PROOF WRITING: ASSIGNMENT 4

Due: Thursday, February 26th

Let \( V \) be a vector space with basis \( B = \{v_1, \cdots, v_n\} \). For \( \pi \in S_n \) define the linear transformation \( T_\pi : V \to V \) by \( T(v_i) = v_{\pi(i)} \) for \( 1 \leq i \leq n \). Let \( M_\pi \) denote the matrix of \( T_\pi \) with respect to the basis \( B \) and let \( M_n \) denote the set of such matrices.

(1) Show that \( M \in M_n \) if and only if there is exactly one 1 in each row and column of \( M \) and zeroes everywhere else.

(2) Show that \( M^T \in M_n \) if and only if \( M \in M_n \).

(3) Let \( M_\pi = (m_{ij})_{1 \leq i,j \leq n} \). Show that

\[
m_{ij} = \begin{cases} 
1, & i = \pi(j) \\
0, & \text{otherwise}
\end{cases}
\]

(4) Show that \( \det(M_\pi) = \text{sign}(\pi) \).

(5) Show that \( M_\pi M_\sigma = M_{\pi \sigma} \). (Hint: If \( AB = C \) then \( c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \).

(6) For \( M_\pi \in M_n \) show that \( M_\pi^T M_\pi = I_n \).