 = 25 \).

If \( x^2 = 25 \), then \( x^2 - 25 = 0 \).

If \( x^2 - 25 = 0 \), then \( (x - 5)(x + 5) = 0 \).

If \( (x - 5)(x + 5) = 0 \), then \( x - 5 = 0 \) or \( x + 5 = 0 \).

If \( (x - 5) = 0 \) or \( (x + 5) = 0 \), then \( x = 5 \) or \( x = -5 \).

Now, by the transitive property of implies (applied 5 times), we have that if \( 5x^2 + 1 = 126 \), then \( x = 5 \) or \( x = -5 \). ■

N.B. Don’t panic. The algebra example was done this way only to illustrate the way transitivity of implies works. You can continue to do routine algebra problems either by inspection, with a calculator, or with a few simple steps. For the problem given,

\[ 5x^2 + 1 = 126 \Rightarrow x^2 = 25 \Rightarrow x = 5 \text{ or } x = -5 \]

would be a sufficient amount of showing. Furthermore, even proofs for eclectic problems will not contain all the “if, then” statements that go into them.

The general format for a direct proof of \( P \Rightarrow Q \) is given in the following.

<table>
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<tr>
<th>Assume P.</th>
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<td>( \circ )</td>
</tr>
<tr>
<td>( \circ )</td>
</tr>
<tr>
<td>{ deductive reasoning steps }</td>
</tr>
<tr>
<td>Therefore, Q.</td>
</tr>
<tr>
<td>Thus, ( P \Rightarrow Q ).</td>
</tr>
</tbody>
</table>

Two propositional forms are logically equivalent if they have the same truth values for all assignments of truth values to its parts; i.e., if they have the same truth tables. When two propositional forms are logically equivalent, they are interchangeable. To find an alternative approach to proving the conditional statement \( P \Rightarrow Q \), all we need to do is find statements that are logically equivalent to it. Towards this end, let’s consider some conditional statements that can easily be derived from \( P \Rightarrow Q \).

Given the conditional statement \( P \Rightarrow Q \), the converse of \( P \Rightarrow Q \) is the conditional statement \( Q \Rightarrow P \); while the contrapositive of \( P \Rightarrow Q \) is the conditional statement \( \neg Q \Rightarrow \neg P \). Fill-in what is missing in the following truth tables.
If you did the fill-ins correctly, the truth tables show that $P \implies Q$ is logically equivalent to its contrapositive $(\neg Q) \implies (\neg P)$. This means that we have found an alternative approach to proving $P \implies Q$. This approach is called a proof by contrapositive. The general format for a proof by contrapositive of $P \implies Q$ is given in the following.

Assume $\neg Q$.
- Deductive reasoning steps
- Therefore, $\neg P$.
Thus, $(\neg Q) \implies (\neg P)$ and we conclude that $P \implies Q$.

The other important observation that follows from the truth tables you completed is that the truth of a conditional statement does not ensure the truth of its converse; that is, the propositional form $P \implies Q$ is not logically equivalent to $Q \implies P$. Because of this, being able to claim a conditional statement and its converse deserves special designation. Given propositions $P$ and $Q$, “$P$ if and only if $Q$” is a proposition that is called a biconditional sentence and is equivalent to the conjunction

$$(P \implies Q) \land (Q \implies P).$$

Symbolically, we may write $P \iff Q$ where the symbol $\iff$ is read as “if and only if.”
Complete the following truth table for the biconditional propositional form.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( P \implies Q )</th>
<th>Q \implies P</th>
<th>((P \implies Q) \land (Q \implies P))</th>
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<tbody>
<tr>
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</table>

If you did the fill-ins correctly, you should be able to see that \( P \iff Q \) is true only when \( P \) and \( Q \) have the same truth values, either both true or both false.

**Problem Set MPB**

1. For each of the following, decide if the given statement is a proposition. State your position clearly. If you believe it is a proposition, indicate its truth value or why you believe there is a definite truth value; if you believe it is not a proposition, indicate why a definite truth value cannot be assigned.

   (a) \( e < \pi \land \ln 2 > e \).
   (b) This sentence is false.
   (c) \( \cos x - \sin x = 0 \).
   (d) Calculus is interesting.
   (e) In the fall quarter of 2048, UCDavis will have exactly 20,000 students enrolled.
   (f) \( \int e^{x^2} \, dx \) is elementary.

2. Suppose that \( P \), \( Q \), and \( R \) are “\( e \) is irrational”, “\( \int_0^{\pi/4} \sin x \, dx < \int_0^{\pi/4} \cos x \, dx \)”, and “\( \sin \frac{\pi}{6} = 2 \left( \cos \frac{\pi}{12} \right) \left( \sin \frac{\pi}{12} \right) \)”, respectively. Form each of the following and decide on a truth value.

   (a) The conjunction of \( P \) and \( Q \).
   (b) The disjunction of \( P \) and \( R \).
   (c) The conditional statement in which \( R \) is the antecedent and \( P \) is the consequent.
   (d) The biconditional statement formed from \( Q \) and \( R \).
   (e) The negation (or, to improve the reading, a denial) of \( P \).
   (f) The negation of \( R \).

3. For each of the following, use a truth table to determine whether or not the given propositional form is a tautology. The letters \( P \), \( Q \), and \( R \) stand for propositions.
4. Indicate the order in which the following conditional statements should be placed in order to apply the “transitive property of implies” to prove a conditional statement. Then explicitly write the conditional statement for which the claims could be used to give a proof.

In what follows, we are working with a right triangle whose hypotenuse has length $c$ and whose legs have lengths $a$ and $b$.

(a) If $\frac{1}{4}c^2 = \frac{1}{2}ab$, then $\frac{1}{4}ab = \frac{1}{4}(a^2 + b^2)$ by the Pythagorean Theorem.
(b) If $2ab = a^2 + b^2$, then $a^2 - 2ab + b^2 = 0$.
(c) If $a = b$, then the triangle is isosceles.
(d) If the area of the triangle is $\frac{1}{4}c^2$, then $\frac{1}{4}c^2 = \frac{1}{2}ab$.
(e) If $(a - b)^2 = 0$ then $a - b = 0$ or $a = b$.
(f) If $\frac{1}{2}ab = \frac{1}{4}(a^2 + b^2)$, then $2ab = a^2 + b^2$.
(g) If $a^2 - 2ab + b^2 = 0$, then $(a - b)^2 = 0$.

Mathematical Writing

The purpose of this section is to provide some guidance on writing mathematics. You might be thinking:

“*We have to WRITE? But this is a MATH class! I’ve NEVER heard of doing THAT before.*”

Welcome to the University! As you expand upon your mathematical knowledge and work towards being able to apply that knowledge, you will need to take more responsibility
for justifying claims that you make. Eventually if you work in an environment where co-workers have different backgrounds, your ability to explain and justify could be essential to making progress in solving various problems.

With the following, we offer both general and specific guidelines, provide examples of good and poor mathematical writing, and define a few commonly used terms. The list is not exhaustive – it is just to help you start thinking in the right direction. The unifying principle of the guidelines is CLARITY. The clearer you can present your work, the more positively disposed the reader will be towards it.

A solution or explanation of mathematics should be self contained, well presented, and complete.

Some specifics:

1. Clearly restate the problem to be solved. Do not assume that the reader knows what you’re talking about, even if “the reader” is your instructor.

2. For long or complicated problems, provide a paragraph or sentence that explains how you will approach the problem. Providing an outline of your solution often helps.

3. Spelling, punctuation, and grammar should be correct. If in doubt, consult a dictionary or grammar handbook. On very long or very important written assignments, ask a friend to proofread your penultimate draft.

4. State the answer to the original problem in a sentence that can stand on its own.
   
   poor: $x = 3$.

   good: Hence, the width that maximizes volume is $x = 3$ cubits.

5. Give acknowledgement where it is due. You should name any people with whom you talked, and you should cite books, computer software, etc., if you got ideas from them. See the Discussion Sets for examples.

   The person(s) reading your mathematics should not have to guess what you mean.

Some specifics:

6. Always define or identify all variables that are being used.

   poor: Distance traveled is $d = rt$. 

   good: Distance traveled is $d = rt$. 

good: Distance traveled is $d = rt$, where $d$ is the distance (in miles), $t$ is time in hours, and $r$ is the constant rate of travel (in miles per hour).

Some judgement may be required to decide whether or not to assign a variable to a quantity. If the quantity is referred to only once, don’t assign a variable to it. If it is referred to two times, use the goals of “clarity” and “smoothness” of exposition to help you decide.

poor: We see that $A = (1/2) \times h \times b$, where $A$ stands for the area of the triangle, $b$ stands for the base of the triangle, and $h$ stands for the height of the triangle, and so $A = (1/2) \times 3 \times 4 = 6$ square inches.

good: We see that the area of the triangle will be half the product of its base and height; hence the area of the triangle is $(1/2) \times 3 \times 4 = 6$ square inches.

7. Always include units of measurement. When you can use words like “of”, “from”, “above”, etc., do so. The more specific you are, the better. This is particularly true of words like “position” (height above the ground? sitting down? political stance?) and “time” (5 o’clock? since a major league baseball game was played? since the experiment started?), which are particularly vague unless you qualify them.

good: Distance traveled on the bicycle ride is $d = rt$, where $d$ is distance from the start (in miles), $t$ is the hours since the start of the ride, and $r$ is the constant rate of travel on the ride (in miles per hour).

8. Clearly state assumptions that underlie any formulas that you use.

Which of the “bicycle” examples is the best? Why? (An answer is given at the end of this section.)

Distance traveled on the bicycle ride is $d = rt$, where $d$ is distance from the start (in miles), $t$ is the time since the start of the ride (hours), and $r$ is the constant rate of travel on the ride (in miles per hour).

Distance traveled on the bicycle ride is $d = rt$, where $d$ is distance from the start (in miles), $t$ is the time since the start of the ride (hours), and $r$ is the rate of travel on the ride (in miles per hour).

9. Clearly label diagrams, tables, graphs, or other visual representations of the mathematics that you do. A picture is worth a thousand words, especially if it’s well labelled. Note that you should still define variables, even if you use them as labels in a diagram.
The presentation of mathematics should fit together well for “smooth” reading, regardless of the inclusion of equations, symbols, diagrams, etc.

10. Like clauses within sentences, mathematical equations require punctuation. Put periods after the end of a computation if the computation ends the sentence and a comma if it doesn’t. One way to check this is to read the sentence out loud. In spoken English, people generally use a falling inflection at the end of a sentence and pause a bit after a comma. Try reading the following two selections aloud.

**good:** The surface area $S$ of a box whose sides have length $l$ inches, width $w$ inches, and height $h$ inches is

$$S = 2lw + 2lh + 2wh \text{ square inches.}$$

So, if $l = 4$ inches, $w = 8$ inches, and $h = 1/2$ inch, substituting yields

$$S = 2(4)(8) + 2(4)(1/2) + 2(8)(1/2) = 76 \text{ square inches,}$$

which is smaller than a breadbox.

**poor:** The surface area $S$ of a box whose sides have length $l$ inches width $w$ inches and height $h$ inches is

$$S = 2lw + 2lh + 2wh \text{ square inches}$$

So if $l = 4$ inches $w = 8$ inches and $h = 1/2$ inch substituting yields

$$S = 2(4)(8) + 2(4)(1/2) + 2(8)(1/2) = 76 \text{ square inches}$$

which is smaller than a breadbox.

Now compare the two selections carefully and notice how they differ. You should find 8 differences (and no, Waldo isn’t in either passage).

11. Very long or important formulas should have their own lines. This is especially helpful if you do additional manipulations on them. A formula placed on its own line is more obvious and easier to spot.

Which of the following looks better? (An answer is given at the end of this section.)

The surface area $S$ of a box whose sides have length $l$ inches, width $w$ inches, and height $h$ inches is $S = 2lw + 2lh + 2wh = 2(lw + lh + wh) \text{ square inches.}$
The surface area $S$ of a box whose sides have length $l$ inches, width $w$ inches, and height $h$ inches is

$$S = 2lw + 2lh + 2wh$$

$$= 2(lw + lh + wh) \text{ square inches.}$$

12. **Connect** or tie computations together with descriptive words or phrases (e.g., “Simplifying, we get...”, “It follows from Theorem 42...”, etc.). Everything should flow smoothly when formulas are replaced by “blah blah blah” (try it on the first breadbox example).

13. Do not use mathematical symbols in place of words. Conversely, don’t use words where mathematical symbols are appropriate.

*poor:* If the weight of the package $> \text{the maximum allowed}$, then it cannot be shipped via carrier pigeon.

*good:* If the weight of the package is greater than the maximum allowed, then it cannot be shipped via carrier pigeon.

Rewrite the following example so that symbols and words are used appropriately. A possible solution is given at the end of this section.

Let $V$ stand for the volume of a bottle and $n = \text{the total \# of bottles produced}$. Then the formula for the total amount of kiwi wine, $K$, produced by the Department of Viticulture and Enology is $K = nV$.

14. **Never** start a sentence with a symbol.

Rewrite the example so that it begins with words, not a symbol. One possible solution is given at the end of this section.

$x^2 - a = 0$ has no real roots if $a < 0.$
Little words in mathematics can be both important and useful.

- “if” vs. “then”: In mathematics, “if” denotes a hypothesis, an assumption, or a condition. On the other hand, “then” denotes a consequence of hypotheses, assumptions, or conditions. The phrase that follows “if” cannot be interchanged with the phrase that follows “then”. Saying, “If it is raining outside, then the sidewalk will get wet” is not the same as saying, “If the sidewalk is wet, then it is raining outside.” The first is true while the second is not necessarily true (your roommates may have had a water balloon fight on the sidewalk).

- “if—then” vs. “if and only if”: The phrase “if and only if” means both “if $A$ is true, then $B$ is true” and “if $B$ is true, then $A$ is true”.

  For example, after doing the work to get from $2x + 1 = 5$ to $x = 2$, it is correct to say, “We have shown that, if $2x + 1 = 5$, then $x = 2$.” It is incorrect to say, “We have shown that, if $x = 2$, then $2x + 1 = 5$.” Consequently, it is incorrect to say, “We have shown that $2x + 1 = 5$ if and only if $x = 2$” unless you have shown both ways.

- In the following list, words within a group are equivalent. Use them in your mathematical writing to avoid repeating the same phrase or word.

  (i) therefore, so, hence, accordingly, thus, it follows that, we see that, from this we get, then.

  (ii) I assume that, assuming, where, let, given

  (iii) if, whenever, provided that, when

  (iv) notice that, note, recall

  The word “since” is not an alternative to “if” in an “if,... then” statement. When opening a sentence with a clause starting with since, the clause should be followed by a comma. “Since $A$, then $B.$” is not well written; it should be corrected to “Since $A$, $B.$”

- Learn a little Latin:

  (i) e.g. – *exempli gratia*; for example

  (ii) i.e. – *id est*; that is

  (iii) q.e.d. – *quod erat demonstrandum*; which was to be demonstrated
(iv) etc. – et cetera; and other things;
(v) n.b. – nota bene; note well.

We leave you with a parting thought:

A good attitude to the preparation of written mathematics is to pretend that it is spoken. Pretend that you are explaining the subject to a friend on a long walk in the woods, with no paper available; fall back on symbolism only when it is really necessary.

–Paul Halmos

Answers and possible solutions

8. The best is the one that states that rate is constant.
11. The first looks better and is easier to read.
13. Let $V$ stand for the volume of a bottle and $n$ be the total number of bottles produced. Then the formula for the total amount of kiwi wine, $K$, produced by the Department of Viticulture and Enology is $K = nV$.
14. The quadratic equation $x^2 - a = 0$ has no real roots if $a < 0$.

Problem Set MPC

1. Correct the following to obtain sentences that satisfy the guidelines discussed in this section.

   (a) since the function $f$ is $> 0$, then $f$ is integrable.
   (b) $f(x) = \sin 2x + \pi x^2$
   (c) Substituting $a = 2c + b$ into the formula $S = a^2 - b^2$ gives $S = (2c + b)^2 + b^2$ which is the same as $S = 4c^2 + 4bc$.

2. The following is a proposed “solution” to the problem: Determine and classify all the relative extrema of $f(x) = (x - 1)^3 (x + 4)^{-4}$. It demonstrates many of the commonly made errors in exposition and in mathematical precision. Analyze the offering, indicating where corrections are necessary.

   The first thing I did was find the derivative of the function. then because $f'(x) = \frac{(x - 1)^2 (-x + 16)}{(x + 4)^5}$ I got that the critical numbers are when $x = 1, x = 16, and x = -4. f'(-10) < 0, f'(0) > 0, f'(10) > 0, and f'(20) < 0$ gave us just that $x = 16$ is a relative maximum
3. For the following, introduce punctuation and capitalization in order to transform what is given into a well written argument.

suppose we have a right triangle with hypotenuse having length $c$ and sides with lengths $a$ and $b$ if the triangle is isosceles then $a = b$ hence the area of the triangle is $a^2/2$ from the Pythagorean Theorem we also have that $a^2 + b^2 = c^2$ it follows from transitivity of equals that $2a^2 = c^2$ or $a^2 = \frac{c^2}{2}$ now substitution yields that the area of the triangle is $\frac{1}{2}a^2 = \frac{1}{2}\left(\frac{1}{2}c^2\right) = \frac{1}{4}c^2$ therefore if the triangle is isosceles then the area is $\frac{1}{4}c^2$

Literature Cited for the Mathematical Writing Section


Mathstories

Annalise Crannell is an assistant professor of mathematics at Franklin and Marshall College. Paul Halmos is a Professor of Mathematics at Santa Clara University and is a renowned advocate for clarity in mathematical exposition. Donald Knuth is a writer of bad puns, aficionado of the arts and letters, and Professor of Computer Science at Stanford University. The typesetting program, \TeX, that Knuth designed has revolutionized technical publication. This document and other course handouts were prepared using \TeX(or one of its modifications).