Specific Topics Reviewed in Detail:

Completing the Square, Using Algebra to Evaluate Limits (functions, sequences), \(\epsilon/N\) Proofs, Solving Trig. Equations, Properties of Functions (domain, range, 1-1, inverses, compositions), Exponential Growth/Decay, Continuity of Function, Fixed Points for Recursion, Formula for Sequences

SELECTED REVIEW 1:

The equation of the circle with radius \(R\) centered about \((a,b)\) is:

\[
(x-a)^2 + (y-b)^2 = R^2
\]

Remark: remember in section how we had to complete the square in order to solve the problem as stated... know how to do this! Let’s practice a couple. Write the following equations as \((x-a)^2 = b\):

\[
(x + 2)^2 = 4
\]

What about this one:

\[
(x - 2)^2 = 4
\]

Write a couple down for yourself. Multiply out the square you get and see that it is correct. Always if you have time go back and check your work.

STUDY TIP: Practice checking your work. When you solve a problem learn how to quickly check your work to verify that you have done things correctly. This skill is very important to getting high scores. Even if you know the material cold, to err is human and we all make little calculation mistakes from time to time.

LINES (2 key forms)

\[
y = mx + b \quad \text{y-intercept form. (} m = \text{slope, } b = \text{y-intercept)}
\]

\[
y - b = m(x - a) \quad \text{Point-slope form. (This is the line through the point (a,b) with slope = m)}
\]

Recall that given two points \((x_1,y_1), (x_2,y_2)\) you can solve for the slope = \(m\) of the line passing between them using:

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}
\]

Parallel lines have the same slopes: \(m_1 = m_2\). Orthogonal (means perpendicular) lines have opposite sign reciprocal slopes: \(m_1 = -\frac{1}{m_2}\).

COMPLEX NUMBERS (a+bi):

\[
i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \quad i^5 = i, \quad i^6 = -1, \text{ etc}....
\]

Remark: There is much to learn about complex numbers, but for our purposes only a minimal understanding is needed. So far you just need to be aware of complex numbers, how they come about (ie. taking the square root of a negative number \(\sqrt{-16} = 4i\)) and understand how to re-write a complex number in the standard form: \(a+bi\) where \(a, b\) are both real numbers.
FUNCTIONS (key concepts so far)

Domain - the set of admissible x-values.
Range - the set of y-values the function can attain.

STUDY TIP - don’t memorize those words (or any others like them). Practice problems, understand what the domain and range are intuitively and practice finding them.

Example 1: What are the domain and range of $f(x) = \ln(x)$? Answer: Domain = $\{ \mathbb{R} : x > 0 \}$, or $\{ x > 0 \}$ or all positive real numbers, Range = $\{ \mathbb{R} \}$, or all real numbers.

Example 2: What are the domain and range of $f(x) = e^x$? Answer: Domain = $\{ \mathbb{R} \}$, Range = $\mathbb{R}$.

Example 3: What are the domain and range of $f(x) = \frac{1}{x-3} + 1$? Answer: Domain = $\{ \mathbb{R} : x \neq 3 \}$ or all real numbers except $x = 3$, Range = $\mathbb{R}$, or all real numbers.

Remark 1 - When finding the domain what we care about is making sure that the function makes sense for a given value of $x$. It does not make sense to ask what is $f(-2) = \ln(-2)$. The natural logarithm or the log with base e - note that this is $\ln(x) = \log_e(x)$ is not defined for $x \leq 0$. Similarly, we say that $\frac{1}{0}$ is undefined. So whenever an x-value results in a function requiring the evaluation of something that looks like $\frac{1}{0}$ you can safely assume that that x-value is not in the domain. As well, for problems involving square roots we are going to require that we never allow the square root of a negative number. So the domain of $f(x) = \sqrt{x+1}$ is all real numbers greater than or equal to -1.

Remark 2 - Notice how I used several different notations. Recall the question asked last week about writing the answers. There are several symbolic and non-symbolic ways to write the correct answer, don’t get caught up learning fancy ways to write the answer or bogged down trying to learn several new notations for writing these sets, just know how to find the domain and range properly and the answer that you give in the notation that you are comfortable with will most likely be correct. If you are not sure about the notation your are using then just ask.

Compositions - The whole trick here is understanding the notation $f(x)$. If I tell you that $f(x) = x^2 + 3$ and then I ask you what is $f(\triangle)$? What do you say? The answer is simple: $f(\triangle) = \triangle^2 + 3$. What we are about to review, formally, is almost identical. Let’s compose the two functions:

$$f(x) = \frac{1}{x-3} \quad \text{and} \quad g(x) = \frac{x}{2x+5}$$

We will find $(f \circ g)(x) = f(g(x))$ and $(g \circ f)(x) = g(f(x))$. Doing this we get the following:

$$(f \circ g)(x) = f(g(x)) = \frac{1}{\frac{x}{2x+5}-3}$$

and

$$(g \circ f)(x) = g(f(x)) = \frac{\frac{1}{x-3}}{\frac{x}{2x+5}+5}$$

One can and should simplify these answers, but that should be a routine calculation. What is most essential now is that you understand how to correctly ascertain the right function composition.

One-to-One property of functions - This means that for each y-value there is at most a single x-value such that $f(x) = y$. We check that a function is one-to-one algebraically by verifying the logical equivalent, namely for two x-values $x_1$ and $x_2$:

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

Pause. This is not a trivial thing. Take a moment to think about the definition and make sure it makes sense to you before reading further. (Aside: This is a deep and abstract mathematical property that is a core element in higher level analysis in which framework it is often referred to as a function being injective.) Graphically, for us, the one-to-one property is equivalent to the function passing the horizontal line test. The test says if you can draw a horizontal line that intersects the graph of the function in 2 or more y-values than the function is not one to one, otherwise it is. If you think about it you will see that the graphical and algebraic definitions are equivalent.

Inverses - the whole idea is to find a function $f^{-1}(x)$ such that $f(f^{-1}(x)) = x$. 
Remark: Note well that we must have that the function we are inverting is one-to-one on an associated domain. If you are asked to find the inverse then you can probably just go ahead and apply this algorithm, but if you are asked to determine if the function is invertible or set conditions on the domain so that the function is invertible and then find its inverse if it exists you should make the connection that you are being asked a question about whether or not the function is one-to-one.

Let's find the inverse of:

\[ f(x) = \frac{x + 1}{2 - x} \]

Begin by writing: \( f(f^{-1}(x)) = x \). Don’t let the notation scare you, this easy… you have the function \( f \), just plug in for \( x \) whatever is in the parenthesis denoting the functions argument just like we did with compositions. When you have done that just solve for \( f^{-1}(x) \) and you are done! So applying this algorithm to our example here’s what we should get:

\[
\begin{align*}
\quad & f(f^{-1}(x)) = \frac{f^{-1}(x) + 1}{2 - f^{-1}(x)} = x \\
\quad \Rightarrow & f^{-1}(x) + 1 = x(2 - f^{-1}(x)) \\
\quad \Rightarrow & f^{-1}(x) + xf^{-1}(x) = 2x - 1 \\
\quad \Rightarrow & f^{-1}(x)(1 + x) = 2x - 1 \\
\quad \Rightarrow & f^{-1}(x) = \frac{2x - 1}{1 + x} \quad \text{DONE!}
\end{align*}
\]

**ABSOLUTE VALUE EQUATIONS** - the goal is to solve for \( x \)

**Example 1.1:** \(|2x - 1| = 3\)
**Solution 1.1:**

\[
\begin{align*}
2x - 1 &= -3 \quad \text{or} \quad 2x - 1 = 3 \\
2x &= -2 \quad \text{or} \quad 2x = 4 \\
&\quad \quad x = -1 \quad \text{or} \quad x = 2
\end{align*}
\]

**Example 1.2:** \(|x - 2| \leq 5\)
**Solution 1.2:**

\[
\begin{align*}
-5 &\leq x - 2 \leq 5 \\
-3 &\leq x \leq 7 \quad \text{(We could represent this answer as: } -3 \leq x \text{ and } x \leq 7) 
\end{align*}
\]

**Example 1.3:** \(|3x - 4| > 11\)
**Solution 1.3:**

\[
\begin{align*}
-11 &> 3x - 4 > 11 \\
-11 &> 3x - 4 \quad \text{or} \quad 3x - 4 > 11 \\
-7 &> 3x \quad \text{or} \quad 3x > 15 \\
-\frac{7}{3} &> x \quad \text{or} \quad x > 5
\end{align*}
\]

**Example 1.4:** \(|x - 3| = |2x + 2|\)
**Solution 1.4:** For a good illustration of how to do this type see Homework 1 ex. 4a. I leave it as an excercise, but be sure to know how to handle it.

Remark: Notice how in example 1.2 we have an ‘and’ condition that bounds admissible \( x \) values between two endpoints while in 1.3 we have an ‘or’ condition which sets up and lower values that \( x \) can take on ranges that extend all the way to \( \pm \infty \).

**TRIGONOMETRIC EQUATIONS** - the goal is to solve for \( \theta \)

**Remark 1:** You must learn how to get the numbers associated with the basic angles: \( 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \text{ etc...} \) Once you know the basic angles understanding where cos and sin are positive and negative respectively will give you all the values you
will need. Practice, practice and practice.

Example: Solve for $0 \pi \leq \theta \leq 2 \pi$ if $\sin^2(\theta) = \frac{3}{4}$. Solution: $\sin(\theta) = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

Remark: Notice that there are infinitely many solutions to $\sin(\theta) = \pm \frac{\sqrt{3}}{2}$. We get our answer by restricting the set of solutions using the condition that $0 \pi \leq \theta \leq 2 \pi$.

LOG STUFF (learn the log rules - they are a very important part of this class):

First note the following important definition (to help me remember I say to myself: the base is the base, the log is the exponent):

$$\log_b(x) = y \iff b^y = x$$

Here are some examples of must know rules:

$$\log_e(x) = \ln(x)$$
$$\log_b(b) = 1$$
$$\log_b(a^x) = x \log_b(a)$$
$$b^{\log_b(x)} = x$$
$$\log_b(xy) = \log_b(x) + \log_b(y)$$
$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Remark: Getting good at applying these rules will help a lot with this course. I strongly recommend that you do enough example problems with log rules until you have an air of dexterity with them.

SELECTED REVIEW 2:

**Exponential Growth and Decay** - a sequential approach...

So far we have been introduced to exponential growth of the form:

$$N_t = N_0 R^t$$

Where, thinking of time discretely as $t = 0, 1, 2, 3, \ldots$ we have $N_t$ as the number of things or population at, or up to, time $t$, $N_0$ is the initial amount of things when $t = 0$ and $R$ is the growth or decay constant. Note carefully the following:

$$0 < R < 1 \Rightarrow \text{The population decays exponentially to 0 as } t \to \infty$$
$$R = 1 \Rightarrow \text{The population is } N_0 \text{ for all time}$$
$$R > 1 \Rightarrow \text{The population grows exponentially to } \infty \text{ as } t \to \infty$$

If you are asked to determine the time at which $N_t$ is a specific value then you just solve for $t$ using your formula for $N_t$ and the log rules. For example: Suppose you have found the equation $N_t = 2(3^t)$ and you are asked how long until $N_t = 3,188,646$? Then just plug the givens into the equation and solve for $t$:

$$3,188,646 = 2(3^t)$$
$$1,594,323 = 3^t$$
$$\log_3(1,594,323) = \log_3(3^t)$$
$$\log_3(1,594,323) = t \log_3(3)$$
$$\log_3(1,594,323) = t$$

There are 2 things that make what I did above the best course of action. The first was dividing by the initial amount $N_0 = 2$ first. The second was choosing log base 3. These are not necessary steps, they just make life easier. You can use log to any base that you want or not do the division by $N_0 = 2$ first, but then you must be more careful with your log rules. For example if I do not first divide and instead of choosing log base 3 I choose log base e which is the natural logarithm denoted:
ln, then I would get:
\[
\begin{align*}
3,188,646 &= 2(3^t) \\
\log_e(3,188,646) &= \log_e(2(3^t)) \\
\ln(3,188,646) &= \ln(2) + \ln(3^t) \\
\ln(3,188,646) - \ln(2) &= t \ln(3) \\
\frac{\ln(3,188,646) - \ln(2)}{\ln(3)} &= t
\end{align*}
\]

Which is the same answer as I got above, just written with log base e. A simple application of the log rules shows us that:
\[
\frac{\ln(3,188,646) - \ln(2)}{\ln(3)} = \log_3(1,594,323)
\]

Remark 1: **CAUTION!** Note that when you use the logarithm on your calculator you have to make sure that you are using log with the right base (usually it is base 10 by default and there is a separate button for log base e). You will lose points if you hastily plug your answer into the calculator incorrectly and give an answer that is wrong. For example: If you wanted to plug the above answer(s) into a calculator to find that \( t = 13 \) then you can use the fact that:
\[
\log_b(x) = \frac{\ln(x)}{\ln(b)} = \frac{\log_{10}(x)}{\log_{10}(b)}
\]

Remark 2: Note the similarity between the limiting behavior of \( N_t \) depending on \( R \) and the limit examples we saw in class:
\[
\begin{align*}
\lim_{n \to \infty} (.9999)^n &= 0 \\
\lim_{n \to \infty} (1)^n &= 1 \\
\lim_{n \to \infty} (1.00001)^n &= \infty
\end{align*}
\]

Note that here the first two converge while the last one diverges. We say that the sequence diverges when the limit equals \( +\infty \) or \( -\infty \). It is important to state that the limit is \( +\infty \) or \( -\infty \) - INCLUDING THE SIGN - and then one can conclude from this further that the limit does not exist, or DNE.

Remark 3: Success on these problems will require that you learn the log rules. Practice them - it will help you a lot during this course.

\( \epsilon \)-N proof

Suppose we face the problem: Give an \( \epsilon \)-N proof that: \( \lim_{n \to \infty} \frac{6n}{11+2n} = 3 \). I will show you first how to solve it, then how to write the solution.

**How to solve it:**

We want to show that given any \( \epsilon > 0 \) we can choose a positive integer \( N \) such that for all integers \( n > N \) we will always have:
\[
\left| \frac{6n}{11+2n} - 3 \right| < \epsilon
\]

The entire challenge for us to figure out what \( N \) should be. Remember: The \( N \) needed is going to depend on \( \epsilon \). The way to think about it is that the smaller \( \epsilon \) is the bigger \( N \) will have to be. In general your choice of \( N \) will be a function of \( \epsilon \).

So let’s start with what we want... and figure out what \( N \) should be in order to make it true for any value of \( n > N \). Thus we start with:
\[
\left| \frac{6n}{11+2n} - 3 \right| < \epsilon \\
\left| \frac{6n}{11+2n} - \frac{11+2n}{11+2n} \right| < \epsilon \quad \text{Know this trick, learn it!} \\
\left| \frac{6n-(33+6n)}{11+2n} \right| < \epsilon \\
\left| \frac{-33}{11+2n} \right| < \epsilon \\
\frac{33}{11+2n} < \epsilon \quad n = 1, 2, 3, \ldots \text{ always positive...}
and having gotten this far the hard work is over. Now we just solve for \( n \):

\[
\frac{33}{11 + 2n} < \epsilon \Rightarrow \frac{11 + 2n}{\epsilon} > \frac{33}{\epsilon} > 1 \\
11 + 2n > \frac{33}{\epsilon} \Rightarrow 2n > \frac{33}{\epsilon} - 11 \\
\Rightarrow n > \frac{33 - 11}{2}
\]

Almost done... Now we know that choosing any integer \( N > \frac{33 - 11}{2} \) will give:

\[
\left| \frac{6n}{11 + 2n} - 3 \right| < \epsilon
\]

So we are done!

How to write your solution:

Solution: Let \( \epsilon > 0 \) be given. Since we have:

\[
\left| \frac{6n}{11 + 2n} - 3 \right| < \epsilon \Rightarrow \left| \frac{6n - (33 + 6n)}{11 + 2n} \right| < \epsilon \\
\left| \frac{-33}{11 + 2n} \right| < \epsilon \\
\frac{33}{11 + 2n} < \epsilon \quad n = 1, 2, 3, \ldots \text{ always positive...}
\]

and solving for \( n \) we get:

\[
\frac{33}{11 + 2n} < \epsilon \Rightarrow \frac{11 + 2n}{33} > \frac{1}{\epsilon} \Rightarrow n > \frac{33 - 11}{2}
\]

Therefore choosing any integer \( N > \frac{33 - 11}{2} \) for all integers \( n > N \) it will follow:

\[
\left| \frac{6n}{11 + 2n} - 3 \right| < \epsilon
\]

\[\Box\]

Remark 1: This type of problem is technically very easy, it is however - at least in my opinion - one of the more difficult conceptual problems that we will encounter this quarter. Do yourself a favor and practice many of these problems! If you are having difficulties with this type of argument make sure that you get your questions asked and answered well before exam time.

Remark 2: There are infinitely many choices for \( N \). For example: \( \frac{100}{\epsilon} > \frac{17}{\epsilon} > \frac{33 - 11}{2} \), but it is customary with this type of proof to give the tighest possible bound on \( N \) in terms of \( \epsilon \).

Using algebra to evaluate limits:

First plug in the limit value. If you get a number (like 2 or \( \pi \)) then that’s your answer. The limit may be plus or minus infinity, like in the case of \( \frac{1}{0^+} \) or \( \frac{1}{0^-} \) respectively (note by \( 0^+ \) we mean that the value approaches 0 from the right - so it is always positive and in the limit it goes to 0). Most often for our problems we will get something that is undefined (like \( 0, \infty, -\infty, \frac{\infty}{0} \)) and then we are going need to do some investigative work to get the real story. There are a couple of standard tricks. Learn them. Remember, sometimes the limit is just a number, sometimes the limit does not exist (write: DNE), sometimes it equals positive infinity and sometimes it is negative infinity.

I recommend doing all the homework problems and all the section problems on limits. If you can do all of them you will probably be in excellent shape for the midterm. Here are a couple of key examples to get you started.

\[
\lim_{x \to 0^+} \frac{1}{x} = +\infty \\
\lim_{x \to 0^-} \frac{1}{x} = -\infty
\]
\[
\lim_{x \to 0} \frac{1}{x} = \text{DNE}
\]

Why does the limit not exist for the last one? This is because the limit as \(x\) approaches 1 from left (\(\lim_{x \to 1^-}\)) does not equal the limit as \(x\) approaches 1 from the right (\(\lim_{x \to 1^+}\)).

**Standard trick 1: FACTOR AND CANCEL:**

\[
\lim_{x \to 4} \frac{x - 4}{\sqrt{x} - 2} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{\sqrt{x} - 2} = \lim_{x \to 4} (\sqrt{x} + 2) = 4
\]

**Standard trick 2: MULTIPLY BY THE CONJUGATE THEN SIMPLIFY:**

\[
\lim_{x \to 0} \frac{\sqrt{x^2 + 16} - 4}{x^2} = \lim_{x \to 0} \frac{(\sqrt{x^2 + 16} - 4)(\sqrt{x^2 + 16} + 4)}{x^2} = \lim_{x \to 0} \frac{x^2 + 16 - 16}{\sqrt{x^2 + 16} + 4} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 16} + 4} = \frac{1}{8}
\]

Notice that when multiplying by the conjugate we are actually just multiplying by 1 and this does not change the value since for any \(a\) we have \(1 \cdot a = a\).

**Remark 1:** Remember to carry the limit notation through to each step. You will lose points if you do not do this!

**Remark 2:** Notice my careful use of parentheses above. Make sure when you do algebra that you use parentheses wisely and well. I probably see more algebra mistakes caused by not carrying parentheses through a calculation from one line to the next than any other error.

**SELECTED REVIEW 3:**

**Find Formula for Sequence**

The whole game here is to write down a formula and then check to make sure it generates the numbers you are given. Most of the sequences we have seen are pretty straightforward. For example starting from \(n = 0\):

\[
\begin{align*}
  -3 & 4 \quad -5 \quad 6 \quad -7 \\
  4 & 5 \quad 6 \quad 7 \quad 8 \quad \ldots
\end{align*}
\]

Would be given by a formula of the form:

\[
a_n = (-1)^{n+1} \frac{n + 3}{n + 4}
\]

Getting more difficult we saw sequences that looked like:

\[-1, 5, -9, 13, -17, 21, \ldots\]

You have to ask yourself first what is happening? Well, without the sign change we are doing something like this:

\[
\begin{align*}
  b_0 &= 1 \\
  b_1 &= 1 + 4 = 1 + 4(1) \\
  b_2 &= 1 + 4 + 4 = 1 + 4(1 + 1) \\
  b_3 &= 1 + 4 + 4 + 4 = 1 + 4(1 + 1 + 1) \\
  \vdots \\
  b_n &= 1 + 4(n - 1)
\end{align*}
\]

So we can see that we should get our sequence by a formula of the form (Starting from \(n = 0\)):

\[
b_n = (-1)^n(1 + 4(n))
\]

Stepping up the difficulty one more notch we saw sequences of the form:

\[1, -8, 27, -64, 125, -216, \ldots\]

Where there was not such a straight-forward pattern, but rather you have to first make the (not obvious) observation that we are adding cubes:

\[1^3, -2^3, 3^3, -4^3, 5^3, -6^3 \ldots\]
to get the sequence. In this case to get the pattern is hard, but once you see it then the formula follows in a straightforward way:

\[ c_n = (-1)^n(n + 1)^3 \]

At the hard end of the spectrum we might see something like this:

1, −5, 10, −16, 23, −31, ...

In this case you again have to ask yourself first what is happening? Well, we are doing something like this (again first thinking without the sign change...):

\[
\begin{align*}
d_0 &= 1 \\
d_1 &= 1 + 4 \\
d_2 &= 1 + 4 + 5 \\
d_3 &= 1 + 4 + 5 + 6 \\
\vdots \\
d_n &= !?
\end{align*}
\]

To me it is not at all obvious how to go about solving this one at first glance. However, a little bit of specialized knowledge can up us here:

\[
1 + 2 + 3 + \ldots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}
\]

If the summation notation is new to you then do not worry about it. The important thing to know for now is the rule: 1 + 2 + ... + n = \(\frac{n(n+1)}{2}\). Using this formula helps us to think of the sequence \(c_n\) as:

\[
\begin{align*}
d_0 &= 1 = 1 + 2 + 3 - 2 - 3 \\
d_1 &= 1 + 4 = 1 + 2 + 3 + 4 - 2 - 3 \\
d_2 &= 1 + 4 + 5 = 1 + 2 + 3 + 4 + 5 - 2 - 3 \\
d_3 &= 1 + 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5 + 6 - 2 - 3 \\
\vdots \\
d_n &= 1 + 4 + 5 + 6 = 1 + 2 + \ldots + n - 5
\end{align*}
\]

So we can exploit our specialized knowledge to get the following formula (remember we are starting from \(n = 0\)...):

\[ d_n = (-1)^n \left( \frac{(n+3)(n+4)}{2} - 5 \right) \]

Whew! That was pretty tricky. Practice it so you are ready and remember that formula!

Remark 1: On the exam I would expect to see one medium but relatively straightforward one and one more difficult one. The trick is to first figure out what the pattern is. Notice how the algorithm I used above helped me to find the general rule. First I wrote out a pattern in terms of \(a_1, a_2, a_3, a_4 \ldots \) and so on, and only after I had the pattern did I write down the general rule for \(a_n\).

Remark 2: Be very careful with your parenthesis when writing down these rules. A misplaced parenthesis is perhaps the easiest way to lose points on these problems.

Remark 3: Always, always, check that your general rule formula works! Double check on alternating sign sequences that you have the first couple signs correct.

Using algebra to evaluate limits:

First plug in the limit value. If you get a number (like 2 or \(\pi\)) then that’s your answer. The limit may be plus or minus infinity and in these cases we will say that the limit does not exist or DNE. Most often for our problems we will get something that is undefined (like \(\frac{0}{0}, \infty - \infty, \frac{\infty}{\infty}\)) at first and then we are going need to do some investigative work to get the real story. There are a couple of standard tricks. Learn them. Remember, the limit is either a number or it does not exist (write: DNE). Sometimes the limit will equal positive infinity or negative infinity and in these cases we will also say that the limit DNE. Trust your intuition, but always use the algebraic rules to prove your intuition is correct. On the exam an answer without an explanation often receives little or no credit!
Standard trick 1: FACTOR AND CANCEL:

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\]

Standard trick 2: MULTIPLY BY THE CONJUGATE THEN SIMPLIFY:

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\lim_{x \to 0} \frac{\sqrt{x^2 + 16} - 4}{x^2} = \lim_{x \to 0} \frac{(\sqrt{x^2 + 16} - 4)(\sqrt{x^2 + 16} + 4)}{x^2(\sqrt{x^2 + 16} + 4)} = \lim_{x \to 0} \frac{x^2 + 16 - 16}{x^2(\sqrt{x^2 + 16} + 4)} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 16} + 4} = \frac{1}{8}
\]

Notice that when multiplying by the conjugate we are actually just multiplying by 1 and this does not change the value since for any \(a\) we have \(1 \cdot a = a\).

Remark 1: Remember to carry the limit notation through to each step. You will lose points if you do not do this!

Remark 2: Notice my careful use of parentheses above. Make sure when you do algebra that you use parentheses wisely and well. I probably see more algebra mistakes caused by not carrying parentheses through a calculation from one line to the next then any other error.

Standard trick 3: DIVIDE BY A POWER OF X OR WELL CHOSEN FUNCTION:
This is a great trick that works for many problems where you want to see the balance of the quotient of two similar functions to different powers. Here are two examples:

\[
\lim_{x \to \infty} \frac{5e^x}{e^x + 100} \quad \text{and} \quad \lim_{x \to \infty} \frac{2 + x^3}{3x + 2}
\]

Remember when we say we are dividing by something we really mean multiplying the whole function by 1. Solving both of these using this trick gives us:

\[
\lim_{x \to \infty} \frac{5e^x}{e^x + 100} = \lim_{x \to \infty} \frac{5e^x}{(e^x + 100)} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \to \infty} \frac{5}{1 + \frac{100}{e^x}} = \frac{5}{1 + \frac{100}{\infty}} = 5
\]

and:

\[
\lim_{x \to \infty} \frac{2 + x^3}{3x + 2} = \lim_{x \to \infty} \frac{(2 + x^3) x^{-3}}{(3x + 2) x^{-3}} = \lim_{x \to \infty} \frac{2}{3} = \frac{2}{3} + \frac{1}{x} = \frac{2}{3} + \frac{1}{\infty} = 0 + 0 = +\infty, \text{ DNE}
\]

Or to solve the latter we could also have done:

\[
\lim_{x \to \infty} \frac{2 + x^3}{3x + 2} = \lim_{x \to \infty} \frac{(2 + x^3) x^{-1}}{(3x + 2) x^{-1}} = \lim_{x \to \infty} \frac{2}{3} + \frac{3}{x} = \frac{2}{3} + \frac{3}{\infty} = 0 + 0 + 0 = +\infty, \text{ DNE}
\]

Standard trick 4: USE THE SQUEEZE THEOREM:
The classic squeeze theorem problems are of the following form: Find the limit:

\[
\lim_{x \to \infty} e^{-x} \sin(x), \quad \lim_{x \to \infty} \frac{2}{x} \cos(x), \quad \text{etc}...
\]

The squeeze theorem gets its name because we are going to squeeze our function between two functions - one that is an upper bound and the other that is a lower bound. The trick is to observe that one of the functions is bounded for all \(x\). Usually this observation requires you to observe (write it down on your test!) that for all \(x \in \mathbb{R}\):

\[-1 \leq \cos(x) \leq 1, \quad \text{or} \quad -1 \leq \sin(x) \leq 1\]

Then you can say that for example:

\[-\frac{1}{e^x} \leq e^{-x} \sin(x) \leq \frac{1}{e^x}\]

So by the squeeze theorem, because

\[
\lim_{x \to \infty} \frac{-1}{e^x} = 0 \quad \text{and} \quad \lim_{x \to \infty} \frac{1}{e^x} = 0
\]

It follows that:

\[
\lim_{x \to \infty} e^{-x} \sin(x) = 0
\]

Key Remark: You must apply the squeeze theorem properly! You will lose points - potentially most - if you do not bound the function between upper and lower bounding functions and explain why you can do that (i.e. write something like:
\(-1 \leq \sin(x) \leq 1\) for all \(x\) and state that you are applying the squeeze theorem. Many times students that just write the answer receive little or no credit, so make sure you show you understand the theorem and apply it when you give your answer.

**Standard trick 5: THE LIMIT DOES NOT EXIST:**
Suppose you are asked to find:

\[
\lim_{x \to 0} \frac{1}{x^2}
\]

In this case the limit DNE (Does Not Exist). Note carefully though that if we asked for the limit from the left or right alone, then the limit also does not exist. In fact:

\[
\lim_{x \to 0^+} \frac{1}{x} = +\infty, \quad \text{(DNE)}\]

\[
\lim_{x \to 0^-} \frac{1}{x} = +\infty, \quad \text{(DNE)}
\]

Moreover, because the limit as \(x\) approaches 1 from left (\(\lim_{x \to 0^-}\)) does not equal the limit as \(x\) approaches 1 from the right (\(\lim_{x \to 0^+}\)) that the limit of \(\frac{1}{x}\) DNE as \(x \to 0\). This is also true because at least one of the limits from the left and from the right DNE. Remember, the conclusion that the limit does not exist is a consequence of it diverging, or in other words, it equaling \(+\infty\) or \(-\infty\). First state the limit is equal to \(+\infty\) or \(-\infty\), and then conclude that the limit DNE.

Key remark: The last idea may have gone unnoticed. It is important. If the limit from the left and from the right at a point do not agree then the limit at that point does not exist.

**Determine that value of some parameter so that the limit exists**

The whole idea here is find the value of some parameter so that the limit from the left and the limit from the right agree. For example, suppose we asked:

Find the value of \(K\) so that the \(\lim_{x \to -1}\) exists:

\[
f(x) = \begin{cases} 
  x^2 + K, & \text{if } x < -1; \\
  x + K^2, & \text{if } x \geq -1,
\end{cases}
\]

So the thing you need to do is find \(K\) such that:

\[
\lim_{x \to -1^-} x^2 + K = \lim_{x \to -1^+} x + K^2
\]

Since there are no discontinuities in the \(f(x)\), straightaway we can plug in \(x = -1\) and solve the resulting quadratic equation for \(K\) to get:

\((-1)^2 + K = -1 + K^2 \Rightarrow K^2 - K - 2 = 0 \Rightarrow (K - 2)(K + 1) \Rightarrow K = -2, 1\)

**Continuity of functions at a point**

There is an important definition to remember for this type of problem, but it should seem natural. We must check that the limit exists from the left and the limit from the right exist and that they are both finite and are the same number. So the definition we give is:

A function \(f(x)\) is continuous at the point \(b\) if and only if:

1. \(f(b)\) exists and is finite.
2. \(\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)\)
3. \(\lim_{x \to b} f(x) = f(b)\)

For this type of problem you just check conditions 1,2, and 3. An important hybrid problem of this type that often shows up on exams involves asking you to find all \(K\) such that the function \(f(x)\) continuous at \(x = -1\)? In that case a correct answer will find \(K = -2\) and \(1\) as before and then check that in both cases conditions 1-3 are satisfied.