PROOF WRITING: ASSIGNMENT 1

Due: Thursday, January 15th

A field \( \mathbb{K} = (S, +, \cdot) \) is a set of objects \( S \) with operations \( + \) (‘addition’) and \( \cdot \) (‘multiplication’), that have the following properties: For \( a, b, c \in \mathbb{K} \),

1. \((a + b) \in \mathbb{K}\).
2. \(a \cdot b \in \mathbb{K}\).
3. \((a + b) + c = a + (b + c)\).
4. \(a + b = b + a\).
5. \(a \cdot b = b \cdot a\).
6. \((a \cdot b) \cdot c = a \cdot (b \cdot c)\).
7. \(a \cdot (b + c) = a \cdot b + a \cdot c\).
8. There exists \(0 \in \mathbb{K}\) such that \(a + 0 = a\).
9. There exists \(1 \in \mathbb{K}\) such that \(1 \cdot a = a\).
10. There exists unique element, \(-a\), such that \(a + -a = 0\).
11. For \(a \neq 0\) there exists unique element, \(a^{-1}\), such that \(a \cdot a^{-1} = 1\).

The set of real numbers, \(\mathbb{R}\), within standard addition and multiplication is a field. The set of integers, \(\mathbb{Z}\), is closed under addition and multiplication but does not contain inverses and hence is not a field.

Suppose \(z_1, z_2 \in \mathbb{R}^2\) with \(z_1 = (x_1, y_1)\) and \(z_2 = (x_2, y_2)\). We can think of \(\mathbb{C}\) as \((\mathbb{R}^2, +, \cdot)\) where

\[
(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)
\]

and

\[
(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).
\]

Problem 1) Show that if \((x_1, y_1) \cdot (x_2, y_2) = (0, 0)\) then either \((x_1, y_1) = (0, 0)\) or \((x_2, y_2) = (0, 0)\).

Problem 2) For any field \(\mathbb{K}\), show that if \(a \cdot b = 0\) then either \(a = 0\) or \(b = 0\).

Problem 3) Suppose we decided that complex multiplication would be replaced with the operation \(\star\) defined as follows:

\[
(x_1, y_1) \star (x_2, y_2) = (x_1x_2 + y_1y_2, x_1y_2 + x_2y_1).
\]

Show that \((\mathbb{R}^2, +, \star)\) is NOT a field.