Chapter 3
Discussion Set Booklet

Introduction

1 Discussion sets are for guided work through which you can practice, explore, and discover, as well as generally enrich your understanding of Calculus and the world of mathematics. These are designed to complement lectures and the text. Each is a self-contained unit, i.e., all new definitions and theorems needed are provided within the body of the discussion set. You are responsible for all material that is treated within discussion sets, even when it is not covered in lecture.

Each set is intended to engender and facilitate group work. More often than not, the discussion sets contain questions for which you are asked to work independently, compare answers with other people, and resolve any differences before going on. In other questions, you are asked to discuss a concept with other students. We believe strongly that such interaction contributes positively to the learning process. Your classmates are your partners in this enterprise.

Discussion sets usually contain questions for which you are asked to write a few sentences or a paragraph of explanation. These are offered to help you learn mathematics by writing. The idea is simple: you must understand a subject in order to write well about it. If you find yourself struggling for words, using vague terms, or avoiding precise language, it may mean that your comprehension is incomplete. Work with others (students, a discussion facilitator, the instructor, etc.) to improve your understanding. The hope is that your learning of mathematics will be better and deeper as a result of writing about it.

Each discussion set has four sections: (1) a Self-Help Background Check; (2) Problems for Guided Discussion or Inquiry; (3) Open Discussion or Group Investigation; and (4) Discussion Set Assignment. The following describes how they fit together and how they might be used in a discussion session with a facilitator or leader.

- The **Self-Help Background Check** offers problems for you to check your recol-

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lection of material or skills that are directly related to the new material within the discussion set. The problems should be review and are to be done **BEFORE** the start of discussion. Answers are provided at the end of the discussion set so that you can check your own proficiency. Discussion session time will not be spent on this section, except for an offer by your facilitator to answer brief questions or to volunteer insights.

- Your discussion facilitator will help you work through the second section, **Problems for Guided Discussion or Inquiry**. In this section, you will be given time to work on a designated problem or problems, followed by a brief wrap-up and opportunity for questions.

- During the **Open Discussion or Group Investigation**, you will continue to work within groups while your facilitator circulates to offer hints and suggestions, as well as to answer questions.

- The **Discussion Set Assignment** provides an opportunity to apply what has been learned or focused on in the discussion set. We encourage you to talk about the problems in the Assignment with others in your group during discussion, but expect the problems to be completed outside of class.

In addition to completing the Self-Help Background Check prior to discussion, it is a good idea to look through the discussion set and to read the Discussion Assignment in advance of discussion. This will help you to know what to expect. It will put you in a better position to know how to use wisely the time that you have with the discussion facilitator.

We hope that you will enjoy the active learning of Calculus fostered by the discussion sets. Calculus is a lively and useful subject; discussion sets illustrate this in many different ways. We wish you a productive experience with differential calculus and many opportunities to develop new insights, to grow intellectually, and to discover new mathematical ideas and new ways of thinking about mathematics.
Discussion Set C1: Surfaces

This discussion set focuses on graphing surfaces in 3-space. The basic technique is to reduce the analysis of surfaces in 3-space to the analysis of curves in 2-space via the use of traces and level curves.

**Definition**  The intersection between a surface $S$ in 3-space and the horizontal plane $z = c$ (where $c$ denotes a constant) is called the trace of $S$ along $z = c$ or a trace of $S$ parallel to the $xy$-plane. If we project this intersection onto the $xy$-plane, then the projection is called the level curve of $S$ corresponding to $z = c$.

(Remark: A similar definition can be made for a trace of $S$ parallel to the $yz$-plane and for a trace of $S$ parallel to the $xz$-plane. In the former case, we use planes of the form $x = a$, whereas in the latter, we use $y = b$, where $a$ and $b$ denote constants.)

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. On the set of axes below, draw the corresponding graph of the given equation in the $xy$-plane. Then give a verbal description of the graph. (e.g. Is it a circle? Hyperbola? Line? etc.)

   (a) $4x^2 + \frac{y^2}{9} = 1$.
   (b) $4x^2 - \frac{y^2}{9} = 1$.
   (c) $4x^2 - \frac{y^2}{9} = 0$.

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2. On the set of axes below, sketch the curves $x^2 - y^2 = k$, for $k = -2, -1, 0, 1, 2$.

3. On the set of axes below, sketch the indicated plane.

(a) $z = 1$.
(b) $y = 1$.
(c) $x = 1$. 
If you missed more than one problem, consult our text on pages S7-S10 and pages 779-784, or if time permits, go the office hours for clarification. In addition, problems 23, 25, and 31 on page S11 provide extra practice.

**Guided Inquiry**

4. Consider the surface $S$ pictured in Figure 1 which is the portion of an inverted circular cone lying above the $xy$-plane.

![Figure 1: An inverted circular cone](image)

Our goal is to examine the traces of $S$ that are parallel to the $xy$-plane. Answer the following questions. Be certain to compare your answers with those of your peers, and resolve any discrepancies that you encounter.

(a) In Figure 1, sketch indications of the planes $z = 1$ and $z = 4$. Briefly describe the nature of the intersection of $S$ with each of these planes?
(b) In general, for what values of $c$ does the plane $z = c$ intersect $S$?

(c) Let’s look at some concrete examples. On Figure 1, sketch enough of the planes $z = 0$, $z = 1$, $z = 2$, and $z = 3$ to show how they cut across the surface. The curves along which these planes and the surface intersect are called the traces of $S$ parallel to the $xy$-plane. Describe these traces.

(d) Of the three traces from part (c), which is largest? The smallest? Briefly describe a relationship between the size of the trace and the position of the corresponding horizontal plane.

(e) Often, it is useful to be able to compare various traces. One way of comparing traces is by sketching the corresponding level curves. Level curves are obtained by projecting the traces onto the appropriate coordinate plane. For example, consider the trace obtained by intersecting the plane $z = 1$ with $S$. Projecting this trace onto the $xy$-plane, we obtain the picture in Figure 2.
Note the label “$c = 1$”, which denotes the fact that this level curve is the projection of the trace of $S$ along the plane $z = 1$. On Figure 2, sketch the level curves of $S$ corresponding to the values $c = 0, 2, 3$. Be certain to properly label the level curves. (Remark: You don’t have to find the $x$- and $y$-intercepts. A rough sketch will do.)

5. Let us again consider the inverted cone from problem 4.
Working individually, sketch the level curves of $S$ that are parallel to the $xz$-plane. Use the set of axes provided below. When you are done, compare your sketches with those of the others in your group; resolve any disputes and/or discrepancies.

6. In the previous problems, you were given the surface and asked to sketch the level
curves. The process is reversible. For example, suppose you were told that the level curves parallel to the $xy$-plane of a certain surface looked like what is shown below.

Working together, sketch what the surface $S$ might look like. Use the set of axes provided below. Be certain to compare your sketch with the others in your group and resolve any disputes and/or discrepancies.
As this last problem illustrates, once we know what the level curves and traces of a surface look like, we can sketch the surface. This observation is the key to sketching a surface given by an equation.

**How to graph a surface described by an equation**

Given an equation in 3-space, one finds a trace parallel to the \( xy \)-plane by setting the variable \( z \) equal to a constant. Varying the constants to which \( z \) is set generates a set of traces. Traces parallel to the \( yz \)-plane are found by setting the variable \( x \) equal to various constants. Finally, traces parallel to the \( xz \)-plane are found by setting the variable \( y \) equal to selected constants. For any given trace, a corresponding level curve is obtained by projecting the trace into the plane to which the trace is parallel. On the other hand, once we know what the level curves look like, we can use these curves to sketch the surface.

**Group Investigation**

7. Consider the surface defined by the equation

\[
\frac{x^2}{4} - y^2 + z^2 = 1.
\]

This surface is an example of a **hyperboloid of one sheet**. In this problem, we will sketch the surface.

(a) Describe the traces parallel to the \( xy \)-plane and the corresponding level curves. In particular, sketch the level curves corresponding to \( c = -1, 0, 1, 2 \).
(b) Describe the traces parallel to the $yz$-plane. In particular, sketch the level curves corresponding to $a = -1, 0, 1, 2$.

(c) Describe the traces parallel to the $xz$-plane. In particular, sketch the level curves corresponding to $b = -1, 0, 1$.

(d) Sketch the surface, using the above information. Be certain to compare your sketch with those of the others in your group.
8. Graph the surface given by the equation \( z = x^2 - y^2 \).

Discussion Set C1 Assignment

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section of Chapter 1 in Cameos for guidance on style. Feel free to discuss the problem with others, BUT your solution should be written, independently, in your own words.

9. For each given equation, describe the trace of its surface along the \( xz \)-plane. (i.e. the intersection of the surface and the \( xz \)-plane.) Then identify the surface (i.e. Is it a sphere? Ellipsoid? What is it?) Please note that some of the surfaces whose equations appear below may not have been described in today’s discussion set. However, they are discussed in the text in sections 14.1 and 14.2.

(a) \( x^2 + 9y^2 + 4z^2 = 25 \)
(b) \( x^2 + 9y^2 - 4z^2 = 25 \)
(c) \( x^2 - 9y^2 - 4z^2 = 25 \)
(d) \( x^2 - 9y^2 - 4z^2 = -25 \)
(e) \( x^2 + 9y^2 + 4z^2 = 0 \)
(f) \( x^2 + 9y + 4z^2 = 0 \)

10. Figure 3 illustrates the level curves parallel to the \( xy \)-plane for a given surface. Use these level curves to sketch the surface. (N.B. This problem was contributed by Prof. Duane Kouba.)

**Brief answers to problems from the Self-Help Background Check:**

1. (a) An ellipse.
   (b) A hyperbola that crosses the \( x \)-axis at the points \((-1/2, 0)\) and \((1/2, 0)\).
   (c) Two lines intersecting at the origin.

2. The graph of \( x^2 - y^2 = 0 \) looks like two lines intersecting at the origin. For the other values of \( k \), the graphs are hyperbolas. If \( k \) is positive, then the hyperbola crosses the \( x \)-axis at the point \((\sqrt{k}, 0)\) and \((-\sqrt{k}, 0)\). If \( k \) is negative, then the hyperbola crosses the \( y \)-axis at the points \((0, \sqrt{|k|})\) and \((0, -\sqrt{|k|})\).

3. (a) A plane parallel to the \( xy \)-plane (i.e. floor).
   (b) A plane parallel to the \( xz \)-axis (i.e. side wall).
   (c) A plane parallel to the \( yz \)-axis (i.e. back wall).

**Mathstory**

One area of research that is currently very popular is the study of **minimal surfaces**. One example of a minimal surface can be obtained by taking a piece of wire, wrapping it into a loop, and dipping it into soapy water. The resulting soap film forms a surface that has a remarkable property: it has the least amount of surface area amongst all those other surfaces having the same wire as their boundary (hence, the adjective **minimal**). The study of minimal surfaces is related to a wide range of applications outside of the bathtub; it plays a role in modern materials science, for example. (Peterson, 1990)
Literature Cited

Discussion Set C2: Branches, Trees, and the Chain Rule

The purpose of this discussion set is to develop skill at constructing and using a “branch” or “tree” diagram to facilitate application of the chain rule for functions of several variables. We begin by taking a slightly different look at our old pal from MAT 21A.

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

Recall the definition of the composition of functions: For the sets X, U, and Y, let g be a function from X to U and f be a function from U to Y. Then the composition of f and g is the function that assigns to each element $x$ in X the element $f(g(x))$ in Y.

1. Write the function $\left(5 + (x^2 - 1)^3\right)^6$ as a composition $(f \circ g \circ h)(x)$ of three functions $f$, $g$, and $h$.

2. Differentiate the function $f(x) = \left(5 + (x^2 - 1)^3\right)^6$.

3. Suppose that $f(x, y) = x^2 + y^2$, where $x$ and $y$ are functions of $t$, given by $x(t) = \cos(t)$, and $y(t) = \sin(t)$.

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(a) By direct substitution, find $f(t)$.

(b) Using part (a), find $\frac{df}{dt}$.

If you missed more than one problem, consult our text on page 149, or if time permits, go to office hours for clarification. In addition, problems 17 and 37 on page 153 provide extra practice.

**Guided Inquiry**

In MAT 21A, the chain rule gave us a means of finding derivatives of functions that could be written as the composition of other functions. We recently saw that the result generalizes to functions of several variables. That is, we have the

**Chain rule:** Let $z = f(x, y)$ have continuous partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Let $x = g(u, v)$ and $y = h(u, v)$ have continuous partial derivatives $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$, and $\frac{\partial y}{\partial v}$.

Then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.$$ 

Before going on to several variables, let’s take a moment to gain some perspective on what you did in the Self-Help Background Check. For the situation $F(x) = (f \circ g \circ h)(x)$, consider the pictorial representation below.

```
f ------ g ------ h ------ x
```

4. The following is a way to read the Chain rule formula for a function from the given representation. One description is: think of yourself as “walking along a path” from $f$ to $x$, taking the rates of change of the functions passed through with respect to the variable that follows.

For the picture above,
we start with \( f \); then passing through \( g \) contributes \( \frac{df}{dg} \) . . .

while passing \( g \) to \( h \) contributes \( \frac{dh}{dg} \) . . . ;

finally, passing through \( h \) gets us to \( x \) which contributes \( \frac{dh}{dx} \).

The chain rule tells us to multiply individual contributions together. With this reading, the representation above leads to

\[
\frac{dF}{dx} = \frac{df}{dg} \frac{dg}{dh} \frac{dh}{dx},
\]

a familiar friend.

Of course, we didn’t need the pictorial representation to get this formula. Our point is to illustrate use of such pictures where we already know what is happening in order to gain insight that will be helpful in a new, more complicated, situation.

What about multivariable functions? Well, you can use the work you did above as a model for writing a definition of the composition of functions of more than one variable. We begin with a definition, followed by doing a problem that motivates the chain rule and makes our idea for using a pictorial representation more useful.

**Let** \( X, U, \) and \( Z \) be sets of real numbers. Let \( g \) and \( h \) be functions from \( X \times X \) to \( U \) and let \( f \) be a function from \( U \times U \) to \( Z \). Then the **composition of \( f \)** with \( g \) and \( h \) is the function that assigns to each pair \((x, y)\) in \( X \times X \) the element \( f(g(x, y), h(x, y)) \) in \( Z \). This composite function also may be written as \( f(u, v) \), where \( u = g(x, y) \) and \( v = \) \( \) \( \).

5. Consider \( z = f(u, v) = 4u + \sin v \), where \( u = g(x, y) = xy \) and \( v = h(x, y) = x^2 + 3y \).

(a) Write \( z \) as a function of \( x \) and \( y \); i.e., \( z = f(g(x, y), h(x, y)) \).

(b) For the rewritten function from (a), compute \( \frac{\partial z}{\partial x} \).
(c) For the original function \( z \), compute \( \frac{\partial z}{\partial u} \) and \( \frac{\partial z}{\partial v} \).

(d) For \( u = xy \) and \( v = x^2 + 3y \), compute \( \frac{\partial u}{\partial x} \) and \( \frac{\partial v}{\partial x} \).

(e) Using parts (c) and (d), express \( \frac{\partial z}{\partial x} \) in terms of \( \frac{\partial z}{\partial u} \), \( \frac{\partial z}{\partial v} \), \( \frac{\partial u}{\partial x} \), and \( \frac{\partial v}{\partial x} \). Compare the result to your answer in (a).

Notice that we can use the analogous steps to compute \( \frac{\partial z}{\partial y} \) in terms of \( \frac{\partial z}{\partial u} \), \( \frac{\partial z}{\partial v} \), \( \frac{\partial u}{\partial y} \), and \( \frac{\partial v}{\partial y} \). If you wish, do this for additional practice.

(f) Speculate on a use for the pictorial representation:

From the given picture, try to come up with a description of a “walk to \( x \)” along the branches of the tree diagram that will lead us to the formula for \( \frac{\partial z}{\partial y} \).

Group Investigation

The point to drawing a tree or branch diagram is that often it can help us to generate the correct form of the chain rule. A tree diagram illustrates the relationship between the variables within a composite function. For example, a tree diagram corresponding to

\[
f(r,s) = e^{2r-s}, \quad r(x,y) = x^2 + y, \quad s(x,y) = \cos(xy),
\]
is shown below on the right.

Each “level” within the diagram denotes a corresponding level of nesting within the composite function. To use this tree diagram to set up the partial derivative formulas with the chain rule, all branches that lead from the independent to the dependent variable must be “traversed.” While on a branch, we take the rate of change of the function passed through with respect to the variable that follows; the resulting set of partials (from one path) is multiplied together. Finally, the products that result from each path to the dependent variable are added.

6. For the given diagram, one path from \( f \) to \( x \) is \( f \to r \to x \) while the other is \( f \to s \to x \). The first leads to the product

\[
\frac{\partial f}{\partial r} \frac{\partial r}{\partial x}
\]

while the second leads to

\[
\frac{\partial f}{\partial s} \frac{\partial s}{\partial x}
\]

Finally, summing gives

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \frac{\partial s}{\partial x}.
\]

The chain rule also can take other forms, depending on \( f \) and its arguments.

7. For another example, suppose that \( z = f(x, y) \), where \( x = g(u, v, w) \), and \( y = k(u, v, w) \). The tree diagram is shown below. Use it to write the form of the chain rule that would be used to compute \( \partial z / \partial u \).

8. Suppose that \( z = f(x, y, t) \), where \( x = g(u, v) \), \( y = h(u, v) \), and \( t = k(u) \). A tree
The form of the chain rule that would be used to compute $\frac{\partial z}{\partial u}$ is

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial t} \frac{\partial t}{\partial u} + \frac{\partial f}{\partial u}.$$  

Notice from the given tree diagram that $t$ is a function of $u$ alone. Therefore, in the chain rule, we use an “ordinary derivative” rather than a partial derivative to express $dt/du$. In general, an ordinary derivative should be written whenever the argument of a function contains only one independent variable.

9. Suppose that $z = f(p, q)$, where $p = g(u, q)$ and $q = h(u, v)$. Draw a tree diagram that represents the set-up and use it to write the form of the chain rule that would be used to compute $\partial z/\partial u$.

10. Given the tree diagram

```
    p
   / \  \\
  s   r
 /   /  \\
q   s
  /   /  \\
 t   t
```

```
    k
   /  \\
  q   s
   /   /  \\
 t   t
```

write a set of relationships in terms of functions of several variables that would have led to this diagram.

**Discussion Set C2 Assignment**

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section of Chapter 1 in *Cameos* for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

11. Let \( x = (\tan uv - v^2)^{-2}, u = rst, \) and \( v = \ln (2s + t^2) \).

(a) Draw a tree diagram that pictorially represents the relationships among the variables.

(b) Compute \( \frac{\partial x}{\partial r}, \frac{\partial x}{\partial s}, \) and \( \frac{\partial x}{\partial t}. \)

By now, you probably have deduced that many composite functions, when written explicitly, can be differentiated without the need to appeal to the chain rule. In these cases, it is possible, but potentially tedious, to grind out the derivatives directly. On the other hand, if a function is given only in general terms, then brute force cannot be used and the chain rule is essential. Usually, this isn’t too difficult – all that is needed is a good understanding of the chain rule and careful bookkeeping.

12. Consider the function \( w = f(x^2 + y^2). \)

(a) To make things conceptually easier, note that \( w = f(x^2 + y^2) \) is equivalent to \( w = f(u) \), where \( u = x^2 + \) (and hence, \( u = g(x, y) \) for the appropriate function \( g \)).

(b) Use the observation in (a) to show that \( y \frac{\partial w}{\partial x} - x \frac{\partial w}{\partial y} = 0. \)

**Brief answers to problems from the Self-Help Background Check:**

(1) \( f(x) = x^6, g(x) = 5 + x^3, \) and \( h(x) = x^2 - 1 \)

(2) \( 36x \left( 5 + (x^2 - 1)^{3/2} \right)^5 \left( x^2 - 1 \right)^2 \)
\[(3a) \ f(t) = 1, \ (3b) \ \frac{\partial f}{\partial t} = 0\]

**Mathstory**

Many mathematicians and historians of science consider Karl Friedrich Gauss (1777-1855) to be one of the greatest mathematicians of all time. It is said that Gauss was three years old when he discovered an error in his father’s payroll accounts (Simmons 1992). From this precocious start, he went onto master and make great contributions to geometry, number theory, algebra, and applied mathematics. Moreover, through his dissatisfaction with the sloppy proofs of some of his predecessors, Gauss created the modern approach to writing mathematics with rigor and elegance. In this sense, Gauss separated modern mathematics from all that went before.

**Literature Cited**

Discussion Set C3: Extrema in Space

The goal of this discussion set is to give you some practice in dealing with optimization problems involving two independent variables. In particular, we address the following question:

Given a function \( f(x, y) \) defined on a planar region \( R \), how do we find the global maximum (or minimum), provided \( f(x, y) \) even has one?

The phrase “provided \( f(x, y) \) even has one” is very important because, in general, global extrema need not exist. (See problem 9a for a counterexample.) However, in certain circumstances, the existence of global extrema is guaranteed.

Theorem 1: Let \( R \) be a bounded planar region that includes the bounded curve that forms its boundary, and \( f(x, y) \) be a function that is continuous on \( R \). Then \( f(x, y) \) attains a global maximum and a global minimum on \( R \).

And when \( f(x, y) \) has continuous partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) throughout the region \( R \), we can even specify where the global extrema occur.

Theorem 2: Let \( R \) be a planar region bounded by a curve, and \( f(x, y) \) be a function that has continuous partial derivatives \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) throughout the region \( R \). Then \( f(x, y) \) attains a global maximum and a global minimum; they occur at one of three places:

- At a **critical point** inside \( R \);
- At a point along the boundary;
- At a vertex along the boundary. (**Vertex** is a fancy word for **corner**)

Recall that finding the **critical points** involves solving the simultaneous system:

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 0 \\
\frac{\partial f}{\partial y} &= 0.
\end{align*}
\]

Once we have the critical points, we then use the discriminant

\[
D = \left( \frac{\partial^2 f}{\partial x^2} \right) \left( \frac{\partial^2 f}{\partial y^2} \right) - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2
\]

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to classify whether the critical points are local maxima or local minima.

As for global extrema that lie on the boundary, in practice they are found by restricting the function \( f(x, y) \) to the boundary and reducing the problem to an extrema problem in one variable.

**Self-Help Background Check**

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. Consider the function
   \[
   f(x) = x^2 - 6x + 18,
   \]
   where we restrict the domain to the interval \([0, 2]\). Find the global minimum and global maximum of \( f \) on this interval.

2. Consider the function
   \[
   f(x, y) = x + y + \frac{1}{xy}.
   \]
   Find and classify all critical points.

3. Parameterize the \( x \) and \( y \) coordinates of the indicated curve. That is, find functions \( f(t) \) and \( g(t) \) so that the given curve is described by the parametric equations \( x = f(t) \) and \( y = g(t) \).
(a) The cardioid \( r = 1 + \cos (\theta) \)

(b) The circle centered at \((0, 0)\), radius 1

4. Let \( f(x, y) = x^2 + y^2 - 6x - 6y + 18 \). Compute \( f_x, f_y, f_{xx}, f_{xy}, f_{yy} \).
If you missed more than one problem, consult our text on pages 846–850, or if time permits, go to office hours for clarification. In addition, problems 1, 3, and 5 on page 855 provide extra practice.

5. Consider the function

\[ f(x, y) = x^2 + y^2 - 6x - 6y + 18, \]

where the points \((x, y)\) are points in the triangular region below:

Note that \(f(x, y)\) is defined on a bounded region \(R\), and so it must have a global minimum. Fill in what’s missing to determine the global minimum of this function on the above triangular region.

(a) The global minimum occurs at one of three possible places:
   - at a critical point inside \(R\); or
   - at _____________________ ; or
   - at _____________________.

(b) Check critical points inside the region. Recall that we find critical points by setting the first partial derivatives of \(f(x, y)\) equal to _____. From problem 4, we know that \(\frac{\partial f}{\partial x} = \) _______ and \(\frac{\partial f}{\partial y} = \) _______ and so we obtain the equations

\[ 2x - 6 = 0 \quad \text{and} \quad \underline{\text{_________}} = 0. \]
Solving for \( x \) and \( y \) in the above equations yields a single critical point, namely, the point _____. However, we note that this critical point cannot yield a global extremum because ________________. Thus, the global minimum cannot occur at a point inside \( R \).

(c) **Check points along the boundary of \( R \).**

We note that the boundary of \( R \) consists of three edges. We examine each edge separately.

i. Along the bottom edge, \( y \) is fixed: \( y = 0 \). Thus, we can express the function \( f \) in terms of the single variable \( x \). Namely, we have

\[
f(x, y) = f(x, 0) = \text{______________________}.
\]

So to determine whether or not the global minimum of \( f(x, y) \) occurs on the bottom edge, we must find the minimum of the function \( x^2 - 6x + 18 \), where \( x \) is in the interval \([__, __]\).

ii. Determine all the points along the bottom edge of the triangle that may yield a global minimum. Be certain to compare your work with the others in your group and resolve any discrepancies.

iii. Determine all the points along the left edge of the triangle that may yield a global minimum. Be certain to compare your work with the others in your group and resolve any discrepancies.
iv. We must also check along the slanted edge of the triangle. Along this edge, the line has a slope of \( \frac{\text{gradation}}{\text{gradation}} \) and a \( y \)-intercept of \( \frac{\text{gradation}}{\text{gradation}} \). Thus, the equation of the slanted edge is \( y = \frac{\text{gradation}}{\text{gradation}} \). Thus, if we restrict \( f(x, y) \) to this line, we can eliminate the variable \( y \) and express \( f(x, y) \) in terms of the single variable \( x \). Namely, we have

\[
f(x, y) = f(x, 4 - 2x) = x^2 + \left( \frac{\text{gradation}}{\text{gradation}} \right)^2 - 6x - 6 \left( \frac{\text{gradation}}{\text{gradation}} \right) + 18 \\
= x^2 + \left( \frac{\text{gradation}}{\text{gradation}} \right) - 6x - 24 + 12x + 18 \\
= \frac{\text{gradation}}{\text{gradation}}
\]

along the slanted edge.

v. Determine all the points along the slanted edge of the triangle that may yield a global minimum. Be certain to compare your work with the others in your group and resolve any discrepancies.

(d) Check vertices

In this particular case, this means we must examine the three points \( \frac{\text{gradation}}{\text{gradation}} \), \( \frac{\text{gradation}}{\text{gradation}} \), and \( \frac{\text{gradation}}{\text{gradation}} \). We evaluate \( f(x, y) \) at these three points:

\[
f(0, 0) = \text{gradation} ; \\
f(2, 0) = \text{gradation} ; \\
f(\text{gradation}, \text{gradation}) = \text{gradation}
\]
(e) In conclusion, we can state that the global minimum occurs at the point ____. At this point, \( f \) attains the value ____.

In short, we have the following method for finding a global maximum (or a global minimum) in the case when \( f(x, y) \) is defined on a bounded region \( R \) and has continuous partial derivatives throughout \( R \).

**HOW TO FIND GLOBAL EXTREMA ON BOUNDED REGIONS**

- Find all the critical points of \( f(x, y) \) by solving the equations

\[
\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.
\]

Discard those critical points that do not lie in \( R \). Keep only those critical points that lie in \( R \).

- Restrict the function \( f(x, y) \) to points along the boundary. This is accomplished by parameterizing the boundary and rewriting the function so that it depends on a single variable only. Once this is done, we apply single-variable techniques to find critical points along the boundary.

- List all vertices of the boundary.

- Evaluate \( f(x, y) \) at each of the above points. We can now simply read off the global maximum and global minimum.

**Group Investigation**

6. Consider the function \( f(x, y) = x^2 + y^2 - 2y \) and the region \( R \) pictured below:

![Graph of a circle with vertices at (2, 2), (-2, 2), (-2, -2), and (2, -2), and a point at (\( \sqrt{2}, \sqrt{2} \)).]

Find the global maximum and global minimum of \( f(x, y) \) on the region \( R \).
Of course, the main reason extrema problems are interesting is because they are so useful in solving practical problems that arise in many different disciplines. The following is such an example.

7. Dr. Volterra, the world-famous marine biologist, needs to build a tank to hold all of the fishes that he is currently studying. He has hired your group to help him. The tank is to be built from steel and reinforced plexiglass, and is to be rectangular with an open top. Reinforced plexiglass and steel are expensive, but cost the same per square meter. The completed tank must hold 1000 cubic meters of seawater. Dr. Volterra wants your group to determine the dimensions of the tank which costs as little as possible.
Discussion Set C3 Assignment

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section in Chapter 1 of Cameos for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

8. Of all parallelograms having a perimeter of 2 meters, find the parallelogram(s) that enclose(s) the largest area $^2$

9. (a) Let $R$ be the entire $xy$-plane. Find a function $f(x, y)$ that is continuous throughout $R$ having at least one relative minimum, but no global minimum. You can represent the function either in terms of a formula or as a graph, whichever you find more convenient.

(b) Does the example you constructed in part (a) contradict Theorem 1 on page 1 of today’s discussion set? Why or why not?

Brief answers to problems from the Self-Help Background Check

1. The global minimum occurs at $(2, 10)$; the global maximum is at $(0, 18)$.

2. $f(x, y)$ has a relative minimum at $(1, 1)$.

3. (a) $x = (1 + \cos t) \cdot \cos t$ and $y = (1 + \cos t) \cdot \sin t$

(b) $x = \cos t$ and $y = \sin t$

4. $f_x = 2x - 6$, $f_y = 2y - 6$, $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 2$.

$^2$For fun, purely as an optional exercise, redo this problem, replacing parallelograms with arbitrary quadrilaterals.
Mathstory

Vito Volterra (1860-1940), a professor of mathematical physics at the University of Rome, is known for two of his contributions. The first is his work in the general theory of integral equations. These are equations that involve integrals of unknown functions (you will see examples of these when you study differential equations in MAT 22B). The second is a pair of differential equations that describes the abundance of interacting species of organisms. These equations, which were investigated independently by the American mathematician Alfred Lotka, have become known as the Lotka-Volterra predator-prey/competition equations. Volterra became interested in modelling predator-prey interactions when his son-in-law, Umberto D’Ancona, brought to his attention the problem of declining fish populations in the Mediterranean Sea that resulted after suspension of commercial fishing during World War I (Hutchinson, 1978).

Literature Cited

Discussion Set C4: Integrals Using Rectangular Coordinates

In MAT 21B, you learned how to compute the area between the x-axis and the graph of a function \( f \) from \( x = a \) to \( x = b \) using a definite integral whose integrand was a function of a single variable; for \( f \) continuous, this is given by the definite integral \( \int_a^b |f(x)| \, dx \). In this discussion set, you will practice computing area in a slightly different way: using a double integral in rectangular coordinates. You also will generalize these ideas to other integration problems. As with definite integrals of a single variable, this process involves determining the integrand and the limits of integration.

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. Consider the region \( R \) where \( R \) is bounded by the graph of \( y = e^x \), the y-axis, and the line \( y = 5 \).

   (a) Draw a picture of \( R \).

   (b) Set up an integral for the area of \( R \) using vertical cross sections. Also draw the appropriate sample unit on \( R \).
(c) Set up an integral for the area of $R$ using horizontal cross sections and draw an appropriate sample unit on $R$.

(d) Evaluate the integrals from parts (b) and (c).

If you missed more than one problem, consult our text on page 462, or if time permits, go to office hours for clarification. In addition, problems 7 and 11 on page 466 provide extra practice.

**Guided Inquiry**

There are many different devices that can be used to come up with the right set-ups for integrals. A useful tool is the “sample unit.” In MAT 21B when we wanted to set up an integral for the area bounded by $f(x)$ for $f(x) \geq 0$ in $a \leq x \leq b$, we drew a rectangle for the sample unit, as shown below. This immediately led us to set up

$$
\int_{a}^{b} f(x) \, dx,
$$

indicating that the sample rectangle is “moved from $a$ to $b$”
to fill the region for which we want the area. The inequality \( a \leq x \leq b \) tells us the bounds of integration. The “\( dx \)” tells us the base of the sample unit and the direction in which we are moving it.

Going to functions of two variables allows us to consider double integrals. This leads us to the possibility of using a “sample unit” that moves in two directions in order to describe a region. Let’s look at an example of describing a region in this new way.

2. Let \( S \) be the region bounded by \( y = \sqrt{x} \), the line \( x = 2 \) and the x-axis

   (a) Sketch a picture of \( S \).

   (b) Inside \( S \), place a small square, oriented so that one side is parallel to the x-axis, in the middle of \( S \) to serve as a sample unit for setting up a double integral. If you move the square vertically, it would go from \( y = \) \( \square \) to \( y = \sqrt{x} \), thus creating a column of squares. In your sketch from (a), draw the column of squares that illustrates this movement of the sample unit.

   (c) Notice that the column of squares that resulted from moving the sample unit vertically corresponds to the sample rectangle that would have been drawn to describe \( S \) by vertical slices. In other words, moving the square first vertically, then moving the column of squares horizontally, corresponds to describing \( S \) as

\[
\begin{align*}
\_\_\_\_\_\_\_\_\_\_ \leq y & \leq \sqrt{x} \\
\_\_\_\_\_\_\_\_\_\_ \leq x & \leq \_\_\_\_\_\_\_\_\_\_.
\end{align*}
\]

These inequalities tell us the bounds of integration for a double integral over \( S \).

(d) Draw another picture of \( S \) with a square for a sample unit. Again, the sample unit is oriented as described in part (b). This time, move the square horizontally first, and then write the corresponding description of \( S \).
When writing iterated (double, triple, quadruple . . .) integrals, remember that the limits on the “innermost” integral can be functions, but the limits on the “outermost” integral must be constants, not functions.

**Group Investigation**

3. Let $R$ be the region bounded by the graph of $y = e^x$, the $y$-axis and the line $y = 5$.

   (a) Work independently to set up the double integral $\int \int dA$ that corresponds to first moving the “square sample unit” vertically.

   (b) Compare your work on (a) with others in your group. When you all agree on the integral, evaluate it and compare the answer you get to the one you obtained in (1d). Discuss this with your group, and then write a few sentences explaining your observations.

   (c) Set up a double integral $\int \int dA$ that would give the area for the region $R$ by first indicating the limits of integration that correspond to moving the “square sample unit” in a horizontal direction. Do not evaluate the integral.

For some double integrals, the order of integration affects the ease with which the integral may be computed. Changing the order of integration usually requires expressing the limits of integration in a different way than was used in the original problem. Note that this entails more than simply reordering the limits.
4. For example, suppose you want to calculate the double integral
\[
\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 \sqrt{1 + x^3} \, dx \, dy.
\]
Fill in what’s missing to evaluate this integral.

(a) First, notice that the function \( \sqrt{1 + x^3} \) does not have an elementary \underline{_________}, hence, we cannot compute the integral as it is written. \underline{When this occurs, reversing the order of integration is worth a try!} The region \( R \) over which the integral is being taken can be defined by the inequalities
\[
0 \leq \underline{____} \leq \sqrt{2}, \quad y^2 \leq \underline{____} \leq \underline{____}.
\]
(b) Sketch the region \( R \), using your answer to (4a), and compare your sketch with your solution to (2d).

(c) In the original integral, with \( R \) described as in (4a), the limits of integration are determined by moving the sample unit (square) first in the \underline{_________} direction. If we reverse the order of integration, this corresponds to moving the sample unit first in the \underline{_________} direction, instead. In this way, \( R \) can be described by
\[
0 \leq \underline{____} \leq 2, \quad 0 \leq y \leq \underline{____}.
\]
Hence, we can rewrite the integral as
\[
\int_0^{\sqrt{2}} \int_{y^2}^{2} y^3 \sqrt{1 + x^3} \, dx \, dy = \int_0^{\sqrt{2}} \int_0^{y^2} y^3 \sqrt{1 + x^3} \, dy \underline{____}.
\]
(d) Work independently to evaluate this integral. Compare answers with others in your group and reconcile any differences before going on.
5. The integral you computed above has a physical interpretation. Think in terms of local approximations and fill in what’s missing.

Let \( z = f(x, y) \) be the nonnegative value of a dependent variable \( z \) (not necessarily the function in problem 4) and let \( dA \) be a square in the \( xy \) plane (with sides \( dx \) and \( dy \) within a region \( R \)).

Draw a picture of \( R \) and a surface whose equation is \( z = f(x, y) \) (any ol’ surface will do). Then in terms of local approximations, \( f(x, y)dA \) represents the (three dimensional) ______ of the solid that lies between the graph of ______ and the region \( dA \) in the \( xy \)-plane. Draw the (three dimensional) sample unit that corresponds to this local approximation. Moving this sample unit first parallel to the \( y \)-axis, then parallel to the \( x \)-axis fills the entire solid.

Now look back at problem (4). Because \( f \geq 0 \), the integral in (4) can be interpreted as the volume of the solid bounded by the surface \( y^3\sqrt{1 + x^2} \) above the \( xy \) plane, and over the region \( R \).

**Discussion Set C4 Assignment**

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section of Chapter 1 in *Cameos* for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

6. Find the volume of the solid whose base is the region \( R \) in the \( xy \)-plane bounded by the graph of \( y = \sqrt{x} \), the \( x \)-axis, and the line \( x = 9 \) if the height of the solid above any point \((x, y)\) in \( R \) is given by \( f(x, y) = \frac{y}{1 + x^2} \).

7. Evaluate the following integral: \( \int_0^4 \int_{\sqrt{x}}^2 \frac{x}{y^5 + 4} \, dy \, dx \).
Brief answers to problems from the Self-Help Background Check

(1a) 

\[
\begin{align*}
\text{(1b) } c(x) &= 5 - e^x \text{ where } 0 \leq x \leq \ln 5 \\
\text{(1c) } c(y) &= \ln(y) \text{ where } 1 \leq y \leq 5 \\
\text{(1d) Area} &= 5 \ln 5 - 4
\end{align*}
\]

Mathstory

In Calculus, for most practical purposes, we use the notion of a definite integral that was developed by Riemann. However, this definition has limitations: for example, it cannot be used to compute a definite integral of a function on an interval when the function is unbounded or discontinuous at a number within that interval. In contrast, a definition of an integral developed by Henri Lebesgue (1875-1941) overcomes these difficulties (Kline 1981). Lebesgue replaced Riemann’s partition of the interval into non-overlapping subintervals with a partition into disjoint sets that satisfy certain technical properties. The theory of integration that resulted from Lebesgue’s work is used widely in applied mathematics, e.g., in quantum physics, probability, and statistics, as well as in pure mathematics. In addition, Lebesgue’s results also advanced a theory that is a foundation for multiple integrals.

Literature Cited

Discussion Set C5: Cylindrical and Spherical Coordinates

The purpose of this discussion set is to improve your understanding of and to help you strengthen your skills in writing integrals over regions in space using cylindrical and spherical coordinates. Use of these coordinate systems can often simplify integrals that are difficult to evaluate in rectangular coordinates.

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. Below, draw the Cartesian coordinates $(x, y)$, and on the same graph draw the polar coordinates $r$ and $\theta$.

2. Find equations for $x$ and $y$ in terms of the polar coordinates $r$ and $\theta$.

3. Convert each of the following equations into equations involving polar coordinates:

   (a) $x^2 + y^2 = 4$
If you missed more than one problem, consult our text on page 519, or if time permits, go to office hours for clarification. In addition, problems 9 and 11 on page 525 provide extra practice.

**Guided Inquiry**

These guidelines will help you to know when to use a coordinate system other than rectangular coordinates.

<table>
<thead>
<tr>
<th>Cylindrical coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>are particularly useful when the region of integration involves circles (in the $xy$ plane) and when $z$ can be written as a function of $x$ and $y$ that is not too complicated. Cylindrical coordinates can be thought of as polar coordinates with a vertical dimension appended. In cylindrical coordinates,</td>
</tr>
<tr>
<td>[ \int_R f(P) dV = \int \int \int f(r, \theta, z) r , dz , dr , d\theta, ]</td>
</tr>
<tr>
<td>where $dV = r , dz , dr , d\theta$.</td>
</tr>
<tr>
<td>To convert from cylindrical to rectangular coordinates, note that</td>
</tr>
<tr>
<td>[ x = r \cos \theta, \quad y = r \sin \theta, \quad z = z. ]</td>
</tr>
<tr>
<td>In addition, $x^2 + y^2 = r^2$, just as in polar coordinates.</td>
</tr>
</tbody>
</table>

The figure below illustrates the relationship between rectangular and cylindrical coordinates. Notice that $x, y, r, \text{ and } \theta$ are related just as they are in polar coordinates, while $z$ is
the same as in rectangular coordinates.

Note: The sample unit that is helpful in setting up the limits of integration in cylindrical coordinates looks like a wedge that has been cut out of a washer – see Figure 10 on p. 920 of our text.

4. Describe and sketch the graph of each of the following equations in cylindrical coordinates.

(a) \( r = 3 \)

(b) \( \theta = \frac{\pi}{4} \)
Spherical coordinates

are particularly useful when the region of integration involves spheres and cones. Spherical coordinates consist of two coordinates that describe angles and one that describes the distance from the pole to a point \(P\) in space. In spherical coordinates,

\[
\int_{R} f(P) dV = \int \int \int f(\rho, \theta, \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,
\]

where \(dV = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta\). To convert from spherical to rectangular coordinates, note that

\[
x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi
\]

expresses the relationship between variables in the two coordinate systems.

The figure on the left illustrates the spherical coordinate system. The figure on the right illustrates the relationship between rectangular and spherical coordinates.

Notice that

\[
r = \rho \sin \phi.
\]

Using the relationship \(x = r \cos \theta\) and \(y = r \sin \theta\), we obtain

\[
x = \rho \sin \phi \cos \theta \quad \text{(3.1)}
y = \rho \sin \phi \sin \theta \quad \text{(3.2)}
z = \rho \cos \phi \quad \text{(3.3)}
\]

Note: The sample unit that is helpful in setting up the limits of integration in spherical coordinates looks like a box whose edges got a bit warped – see Figure 10 on p. 924 of our text. Alternatively, you can think of it as a chunk of an orange peel.
5. Describe and sketch the graph of each of the following equations in spherical coordinates. You may need to use the equations given above for converting from spherical to rectangular coordinates.

(a) \( \phi = \frac{\pi}{4} \)

(b) \( \rho = 5 \csc \phi \)

Finally, comparable to setting up an integral in rectangular coordinates, when writing an integral in cylindrical or spherical coordinates, the limits of integration cannot involve a variable with respect to which an integration has already been performed. For example, if \( dz \) is the innermost differential, then the outside and middle limits of integration cannot involve the variable \( z \).

6. The region \( R \) is below the sphere \( x^2 + y^2 + z^2 = 36 \) and above the cone \( z = \sqrt[3]{x^2 + y^2} \)

(a) Sketch a graph of \( R \).
(b) Find the equation of the level curve where the sphere and cone intersect. Sketch its projection in the $xy$ (i.e., the $z = 0$) plane.

(c) Convert the equations of the surfaces that bound $R$ into cylindrical coordinates.

(d) Fill in what’s missing to write an integral for the mass of $R$, if the density at a point within $R$ is twice the distance from the point to the $z$ axis.

Recall that mass $= \int \delta(P) dV$, where $\delta(P)$ is the density at the point $P$. First, note that we can write the desired integral either by writing it first in rectangular coordinates and converting it to cylindrical coordinates, or by “thinking cylindrically” and writing the integral directly. Let’s use the latter approach here.

Because the density at the point $(x, y, z)$ within $R$ is twice the distance from $(x, y, z)$ to the $z$ axis, we can write an expression for density as

$$\delta(P) = 2r.$$ 

To describe the region $R$ in cylindrical coordinates, it is usually easiest to fix $r$ and/or $\theta$, then write $z$ as a function of $r$ and $\theta$. Hence, from the projection on the $z = 0$ plane,

$$0 \leq r \leq \___ \ and \ \___ \leq \theta \leq 2\pi.$$ 

Now the $z$ coordinate extends from the cone up to the hemisphere. Thus, $z$ must lie between the equations for the two surfaces. Solving for $z$ in these equations, we obtain

$$z = \sqrt{36 - \___} \text{ for the sphere, and}$$

$$z = \___ \text{ for the cone.}$$

Therefore, the inequality that describes the range for $z$ is

$$\sqrt{3}r \leq z \leq \___.$$
The integral in cylindrical coordinates becomes

\[ \int_0^{2\pi} \int_0^{\sqrt{36}} \int_{r}^{\sqrt{36}} 2r \, dz \, d\theta \, dr. \]

**Group Investigation**

Let’s turn to spherical coordinates and do some more work with the region that was given in (6).

7. The region \( R \) is bounded above by the sphere \( x^2 + y^2 + z^2 = 36 \) and below by the cone \( z = \sqrt{3} \sqrt{x^2 + y^2} \).

   (a) Describe \( R \) using spherical coordinates. Check your answer with your discussion facilitator before going on.

   (b) Set up an integral to find the volume of \( R \) using spherical coordinates.

**Discussion Set C5 Assignment**

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section in Chapter 1 of *Cameos* for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

For the following problems, set up an integral to describe the desired quantity. For each problem, draw the region, choose a coordinate system (rectangular, cylindrical, or spherical) so that the region is easy to describe and the integral is easy to compute. Also show the sample unit related to the coordinate system that you are using. Then, write a paragraph explaining your choice of coordinate system, and the consequences of using other coordinate systems.

8. Set up an integral for the volume of the solid that lies above the \( xy \)-plane, below the plane \( x + z = 5 \) and inside the cylinder \( x^2 + y^2 = 9 \).
9. Set up an integral for the mass of $R$, where $R$ is the region bounded above by the sphere $x^2 + y^2 + z^2 = 36$ and bounded below by the cone $z = \sqrt{3} \sqrt{x^2 + y^2}$, and the density at a point $P$ within $R$ is the distance from $P$ to the point $(0, 0, 0)$.

10. Suppose you wish to compute

$$\int_{R} f(P)\,dV,$$

where $R$ is the solid bounded between the paraboloid $z = 6 - (x^2 + y^2)$ and the cone $z = \sqrt{3} \sqrt{x^2 + y^2}$. Without knowing $f(P)$, discuss the advantages and disadvantages of using

(a) cylindrical

(b) spherical

coordinates for this computation.

**Brief answers to the problems from the Self-Help Background Check:**

(1a)

(2) $x = r \cos \theta$, and $y = r \sin \theta$

(3a) $r^2 = 2$

(3b) $r = \sec \theta$

(3c) $r = \sec \theta + \csc \theta$
Mathstory

Richard Courant (1888 - 1972) was a mathematician who worked in function theory and a branch of applied mathematics called the calculus of variations. He received his doctorate from the University of Göttingen in 1910, and later became founder and director of its institute of mathematics. During that time, he became close friends with David Hilbert and coauthored a two-volume treatise, Methods of Mathematical Physics, that today remains a classic and important reference work. Courant, like many of his peers, fled Germany in 1933. He became a professor of mathematics at New York University and was one of the founders of a mathematical research institute at NYU, which was named after him posthumously.

Literature Cited

Discussion Set C6: Comparison Tests

There are a variety of tests one can use to determine the convergence or divergence of an infinite series. In this discussion set we will study the Comparison Test and its close cousin, the Limit Comparison Test. Both apply only to series with all positive terms.

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. Determine which of the functions in each pair is larger (for $x$ sufficiently large). Circle the one that is larger (i.e., greater than or equal to the other).

   a. $\cos x$ or $1$
   b. $x - 1$ or $x - 4$
   c. $x \ln x$ or $x^2$
   d. $x^{27}$ or $27^x$
   e. $\sqrt{x}$ or $\ln x$
   f. $x^{10}$ or $\ln x$
   g. $n^n$ or $n!$ if $n$ is an integer

2. Determine whether each series converges or diverges and give the name of the test you used. For this problem, you need not show your work.

   a. $\sum_{n=1}^{\infty} \frac{1}{n}$ by the ___________ Test;
   b. $\sum_{n=1}^{\infty} e^{-n}$ by the ___________ Test;
   c. $\sum_{n=1}^{\infty} \pi^n$ by the ___________ Test;
   d. $\sum_{n=1}^{\infty} n^{-1.314159}$ by the ___________ Test.

If you missed more than one problem, consult our text on 578ff and 586ff, or if time permits, go to office hours for clarification. In addition, problems 15, 17, and 19 on page 590 provide extra practice.
Guided Inquiry

You will use the work you did above in determining convergence or divergence of series with comparison tests. One of the most important tests for convergence or divergence of a positive-term series is the Comparison Test. In addition to being the best to use for many infinite series, the Comparison Test provides the justification for two other important tests which you’ll see later in the course, the Ratio Test and the Root Test.

The Comparison Test: Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be positive-term series.

If \( a_n \leq b_n \) for \( n \geq k \) (for some \( k \)) and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

If \( a_n \geq b_n \) for \( n \geq k \) (for some \( k \)) and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

3. (a) Why can’t you use the Comparison Test on the series \( \sum_{n=1}^{\infty} \frac{\cos n}{n} \)?

(b) If you were comparing a given series to \( \sum_{n=1}^{\infty} \frac{1}{n} \), would you be trying to show that the given series converged or diverged?

(c) If you were comparing a given series to \( \sum_{n=1}^{\infty} e^{-n} \), would you be trying to show that the given series converged or diverged?

Some miscellaneous comments:

(I) The series \( \sum_{n=1}^{\infty} b_n \) is the one to which the series \( \sum_{n=1}^{\infty} a_n \) is compared; it should be a series whose convergence is easy to determine. In practice, usually we will use either a \( p \)-series or a geometric series.
(II) If \( a_n \) is given by a quotient involving functions of \( n \), we can generally find a series \( \sum_{n=1}^{\infty} b_n \) to use for comparison by taking

\[
b_n = \frac{\text{the largest term in the numerator of } a_n}{\text{the largest term in the denominator of } a_n}
\]

and simplifying the result.

(III) The Comparison Test gives **no conclusion** if \( \sum_{n=1}^{\infty} b_n \) converges with \( a_n \geq b_n \), or if \( \sum_{n=1}^{\infty} b_n \) diverges with \( a_n \leq b_n \).

4. Fill in the missing information in the example below.

From miscellaneous comment (II), for

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n^2 + 4}
\]

a good candidate to try to compare the series to is

\[
\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n},
\]

which ___________ since it is the harmonic series. Because the series to which we are comparing diverges, we would like to show that

\[
\frac{n + 1}{n^2 + 4} \geq \frac{1}{n}
\]

for \( n \) sufficiently large. Cross-multiplying gives

\[
n(n + 1) \geq ___________
\]

\[
n^2 + ____ \geq ____ + 4,
\]

which is equivalent to \( n \geq ____ \). Therefore, for \( n \geq 4 \), each term in the series \( \sum_{n=1}^{\infty} \frac{n + 1}{n^2 + 4} \) is ___________ than the corresponding term in a divergent series. We conclude that the series

\[
\sum_{n=1}^{\infty} \frac{n + 1}{n^2 + 4}
\]

__________ by the Comparison Test.
Notice that in the preceding problem, the argument we gave consisted of 4 steps:

i. Choose the series $\sum b_n$ to which you are comparing.
ii. Explain why the series $\sum b_n$ converges or diverges.
iii. Verify that the inequality needed is satisfied for $n$ sufficiently large.
iv. State the conclusion which follows from the Comparison Test.

In general, we use this procedure anytime divergence or convergence is determined by comparison.

5. Test the series $\sum_{n=1}^{\infty} \frac{1}{n \ln n + n^2}$ for convergence or divergence. Check to make sure that your group is in agreement concerning the conclusion.

6. Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{n + 3^n}{n^3 + 7^n}$.

   (a) Specify a choice of $\sum_{n=1}^{\infty} b_n$ that you will use for comparison.

   (b) Is the necessary inequality satisfied for this choice of $\sum_{n=1}^{\infty} b_n$? Explain and justify the position taken.
In some cases, the inequality we need to verify in using the Comparison Test is not actually satisfied. In such a case, we could change our choice of \( \sum_{n=1}^{\infty} b_n \). An easier way to deal with this situation, though, is to use the following variant of the Comparison Test:

**Limit Comparison Test**: Let \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) be positive-term series, and suppose

\[
\lim_{n \to \infty} \frac{a_n}{b_n} = L \quad \text{where} \quad 0 \leq L.
\]

If \( L \) is finite and \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges. If \( L > 0 \) or if the limit approaches infinity, and \( \sum_{n=1}^{\infty} b_n \) diverges, then \( \sum_{n=1}^{\infty} a_n \) diverges.

Notice that the Limit Comparison Test yields no conclusion if the limit is infinite and \( \sum_{n=1}^{\infty} b_n \) converges, or if \( L = 0 \) and \( \sum_{n=1}^{\infty} b_n \) diverges.

7. Fill in the missing information for the following example: To test the series

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n} + 1}{n^2 + 5}
\]

using the Limit Comparison Test, we can compare it to

\[
\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}},
\]

which \( \frac{\sqrt{n} + 1}{n^2 + 5} \) since it is a \( p \)-series with \( p = \frac{3}{2} \). Then

\[
\lim_{n \to \infty} \frac{\sqrt{n} + 1}{n^2 + 5} \div \frac{1}{n} = \lim_{n \to \infty} \frac{n^{3/2}}{n^2 + 5} = \lim_{n \to \infty} \frac{1 + \frac{5}{n}}{1 + \frac{5}{n^2}} = 1 > 0.
\]

We conclude that

\[
\lim_{n \to \infty} \frac{\sqrt{n} + 1}{n^2 + 5}
\]

by the Limit Comparison Test.

**Group Investigation**

8. In problem (6) it is possible to use the Comparison Test (by being a little clever). Miscellaneous comment (II) probably led you to try

\[
\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{3^n}{7n} = \sum_{n=1}^{\infty} \left( \frac{3}{7} \right)^n
\]
which does converge, but unfortunately is not larger than the original series. All is not lost! Use the Limit Comparison Test to determine convergence.

9. Suppose that you were testing a series $\sum_{n=1}^{\infty} a_n$, with all negative terms, for convergence or divergence.

(a) Could you use the Comparison Test or Limit Comparison Test directly to do this? Why or why not?

(b) Consider the series $\sum_{n=1}^{\infty} (-a_n)$. Discuss in your group how this related series might help you determine the convergence or divergence of the original one.

Discussion Set C6 Assignment

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems, Refer to the “Mathematical Writing” section in Chapter 1 of Cameos for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

10. Use the Comparison or Limit Comparison Test to determine convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{3n}{(n + 6)^2}$.

11. (a) Use the Comparison Test to determine convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\cos^4 n}{n^2 + 1}$. 
(b) Explain what happens if you try using the Limit Comparison Test on the series in (11a).

12. Prove or find a counterexample:

If \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are both divergent positive term series, then \( \sum_{n=1}^{\infty} (a_n - b_n) \) diverges.

**Brief answers to problems from the Self-Help Background Check**

1. Larger terms: 1, \( x - 1 \), \( x^2 \), 27\(x\), \( \sqrt{x} \), \( x^{1/10} \), \( n^n \)
2a. Diverges by the Integral Test.
2b. Converges by the Geometric Series Test, or Integral Test.
2c. Diverges by the Geometric Series Test, or \( n^{th} \) Term Test.
2d. Converges by the Integral Test, or \( p \)-series Test.
Discussion C7: What to Do in the Face of a Series

In today’s discussion, we will practice classifying an infinite series \( \sum_{n=1}^{\infty} a_n \) into one of three categories:

- **Divergent**: \( \sum_{n=1}^{\infty} a_n \) diverges

- **Conditionally convergent**: \( \sum_{n=1}^{\infty} a_n \) converges, but \( \sum_{n=1}^{\infty} |a_n| \) diverges.

- **Absolutely convergent**: \( \sum_{n=1}^{\infty} |a_n| \) converges. (Note: In this case, since \( \sum_{n=1}^{\infty} |a_n| \) converges, so does \( \sum_{n=1}^{\infty} a_n \), by the Absolute Convergence Test.)

An important distinction between an absolutely convergent series and a conditionally convergent series is the fact that the terms of an absolutely convergent series can be rearranged in any order without affecting the sum of the series, whereas the terms of a conditionally convergent series can be rearranged so that the resulting series converges to any specified number or even diverges.

The following is an outline of a basic strategy for determining whether a given series \( \sum_{n=1}^{\infty} a_n \) converges absolutely, converges conditionally, or diverges.

**BASIC STRATEGY**

- As a general rule of thumb, try the nth Term Divergence Test first.
  - If \( \lim_{n \to \infty} a_n \neq 0 \), then we can immediately conclude that \( \sum_{n=1}^{\infty} a_n \) diverges.
  - On the other hand, if \( \lim_{n \to \infty} a_n = 0 \), then no conclusion can be drawn at this point.

- If the nth Term Divergence Test is inconclusive, the next step is to examine the corresponding series \( \sum_{n=1}^{\infty} |a_n| \). At this point, one has a variety of tests that may or may not apply to \( \sum_{n=1}^{\infty} |a_n| \).

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2 © 1994 by CaRP, Department of Mathematics, University of California, Davis
– **Integral Test**: especially useful if \(\{|a_n|\}_{n=1}^{\infty}\) is eventually decreasing and the function that is obtained by substituting \(x\) for each \(n\) in \(|a_n|\) is easy to integrate.

– **Limit Comparison Test**: especially useful if \(|a_n|\) is a rational function in \(n\) or in some root of \(n\).

– **Comparison Test**: especially useful if \(|a_n|\) involves either a sine or a cosine.

– **Ratio Test**: especially useful if \(|a_n|\) has factorials in products somewhere, or if it looks like the product of a rational function and an exponential.

– **Root Test**: especially useful if it’s easy to calculate the \(n\)th root of \(|a_n|\).

Depending on the results of these tests, we can make the following assertions:

– If \(\sum_{n=1}^{\infty} |a_n|\) converges by any of the above tests, then by an application of the **Absolute Convergence Test**, we know that \(\sum_{n=1}^{\infty} a_n\) converges as well; in this case, we conclude that \(\sum_{n=1}^{\infty} a_n\) **converges absolutely**, and we are done.

– If \(\sum_{n=1}^{\infty} |a_n|\) diverges by use of the **Ratio Test** or the **Root Test**, then \(\sum_{n=1}^{\infty} a_n\) also **diverges**. (Think about it. Applying the Ratio or Root Test to \(\sum_{n=1}^{\infty} |a_n|\) is the same as applying the Absolute Ratio or Absolute Root test to \(\sum_{n=1}^{\infty} a_n\).) We are done.

– But if \(\sum_{n=1}^{\infty} |a_n|\) diverges by the **Integral, Comparison**, or **Limit Comparison Test** then the best possible conclusion is that \(\sum_{n=1}^{\infty} a_n\) cannot converge absolutely; in this case, we can assert that either \(\sum_{n=1}^{\infty} a_n\) converges conditionally or diverges. Since we don’t know which, we aren’t done yet.

- If an analysis of \(\sum_{n=1}^{\infty} |a_n|\) yields ambiguous results about \(\sum_{n=1}^{\infty} a_n\) (i.e. that either \(\sum_{n=1}^{\infty} a_n\) diverges or converges conditionally), then we must examine \(\sum_{n=1}^{\infty} a_n\) itself. For example, if \(\sum_{n=1}^{\infty} a_n\) is an alternating series, \(\lim_{n \to \infty} a_n = 0\), and \(|a_n|\) decreases, then the **Alternating Series Convergence Test** applies.
As the above strategy indicates, to be successful at classifying a given series, you must know your series tests. The problems in the self-help section are designed to refresh your memory with regards to the various tests.

**Self-Help Background Check**

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. Consider the series \( \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \). In each case, determine if the indicated test tells us whether the series converges or diverges or yields inconclusive results.

   (a) The nth term divergence test.

   (b) The Integral Test.

   (c) The Comparison Test. (Compare to the series \( \sum_{n=1}^{\infty} \frac{1}{n} \).)

   (d) The Limit Comparison Test. (Compare to the series \( \sum_{n=1}^{\infty} \frac{1}{n} \).)
(e) The Ratio Test.

2. Consider the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \). Determine whether the indicated test tells us the series converges or diverges or yields inconclusive results.

(a) The Alternating Series Test.

(b) The Absolute Convergence Test.

If you missed more than one problem, consult our text on page 617, or if time permits, go to office hours for clarification. In addition, problems 6 and 8 on page 618 (GUIDE QUIZ) provide extra practice.

**Guided Inquiry**

3. Consider the series \( \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2} \).

Fill in what’s missing to determine whether it diverges, converges absolutely, or converges conditionally.

(a) The first test we should try is the \text{______} Test. We see that \( \lim_{n \to \infty} a_n = \) \text{______} since the degree of the numerator is less than the degree of the denominator. Thus, the \text{nth Term Divergence Test} is inconclusive.
(b) Since the nth Term Divergence Test is inconclusive, we should examine the corresponding series \( \sum_{n=1}^{\infty} |a_n| \). In this case, we have that

\[
\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 2}.
\]

Since this series is a rational function in \( n \), the most useful test is the \textit{Comparison Test}. Comparing \( \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 2} \) to the series \( \sum_{n=1}^{\infty} \frac{1}{n} \), we note that

\[
\lim_{n \to \infty} \frac{\frac{n^2 + 1}{n^3 + 2}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^3 + 2}{n^3} = 1.
\]

Since this limit is finite and nonzero and since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges (by the \textit{Comparison Test}), we conclude that the series \( \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 2} \) must also converge, by the \textit{Comparison Test}. Therefore,

\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2}
\]

cannot converge absolutely; it must either diverge or converge \textit{conditionally}.

(c) Finally, we apply the \textit{Alternating Series Test} to the series \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2} \). First, however, we must verify that this series satisfies the hypotheses of the alternating series test. To refresh our memory, let’s restate the hypotheses of the test:

- \( \lim_{n \to \infty} a_n = 0 \)
- The sequence \( \{|a_n|\} \) is \textit{decreasing}, for \( n \) sufficiently large.
- \( \sum_{n=1}^{\infty} a_n \) is an alternating series.

Now, examine each of these for the series under consideration.

We already checked in part (a) that \( \lim_{n \to \infty} \frac{n^2 + 1}{n^3 + 2} = 0 \). Moreover, the sequence

\[
\left\{ (-1)^n \frac{n^2 + 1}{n^3 + 2} \right\}
\]

clearly alternates because \((-1)^n\) alternates in sign and \( \frac{n^2 + 1}{n^3 + 2} \)
is always positive (since \( n \geq 1 \)). Thus, it suffices to check that the sequence 
\[
\left\{ \frac{n^2 + 1}{n^3 + 2} \right\}
\]
is decreasing.

One way to do this is to show that the function 
\[
f(x) = \frac{x^2 + 1}{x^3 + 2}
\]
is decreasing for \( x \) sufficiently large, or equivalently that 
\[
f'(x) = \frac{(x^3 + 2) \cdot \left( \frac{x^2 + 1}{x^3 + 2} \right) - (x^3 + 2)^2 \cdot 3x^2}{(x^3 + 2)^2}
\]
\[
= \frac{x \cdot \left( \frac{x^2 + 1}{x^3 + 2} \right)}{(x^3 + 2)^2}
\]
for \( x \) sufficiently large. But it is easy to see that for \( x \geq 2 \), we have 
\[
f'(x) < 0.
\]
Thus, \( f(x) \) decreases. So the sequence 
\[
\left\{ \frac{n^2 + 1}{n^3 + 2} \right\}
must also decrease. Since 
\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2}
\]
satisfies all the conditions to the Test, we assert that this series converges.

\[
\sum_{n=1}^{\infty} \left| a_n \right| = \sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 + 2}
\]
diverges; but in part (c), we showed 
\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2}
\]
converges. So, by definition, we conclude that
\[
\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 + 2}
\]
converges 

Group Investigation

If you want to be successful in dealing with problems involving absolute and conditional convergence, you must learn to recognize which convergence tests apply in which situations. The following problem is designed to give you some practice in quickly identifying the most effective convergence/divergence test.

4. Using the guidelines outlined in the basic strategy, discuss with the other members of your group which test you would use for each of the following series and whether you think the series converges or diverges.
DS C7: WHAT TO DO IN THE FACE OF A SERIES

(a) \[ \sum_{n=1}^{\infty} \frac{n^3 + 2}{n^5 - 4n^2 + 1} \]

(b) \[ \sum_{n=1}^{\infty} \frac{n^2}{10^n} \]

(c) \[ \sum_{n=1}^{\infty} \frac{n!}{10^n} \]

(d) \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{n(n + 4)} \]

(e) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1 - 1/n}} \]

(f) \[ \sum_{n=1}^{\infty} \frac{1}{n(\ln n)^2} \]

Discussion C7 Assignment

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section in Chapter 1 of Cameos for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

5. Determine if the given series diverges, converges absolutely, or converges conditionally.

(a) \[ \sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{\ln n}} \]

(b) \[ \sum_{n=1}^{\infty} (-1)^n \frac{n^{3n}}{4^n (n!)^3} \]

6. Let \( \sum_{n=1}^{\infty} a_n \) be a series whose terms may be either positive or negative.

(a) Suppose \( \sum_{n=1}^{\infty} a_n \) converges absolutely. Is it necessarily true that \( \sum_{n=1}^{\infty} a_n^2 \) converges absolutely? If so, provide a proof of this claim. Otherwise, provide a counterexample.

(b) Suppose \( \sum_{n=1}^{\infty} a_n \) converges conditionally. Is it necessarily true that \( \sum_{n=1}^{\infty} a_n^2 \) converges conditionally? If so, provide a proof of this claim. Otherwise, provide a counterexample.
**Brief answers to problems from the Self-Help Background Check**

1. (a) No conclusion can be made since \( \lim_{n \to \infty} \frac{n}{n^2 + 1} = 0 \)
   
   (b) Diverges.
   
   (c) Since \( \frac{n}{(n^2 + 1)} < \frac{1}{n} \) and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, no conclusion can be made.
   
   (d) Since \( \lim_{n \to \infty} \frac{n}{(n^2 + 1)} = 1 \) and \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, we conclude that the given series also diverges.
   
   (e) Ratio Test yields no conclusion.

2. (a) Converges.
   
   (b) No conclusion can be made.

**Mathstory**

The Bernoulli brothers, Jakob (1654-1705) and Johann (1667-1748), played central roles in the development of Calculus. Both were self-taught mathematicians who corresponded with Leibniz, became his students, and subsequently championed Leibniz’ view of Calculus. Jakob rose to appointment as a professor of mathematics in the delightful and beautiful Swiss city of Basel; Johann became a professor of mathematics at Groningen in Holland. After Jakob’s death, Johann succeeded him in the chair at Basel. Jakob’s work included the invention of polar coordinates, the study of curves such as the lemniscate (given by \( r = 1/\cos \theta \)) and exponential spiral, and infinite series. He established the divergence of the harmonic series and the convergence of the \( p \) series, where \( p = 2 \). The actual sum of the latter series was a source of much discussion; it was not until 1736 that Johann’s student, Euler, discovered that it converges to \( \pi^2/6 \).

**Literature Cited**

Discussion Set C8: Taylor Series and Taylor Polynomials

The notion that we can approximate non-linear functions by polynomials and use these simpler forms in computations is critically important in applied mathematics and in many disciplines that use mathematics. For many applications, polynomials give us results that are “close enough” to what we would get by using the actual function and are easier to use. The polynomials that best describe the function are called Taylor polynomials.

Self-Help Background Check

In order to get the most benefit from the discussion period, do the following problems before the start of discussion. You can check your answers with those given after the end of the discussion assignment.

1. (a) Evaluate \(0!\), \(1!\), \(2!\), \(3!\), \(4!\).

(b) What does the notation \(f^{(n)}(x)\) mean?

(c) If \(f(x) = \sin x\), find \(f^{(0)}(\frac{\pi}{2})\), \(f^{(1)}(\frac{\pi}{2})\), \(f^{(2)}(\frac{\pi}{2})\), \(f^{(3)}(\frac{\pi}{2})\), \(f^{(4)}(\frac{\pi}{2})\).

(d) Write a general formula for \(f^{(n)}(\frac{\pi}{2})\).
2. Recall that the linearization of a function \( f(x) \) at a point \( a \) is \( L(x) = f(a) + f'(a)(x - a) \). We called \( L \) the best fitting line (at the point \( a \)) because it is the equation of a line that agrees with \( f \) and has a first derivative that agrees with the first derivative of \( f \) at the point \( a \), i.e. \( L(a) = f(a) \) and \( L'(a) = f'(a) \).

Find the linearization of the function \( f(x) = \sin x \) at the point \( a = \frac{\pi}{2} \).

3. Sketch the graph of \( f(x) = x^2 \), \( g(x) = x^2 + c \), \( c > 0 \), \( h(x) = (x - a)^2 \), \( a > 0 \), and \( k(x) = (x - a)^2 + c \), \( a > 0 \), \( c > 0 \), with as little effort as possible (Note: This does not require calculus).

If you missed more than one problem, consult our text on page 231, or if time permits, go to office hours for clarification. In addition, problems 37 and 38 on page 233 provide extra practice.

**Approximating Functions**

Linearization is a very important approximation tool for at least three reasons:

1. Because \( L \) is a polynomial, it is very easy to integrate, differentiate, and evaluate.

2. The function \( L \) is a good approximation to \( f \), **provided that we are near enough to the point \( a \)**.

3. The function \( L \) is the “best linear approximation” to the original function \( f \) in the sense that it matches the function in the zeroth derivative and the first derivative, i.e.
As nice as the linearization is, often it is desirable to have an even better approximation to a function. In our search for something better, we don’t want to abandon property (1); i.e., we want our new, improved approximation, say \( p \), to be a polynomial so that it is still easy to work with. We can relax the “nearness to \( a \)” required in property (2), if we can improve upon property (3), the “quality” of the approximation. The way to do this is to go for a polynomial \( p \) with degree greater than one and demand that more of the derivatives of \( p \) and \( f \) match; that is,

\[
\begin{align*}
p(a) &= f(a) \\
p'(a) &= f'(a) \\
p''(a) &= f''(a) \\
&\vdots \\
p^{(n)}(a) &= f^{(n)}(a).
\end{align*}
\]

When we do this, we obtain a polynomial called the the \( n^{th} \) order Taylor Polynomial for \( f \) at \( a \) which we write as \( P_n(f(x); a) \). It is also possible to match ALL derivatives of \( f \); in that case we get a power series called a Taylor Series.

**Guided Inquiry**

We begin by giving a few definitions:

If \( f \) is a function that has \( n \) derivatives at the number \( a \), then the \( n^{th} \) order Taylor polynomial for \( f \) at \( a \) is the polynomial \( P_n(f(x); a) \) defined by

\[
P_n(f(x); a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k \\
= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n.
\]
If \( f \) is a function which is infinitely differentiable at the number \( a \), then the **Taylor series for \( f \) at \( a \)** is the power series

\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x - a)^k
\]

\[= f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots.\]

The Taylor series for \( f \) at \( a = 0 \) also is called the **Maclaurin series** for \( f \). (N.B. We have not set the series equal to the function that gave rise to it.)

At first glance, it appears that a Taylor series is nothing more than a Taylor polynomial with the upper limit \( n \) replaced with \( \infty \), and in some senses that is true.\(^3\) The major difference is that Taylor polynomials always make sense no matter where we evaluate them, whereas when we work with Taylor Series we must always worry about where they converge. (In fact, we also have to worry about whether or not the series has a positive radius of convergence.) For the moment we will concentrate on simply using the formulas given above.

4. Approximate the function \( f(x) = \sin x \) around the point \( a = \frac{\pi}{2} \).

(a) Use your work from problem (1c) to fill in what is missing in the expressions for \( P_0(\sin(x); \frac{\pi}{2}) \), \( P_1(\sin(x); \frac{\pi}{2}) \), and \( P_2(\sin(x); \frac{\pi}{2}) \).

\[
P_0(\sin(x); \frac{\pi}{2}) = \text{________}
\]

\[
P_1(\sin(x); \frac{\pi}{2}) = \text{________} + \text{________}(x - \frac{\pi}{2})
\]

\[
P_2(\sin(x); \frac{\pi}{2}) = \text{________} + \text{________}(x - \frac{\pi}{2}) + \frac{1}{2!}(\text{__________})
\]

(b) Graph \( \sin(x) \), \( P_0(\sin(x); \frac{\pi}{2}) \), \( P_1(\sin(x); \frac{\pi}{2}) \), and \( P_2(\sin(x); \frac{\pi}{2}) \) on the same set of axes below. Which one looks most like \( \sin(x) \)\

---

\(^3\)Some have said that mathematicians can only count using 3 numbers: 1, 2, and \( \infty \). In the case of Taylor Polynomials, this does not cause a problem because \( P_1(f(x); a) \) is the linearization, \( P_2(f(x); a) \) is the often used quadratic approximation, and \( P_{\infty}(f(x); a) \) is the Taylor Series.
(c) Make a guess as to what $P_4(f(x); \frac{\pi}{2})$ would look like and graph it above.
(d) Use your work from problem (1d) to write out the Taylor series for $f(x)$.

5. One of the most important functions in mathematics is the exponential, $f(x) = e^x$. Similarly, its polynomial form is an approximation that is used widely.

(a) Find the $5^{th}$ order Taylor polynomial $P_5(e^x; 0)$.

(b) Find the Maclaurin series for $f(x) = e^x$. 
(c) Use the Taylor polynomial found in (a) to approximate the number $e$. Compare this to the value stored in your calculator.

6. The figure below shows the graphs of $f(x) = e^x$, $P_1(e^x; 0)$, $P_2(e^x; 0)$, and $P_3(e^x; 0)$.

(a) Identify the graphs given in the figure, i.e., which is $e^x$, $P_1(e^x; 0)$, etc.…

(b) Discuss in your group your observations about the graphs of the polynomials and how well they match the graph of $f(x) = e^x$. Summarize your observations here.

(c) For all practical purposes, if you wished to approximate $f(x) = e^x$ near $x = 0$, which of the Taylor polynomials would you use? Discuss pros and cons of your
Group Investigation

A Shortcut
Suppose we have a Maclaurin series for $f(x)$ and want to find a Maclaurin series for $f(\pm x^n)$, where $n$ is an integer. In general, we can replace $x$ in the Maclaurin series for $f$ with $\pm x^n$ to obtain a Maclaurin series for $f(\pm x^n)$ (it’s not difficult to prove this – see what happens when you differentiate $f(\pm x^n)$ with the chain rule).

A Warning
However, simple substitution of $(x - a)$, for $a \neq 0$, in the Maclaurin series for a function $f$ will not give you the Taylor series around $x = a$. Usually, it is easiest to derive the desired Taylor series directly.

Let’s work out an example of how to use the shortcut and to justify the warning.

7. Find the Maclaurin series for $f(x) = e^{-x^2}$.

(a) Drawing on your computations from problem (5) write down the Maclaurin series for $e^x$.

(b) Now find the Maclaurin series for $e^{-x^2}$ by replacing every occurrence of $x$ in the Maclaurin series above with $-x^2$. 
(c) Let $p_2(x) = P_2(e^{-x^2}; 0)$. Find $p_2(x - 1)$.

(d) Find $P_2(e^{-x^2}; 1)$ and compare it to $p_2(x - 1)$. Discuss the comparison in your group and write a brief statement relating your conclusions to the warning.

Convergence of Taylor Series

We mentioned that convergence of the Taylor Series was an important issue and now we need to address it. Consider the function $\frac{1}{1-x}$. Its Taylor Series at $a = 0$ is

$$1 + x + x^2 + x^3 + x^4 + \cdots$$

Evaluating $\frac{1}{1-x}$ at $x = 2$ yields the value $-1$ while evaluating the Taylor series at $x = 2$ gives an infinite result! Hence, when $x = 2$, we certainly can not set $\frac{1}{1-x}$ equal to $\sum_{n=0}^{\infty} x^n$. On the other hand, when $x = \frac{1}{2}$, we see that $\frac{1}{1-x} = 2$ and $\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1/2}{1 - (1/2)} = 2$; that is, when $x = \frac{1}{2}$, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. This motivates our desire to know when a given Taylor series converges.

In short, we want to find the largest number, $R$, called the radius of convergence, so that the Taylor Series for $f$ at $a$ is guaranteed to converge for every $x$ in the interval of convergence $(a - R, a + R)$. For each $c$ in such an interval, the series will converge to $f'(c)$. We usually use the Ratio Test to determine the value $R$.

8. Suppose a function has a Taylor Series at $a = 4$ given by

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{3n-1}}(x - 4)^n.$$

Fill in what is missing to determine the radius of convergence and the interval of convergence.
To make things clearer, we label the expression:

\[ \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{2^{3n-1}} (x - 4)^n. \]

Now to use the Ratio Test, consider

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x - 4)^{n+1} 2^{3n-1}}{2^{3(n+1)-1} (x - 4)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(x - 4)^{n+1} 2^{3n-1}}{2^{3n+2} (x - 4)^n} \right|
\]

\[
= \lim_{n \to \infty} \left| \frac{(x - 4)}{2^3} \right|
\]

\[
= |x - 4| \cdot \frac{1}{8}.
\]

The Ratio Test says that if the limit of the ratio is \[ \frac{1}{8} \] than 1, then the series converges. To determine when the limit is less than 1, we just solve the inequality

\[ |x - 4| \cdot \frac{1}{8} < 1 \]

\[ |x - 4| < 8. \]

The series is therefore guaranteed to converge whenever \( x \) is in the open interval \(-8, 8\) and the radius of convergence is 8.

Discussion Set C8 Assignment

At the start of class on the day indicated by your instructor, hand in your well-written work for the following problems. Refer to the “Mathematical Writing” section in Chapter 1 of Cameos for guidance on style. Feel free to discuss the problems with others, BUT your solutions should be written independently in your own words.

9. Let \( f(x) = (1 - x)^{-1/2} \).

   (a) Find the Maclaurin series for \( f(x) = (1 - x)^{-1/2} \).
(b) Is it possible to use the root or ratio tests to determine the interval of convergence for this series? If so, find the interval of convergence. If not, explain why not.

(c) Use your answer to (a) to find the Maclaurin series for \( g(x) = (1 - x^2)^{-1/2} \).

(d) Use your answer to (c) to find the Maclaurin series for \( h(x) = \arcsin x \).

(e) Approximate \( \pi/6 \) using the first 4 terms of your answer to (d).

**Brief answers to problems from the Self-Help Background Check**

1a. 1, 1, 2, 6, 24 lb. The \( n^{th} \) derivative of \( f(x) \). 1c. 1, 0, \(-1\), 0, 1

1d. \( f^n(x) = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^{n/2} & \text{if } n \text{ even} \end{cases} \)

2. \( L(x) = 1 \)

**Mathstory**

As we noted earlier, in many scientific fields, non-linear functions routinely are approximated near \( x = a \) using the Taylor polynomial \( P_n(x, a) \). Linear models are easy to work with and have the property that future values of \( x(t) \) are completely “predictable.” However, recent work in differential equations has led scientists to become much more cautious in using this method: for many differential or difference equations, omission of non-linear terms greatly alters the behavior of solutions that satisfy the differential equation. This was noted by the meteorologist **E. M. Lorenz** in the late 1950’s (Peitgen \textit{et al.} 1992). His work has blossomed into the field known today as non-linear dynamics, and, to the general public, as the study of chaos.

You can examine a non-linear model with complex dynamics yourself by iterating a difference equation, the \textit{logistic map}, on a hand-held calculator or spreadsheet: compute values of

\[
x_{n+1} = x_n + rx_n(1 - x_n)
\]

using \( r = 3 \) and \( x_0 = 0.01 \). Iterate 100 times and record the results for \( x_n \), \( n = 10, 20, \ldots, 100 \). Then repeat, but this time, use \( x_0 = 0.011 \). Because the two \textit{initial} values \( x_0 = 0.01 \) and \( x_0 = 0.011 \) are very similar, we’d expect the two \textit{final} values to be similar. Is this what you observe? This phenomenon first was described by the ecologist **Robert May** (May 1976).

**Literature Cited**
