Chapter 4

Model Reduction for Control Design for Distributed Parameter Systems

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4.1 Introduction

During the past decades, considerable advances have been made in the numerical simulation of controlled distributed parameter systems (DPS). In the opinion of this author, this numerical sophistication has not been matched by the theoretical understanding of the approximation processes involved. The aim of this chapter is to shed a little light on some of the system theoretic properties which determine the suitability of an approximation scheme for control design of DPS. At the same time, a new robust control design is proposed which leads to robust, low-order controllers. It is shown that, at least for the class of exponentially stabilizable and detectable state linear systems with bounded and finite-rank input and output operators, this design always leads to a low-order controller which stabilizes not only the original system but also a large class of perturbations. This robustly stabilizing controller also guarantees bounds on the main performance indices.

The class of systems considered in this chapter is that of the exponentially stabilizable and detectable state linear systems \( \Sigma(A, B, C) \) on the Hilbert space \( Z \), where \( A \) is the infinitesimal generator of the strongly continuous semigroup \( T(t) \) on \( Z \), and the operators \( B \) and \( C \) are finite-rank and bounded; \( B \in \mathcal{L}(\mathbb{C}^m, Z) \), \( C \in \mathcal{L}(Z, \mathbb{C}^k) \). The basic properties required of a finite-dimensional controller for this system are:

(P1) The controller stabilizes \( \Sigma(A, B, C) \).

(P2) The controller is robust so that it will have a chance of stabilizing the actual physical plant.

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(P3) The performance of the controller is reasonable, e.g., with respect to disturbance rejection, sensitivity of the output, etc.

If one assumes that a transfer function model of the system is given, there exist several robust control designs for larger classes of systems than $\Sigma(A, B, C)$ which satisfy the above properties (P1)–(P3) (see Curtain and Zwart [7], Chapter 9 and other references in Section 9.7). However, the present mode of modeling physical linear DPS rarely leads to a nice compact transfer function model, but rather to a system of partial differential equations (PDE) or to a linear state model of extremely high dimensions (in finite-element form). This is the starting point taken in this chapter. It is appropriate to start with a review of two popular approaches used to design finite-dimensional controllers for state-space models of DPS.

4.1.1 Trotter-Kato Semigroup Approximations

The system $\Sigma(A, B, C)$ is approximated by a sequence of finite-dimensional approximating systems $\Sigma(A^n, B^n, C^n)$ which converges to the original system in some sense (see Theorem 4.10). One designs a controller $K^n$ for $\Sigma(A^n, B^n, C^n)$ and uses it to stabilize the original infinite-dimensional system. Favorite choices for the controller design are the linear quadratic Gaussian (LQG) and min-max designs and various $H_{\infty}$ designs. Initially, the focus was on obtaining good numerical approximations of the operator solution to the standard Riccati equation. It is only relatively recently that researchers seriously addressed even the first requirement (P1) (see Ito [14, 15] and Morris [28, 29]). The approach of Morris is particularly nice in that, rather than listing various technical conditions, she emphasized the crucial property which is necessary (but not sufficient) for these approximations to satisfy the requirements (P1)–(P3); namely, the approximating transfer functions must converge in the gap topology (or graph topology, as in Vidyasagar [42]). If $G^n$ is a sequence of stable transfer functions (i.e., $G^n \in H_{\infty}$), then convergence in the gap topology is equivalent to convergence in the $H_{\infty}$-norm. For unstable state linear systems the definition is in terms of coprime factorizations (see Curtain and Zwart [7], Definition 7.2.7). The following is an equivalent definition from Vidyasagar [42] Lemma 20, p. 238, and for more background on the gap and the graph topology see Zhu [44].

**Definition 4.1.** Let $\Sigma(A, B, C)$, $\Sigma(A^n, B^n, C^n)$ have the transfer functions $G$ and $G^n$, respectively, and suppose that $G$ and $G^n$ have left-coprime factorizations

$$G = \tilde{M}^{-1}\tilde{N}, \quad G^n = (\tilde{M}^n)^{-1}\tilde{N}^n.$$ 

Then $G^n$ converges to $G$ in the gap topology if and only if

$$\|\tilde{N}^n - \tilde{N}\|_{\infty} \to 0, \quad \|\tilde{M}^n - \tilde{M}\|_{\infty} \to 0 \text{ as } n \to \infty. \quad (4.1)$$

The above definition is independent of the left-coprime factorizations chosen and, if (4.1) holds for one choice, it holds for all left-coprime factorizations. An
equivalent definition can be given in terms of right-coprime factorizations. Sufficient
conditions on the approximating systems \( \Sigma(A^n, B^n, C^n) \) to guarantee convergence
in the gap topology are given in [29]. For properties (P1), (P3) to hold, even
stronger conditions are needed, and these are discussed by Morris in [28]. As an
application, she gives conditions under which the standard LQG design on the
system \( \Sigma(A^n, B^n, C^n) \) will stabilize \( \Sigma(A, B, C) \). Similar results for the \( H_\infty \) design
are in Morris [27].

4.1.2 Proper Orthogonal Decomposition Reduced-Order Models

Proper orthogonal decomposition (POD) is a technique for obtaining reduced-order
models from data collected from nonlinear partial differential equations. Initially it
was applied with considerable success to obtain low-order models for uncontrolled
dynamical systems, but more recently it has been applied to controller design. The
controller is designed to control the reduced-order model with the hope that it will
perform well on the original nonlinear partial differential equation. The literature
on POD is extensive and there are several different approaches. One well-known
approach is the “Method of Snapshots” (see Atwell and King [1, 2], Banks, del
Rosario and Smith [3], Kepler, Tran and Banks [19], Ly and Tran [23, 22] and
Kunisch and Volkwein [20]). Another approach stems from Principle Component
Analysis, as in the recent paper by Lall, Marsden and Glavaški [21]. In their paper
an attempt is made to justify a methodology of controller design based on empir-
ically obtained POD reduced-order models. In particular, the important point is
made that the requirements for POD models for control design are different than
those for open-loop uncontrolled systems. The main theoretical result is that for
the class of finite-dimensional linear systems their POD reduced-order modeling
scheme is theoretically equivalent to obtaining balanced truncations (see Moore
[26]). Although my knowledge of the POD literature is limited, my knowledge of
the literature on balanced realizations and their truncations is considerable. So
I focus my attention on the specific POD scheme treated in [21] applied not to
nonlinear finite-dimensional systems as they do, but to linear infinite-dimensional
systems (after all, partial differential equation models are infinite-dimensional). I
analyze the use of POD reduced-order models for control design in light of the exist-
ing theory of balanced realizations and truncations for infinite-dimensional systems
from Curtain and Glover [5] and Glover, Curtain and Partington [10]. The first
thing which is clear is that the theory of balanced realizations exists only for lin-
ear stable systems and consequently this applies to the POD method as well – a
significant limitation. Also note that balanced realizations are determined only by
the transfer function and so are independent of the initial state. So to calculate
POD models from data one should set the initial conditions to zero. Now suppose
that \( \Sigma(A, B, C) \) is exponentially stable, and that \( \Sigma(A^n, B^n, C^n) \) corresponds to a
sequence of balanced truncations. As already explained, for convergence in the gap
topology of stable systems we require that

\[ \|G - G^n\|_\infty \to 0 \quad \text{as} \quad n \to \infty. \]
Now in [10], various bounds on the $H_\infty$-errors of the balanced truncated approximations are obtained. These depend on the Hankel singular values $\sigma_i$, $i \geq 1$, which are invariants of $\Sigma(A, B, C)$ (see Section 4.2). Since $\Sigma(A, B, C)$ is exponentially stable, it is known that

$$\|G - G^n\|_\infty \leq 2 \sum_{i=n+1}^{\infty} \sigma_i < \infty. \quad (4.2)$$

Even for finite-dimensional systems there are examples for which the tail does not drop off rapidly, which suggests that for a POD reduced-order model approach to controller design to have a good chance of success the original system should satisfy $\sum_{i=1}^{\infty} \sigma_i < \infty$; i.e., the transfer function should be nuclear – another limitation. Systems with an $A$ operator which is only strongly stable or with $B$ or $C$ unbounded may or may not be nuclear (see Sasane and Curtain [38]). It is known that many systems with infinitely many eigenvalues which asymptote to the imaginary axis at infinity will not be nuclear. This is often the case for many PDE models of undamped flexible systems (see Oostveen [34]). Still, the above discussion has illuminated one positive result: for an exponentially stable state linear system with finite-rank and bounded input and output operators, the balanced truncations converge in the gap topology and so do the POD approximations. Although the above discussion was limited to one POD scheme, in view of the underlying similarity of the approaches, it seems likely that many of the above comments may apply to other POD schemes. The implications for controller design are taken up in Section 4.2.

Although the two above control design procedures have been successfully applied in many simulations of DPS, from a control theoretic viewpoint, they both have shortcomings. The connection of POD approximations to balanced truncations suggests that they are not suitable for unstable systems. In contrast, the Trotter-Kato approach is applicable to unstable systems, and there is a theory for testing whether properties (P1), (P3) hold. The weak point is, however, the robustness property (P2); it is known that, even in finite dimensions, the LQG design gives no guarantee of robustness (see Doyle [8]). Since no mathematical model can exactly match a physical model, if one is really interested in controlling the physical model, and not some sophisticated simulation of it, it seems to this author that the robustness issue is of paramount importance. Furthermore, the finite-dimensional theory demonstrates that the approximation procedure should be done in closed-loop; i.e., the type of approximations should match the robust control design. In my opinion, there is need for more research into designing robustly stabilizing finite-dimensional controllers for DPS incorporating the types of modeling errors and the choice of the approximating systems explicitly into the design procedure.

In this chapter we elaborate further on some of the above issues for the special class of exponentially stabilizable and detectable state linear systems $\Sigma(A, B, C)$ with bounded, finite-rank input and output operators. In Section 4.2 the theory of balanced truncations is reviewed and implications for control design are discussed. In Section 4.3, a sequence of approximations is proposed which is suitable for unstable systems. It is called the $LQG$-balanced truncations and it was introduced in the finite-dimensional literature by Jonckheere and Silverman [17] and other in-
interpretations followed in Meyer [25] and Ober and McFarlane [33]. We show that
LQG-balanced realizations and truncations exist for our special class of systems. In
Section 4.4 the numerical computation of balanced and LQG-balanced truncations
is treated. In Section 4.5, a new algorithm is proposed for designing a low-order
finite-dimensional controller which has the properties (P1)–(P3). It is shown that it
is always possible to find a robustly stabilizing, low-order controller for the original
system and that this controller satisfies certain performance bounds.

Finally, Section 4.6 contains my conclusions and suggestions for future re-
search.

4.2 Approximating via Balanced Truncations

First we remark that only stable transfer functions \( G \in H_\infty \) can possess balanced
realizations. We review the theory of balanced realizations from Curtain and Glover
[5] applied to the special class of exponentially stable state linear systems \( \Sigma(A, B, C) \)
on the Hilbert space \( Z \), where \( A \) generates the exponentially stable semigroup \( \tilde{T}(t) \)
on \( Z \), and the operators \( B, C \) are finite-rank and bounded; \( B \in \mathcal{L}(C^m, Z), C \in \mathcal{L}(Z, C^k) \). \( \Sigma(A, B, C) \) has the transfer function \( G(s) = C(sI - A)^{-1}B \), but there
are infinitely many other triples of operators \( A, B, C \) which define the same transfer
function, i.e., infinitely many realizations. Many of these will have very unbounded
\( B \) and \( C \) operators, but for our purposes, we only need to consider realizations
which define a Pritchard-Salamon (PS) system with finite-dimensional inputs and
outputs.

Definition 4.2. \( \text{PSE}(\tilde{\Lambda}, \tilde{\Phi}, \tilde{\Gamma}) \) is a Pritchard-Salamon (PS) system with respect to
the Hilbert spaces \( W, V \) if the following hold:

(i) \( W \hookrightarrow V \).

(ii) \( \tilde{\Lambda} \) is the infinitesimal generator of a strongly continuous semigroup \( \tilde{T}(t) \) on \( V \)
which restricts to a strongly continuous semigroup on \( W \).

(iii) \( \tilde{\Phi} \in \mathcal{L}(C^m, V) \) and there exist \( t_1, \alpha > 0 \) such that
\[
\int_0^{t_1} \tilde{T}(t_1 - \tau)\tilde{\Phi}u(\tau) \, dt \in W \quad \text{and}
\]
\[
\left\| \int_0^{t_1} \tilde{T}(t_1 - \tau)\tilde{B}u(\tau) \, dt \right\|_W \leq \alpha \| u \|_{L^2(0,t_1; C^m)} \quad \text{for all } u \in L^2(0,t_1; C^m).
\]

(iv) \( \tilde{\Gamma} \in \mathcal{L}(W, C^k) \) and there exist \( t_2, \beta > 0 \) such that
\[
\left\| \tilde{\Gamma}\tilde{T}(\cdot)z \right\|_{L^2(0,t_2; C^k)} \leq \beta \| z \|_V \quad \text{for all } z \in W.
\]

All of the nice system theoretic concepts and properties of state linear systems
given in Curtain and Zwart [7] extend in a natural way to the PS-class (see van
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Keulen [41] and Curtain et al. [6]). In particular, the transfer function is given by \( G(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} \). We denote the PS-system by \( \text{PS}\Sigma(\hat{A}, \hat{B}, \hat{C}) \) to distinguish it from a state linear system with bounded input and output operators.

By a PS realization of a transfer function \( G \) we mean a PS-system \( \text{PS}\Sigma(\tilde{A}, \tilde{B}, \tilde{C}) \) which has the transfer function \( G \). For stable systems we shall be concerned with the balanced realization which was constructed by Curtain and Glover [5] who further studied it in [10] for a class of stable transfer functions with finite-dimensional inputs and outputs. Later, Ober and Montgomery-Smith [32] showed that most stable systems possess (par)balanced realizations.

**Definition 4.3.** Let \( G \in H_\infty(\mathbb{C}^{k \times m}) \). A PS-system \( \text{PS}\Sigma(\tilde{A}, \tilde{B}, \tilde{C}) \) is called a balanced realization of \( G \) if its transfer function is \( G \) and the controllability and observability gramians are both equal to the same diagonal operator.

The controllability gramian \( L_B \) and the observability gramian \( L_C \) of an exponentially stable state linear system \( \Sigma(A, B, C) \) are defined in Definition 4.1.20 of [7], and in Lemma 4.1.24 of [7] it is proven that they are the unique self-adjoint, non-negative definite solutions of their respective Lyapunov equations

\[
AL_B z + L_B A^* z = -BB^* z \quad \text{for} \quad z \in D(A^*)
\]

\[
A^* L_C z + L_C A z = -C^* C z \quad \text{for} \quad z \in D(A).
\]

The above-mentioned theory extends to a much larger class of systems, but here we restrict our remarks to PS-systems. In particular, the theory extends to allow for PS-systems which are not necessarily exponentially stable, but \( B \) and \( C \) are infinite-time admissible, i.e., conditions (iii) and (iv) of Definition 4.2 hold with \( t_1 = t_2 = \infty \). This is sufficient to guarantee the existence of bounded controllability and observability gramians which satisfy their respective Lyapunov equations. If, however, no explicit assumptions on the stability of the semigroup are made, the gramians are not necessarily the only solutions of the Lyapunov equations (see Grabowski [12] and Hansen and Weiss [13]). In the special PS-case, the gramians are well-defined bounded operators in \( \mathcal{L}(V) \cap \mathcal{L}(W) \) which satisfy their respective Lyapunov equations considering either \( W \) or \( V \) as the state space (see van Keulen [41], Chapter 2).

The construction of the balanced realizations in [5, 10] is based on the singular values and Schmidt vectors of the Hankel operator of the system. Suppose that the transfer function \( G \) is the Laplace transform of \( h \in L_1(0, \infty; \mathbb{C}^{m \times k}) \). The Hankel operator with symbol \( G \) is the bounded operator \( \Gamma : L_2(0, \infty; \mathbb{C}^m) \rightarrow L_2(0, \infty; \mathbb{C}^k) \) defined by

\[
(\Gamma u)(t) = \int_0^\infty h(t + s)u(s) \, ds, \quad h(t) = CT(t)B
\]

for all \( u \in L_2(0, \infty; \mathbb{C}^m) \). \( \Gamma \) is compact and has countable many singular values \( \{\sigma_i; i = 1, \ldots, \infty\} \) which are called the Hankel singular values of \( G \) or of the system with this transfer function (see [7], Chapter 8 for more details).
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The balanced realization has the controllability and observability gramians both equal to diag \((\sigma_1, \sigma_2, \ldots, \sigma_r, \ldots)\). In Curtain and Sasane [39] it is shown that an exponentially stable state linear system with finite-rank input and output operators has a nuclear Hankel operator; i.e., its singular values satisfy

\[
\sum_{i=1}^{\infty} \sigma_i < \infty. \tag{4.6}
\]

In the following lemma, we collect the results on the existence (via construction) of a balanced realization from [5, 10] applied to our special class.

**Lemma 4.4.** Let \(\Sigma(A,B,C)\) be an exponentially stabilizable and detectable state linear system with bounded, finite-rank inputs and outputs. The transfer function \(G(s) = C(sI - A)^{-1}B\) has a balanced realization \(\text{PS} \Sigma(A^{\text{bal}}, B^{\text{bal}}, C^{\text{bal}})\) on the state space \(l_2\) which is a PS-system with respect to the spaces \(V, W\), where

\[
W = \Sigma^{-\frac{1}{2}}l_2 \leftrightarrow l_2 \leftrightarrow \Sigma^{\frac{1}{2}}l_2 = V.
\]

Its controllability and observability gramians both equal \(\Sigma = \text{diag}(\sigma_i)\) and they satisfy their respective Lyapunov equations.

**Proof.** Most of the results are proven in [5], but others are more easily deduced from results on the output normal realization \(\text{PS} \Sigma(A^{\text{out}}, B^{\text{out}}, C^{\text{out}})\) which is constructed in [10]. The output normal realization has diagonal gramians and the observability gramian is the identity. It is closely related to the balanced one via

\[
\text{PS} \Sigma(\Sigma^{-\frac{1}{2}}A^{\text{bal}}\Sigma^{\frac{1}{2}}, \Sigma^{-\frac{1}{2}}B^{\text{bal}}, C^{\text{bal}}\Sigma^{\frac{1}{2}}).
\]

Notice that the extension of the balanced semigroup on \(V\) corresponds to the output normal semigroup on \(l_2\), while the restriction of the semigroup to \(W\) corresponds to the input normal semigroup on \(l_2\). (The input normal realization is the dual of the output normal one; it has the controllability gramian \(I\) and the realization \(\text{PS} \Sigma(\Sigma^{-\frac{1}{2}}A^{\text{out}}\Sigma^{\frac{1}{2}}, \Sigma^{-\frac{1}{2}}B^{\text{out}}, C^{\text{out}}\Sigma^{\frac{1}{2}})\).)

We note that balanced realizations are unique up to a state space transformation by a unitary operator [32]. When we write *the* balanced realization we refer to the particular one constructed in [5].

A rather strange feature is that even though the original realization has bounded input and output operators, in the balanced realization they will in general be unbounded. In the output normal realization the input is bounded, but the output is usually unbounded. Moreover, the semigroups are usually not exponentially stable: \(A^{\text{bal}}\) and \(A^{\text{out}}\) both generate contraction semigroups, but only \(A^{\text{out}}\) is guaranteed to generate a strongly stable semigroup. Nonetheless, both realizations have well-defined controllability and observability gramians which satisfy their Lyapunov equations.

**Balanced truncations** are constructed from the balanced realizations in the following manner. Choose an integer \(r\) such that \(\sigma_r > \sigma_{r+1}\) and partition \(\text{PS} \Sigma(A,B,C)\).
compatibly with the partition $L_B = L_C = \text{diag}(\Sigma_r,*)$, where $\Sigma_r = \text{diag}(\sigma_1, \ldots, \sigma_r)$:

$$
\hat{A} = \begin{bmatrix} \hat{A}_{11}(r) & \ast \\ \ast & \ast \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_1(r) \\ \ast \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \hat{C}_1(r) & \ast \end{bmatrix}.
$$

The $r$th order balanced truncation of $\Sigma(A, B, C)$ (or of $G$) is the finite-dimensional system $\Sigma(\hat{A}_{11}(r), \hat{B}_1(r), \hat{C}_1(r))$. Note that this system is stable and balanced and its Hankel singular values are $\sigma_1, \sigma_2, \ldots, \sigma_r$. Denote its transfer function by $G_r$. There holds

$$
\|G - G^r\|_\infty \leq 2 \sum_{i=r+1}^{\infty} \sigma_i,
$$

and so the transfer functions of the balanced truncations converge in the $H_\infty$-norm and so in the gap topology as $n \rightarrow \infty$. In fact, the convergence holds in the stronger nuclear norm (see [10]). Convergence in the $H_\infty$-norm is a very strong type of convergence which is only guaranteed to hold for nuclear systems, a restricted class of infinite-dimensional systems. In particular, systems with $C = I$ (the whole state is measured or estimated) are unlikely to be nuclear. This does not mean that the balanced truncations will not converge in some weaker norm; simulations of the open-loop balanced truncations can still produce good approximations to the original system.

Convergence in the $H_\infty$-norm becomes important when one uses the balanced truncations to design controllers to achieve some performance objective, such as tracking, reduction of sensitivity to disturbances or increasing the stability margin. There are many finite-dimensional designs which produce a controller $K_r$ which achieves the chosen performance objective for $G_r$ and produces a stable closed-loop system. However, to ensure that $K_r$ applied to $G$ will have the same effect requires a careful analysis. For example, in Morris [28] it is shown that this approach succeeds for one $H_\infty$-design (weighted mixed sensitivity), but fails for another. Two other control designs which will be successful are the robust stabilization under additive uncertainty in Curtain and Zwart [7], Section 9.3 (for increasing the stability margin) and robust stabilization with respect to normalized left-coprime factor perturbations in Section 9.4.

Note that in the above discussion we have tacitly assumed that the truncated balanced realizations are available. Of course, one needs to compute these numerically. If one has available an explicit expression for the transfer function, then a first step would be to approximate it closely in the $H_\infty$-norm by a high order rational transfer function and then use MATLAB® to compute its balanced truncations. These will be good low-order approximations to the original transfer function in the $H_\infty$-norm. Most often the transfer function is not readily available and so POD methods are used to estimate the system. In Corollary 4.11 of Section 4.4 we give an alternative approximation approach based on a state-space description of the system.
4.3 LQG-Balancing

As we have already emphasized, balanced realizations and truncations only make sense for stable systems. In this section, we investigate the existence of an LQG-balanced realization of the exponentially stabilizable and detectable state linear system \( \Sigma(A, B, C) \) on the Hilbert space \( Z \), where \( A \) is the infinitesimal generator of the strongly continuous semigroup \( T(t) \) on \( Z \), and the operators \( B \) and \( C \) are finite-rank and bounded; \( B \in \mathcal{L}(\mathbb{C}^m, Z) \), \( C \in \mathcal{L}(Z, \mathbb{C}^k) \). To do this we make use of the unique self-adjoint, nonnegative definite, stabilizing solutions \( Q \) and \( P \) of the control and filter Riccati equations, respectively,

\[
A^*Qz + QAz - QBB^*Qz + C^*Cz = 0, \quad z \in \mathcal{D}(A)
\]

\[
APz + PA^*z - PC^*CPz + B^*Bz = 0, \quad z \in \mathcal{D}(A^*).
\]

“Stabilizing” refers to the fact that \( A_Q = A - BB^*Q \) and \( A_P = A - PC^*C \) generate the exponentially stable semigroups \( T_Q(t) \) and \( T_P(t) \), respectively, on \( Z \) (see Curtain and Zwart [7], Chapter 6). The linear quadratic Gaussian (LQG) control design is based on \( P \) and \( Q \), although one usually includes weighting matrices in the quadratic terms, e.g., \( QBR^{-1}B^*Q \). It is easy to allow for this modification by redefining \( B \) as \( BR^{-\frac{1}{2}} \).

**Definition 4.5.** The PS-system \( \text{PS} \Sigma(\hat{A}, \hat{B}, \hat{C}) \) is called an LQG-balanced realization (of its own transfer function) if there exist bounded, self-adjoint, nonnegative solutions \( \hat{Q}, \hat{P} \) to its control and filter Riccati equations such that \( \hat{Q} = \hat{P} = \hat{\Lambda} \) is a diagonal operator.

In finite dimensions it is known that every transfer function possesses an LQG-balanced realization and the eigenvalues of \( \hat{Q} \hat{P} \), \( \{\hat{\mu}_i^2\} \) are invariants of the transfer function and \( \hat{\Lambda} = \text{diag}(\hat{\mu}_i) \) (see Jonckheere and Silverman [17]). It is our aim to investigate the viability of this concept in infinite dimensions (we are unaware of any theory for the infinite-dimensional case).

The key to our analysis is to exploit the one-one relationship between the exponentially stabilizable and detectable state linear system \( \Sigma(A, B, C) \) and the exponentially stable state linear system \( \Sigma(A_P, [B, -PC^*], C, [0, I]) \), as is done in Meyer [25].

\( \Sigma(A_P, [B, -PC^*], C, [0, I]) \) is a state-space realization of the normalized left-coprime factor system of \( G \), i.e., of the transfer function \( [\hat{N}(s), \hat{M}(s)] \) given by

\[
\hat{N}(s) = C(sI - A_P)^{-1}B, \quad \hat{M}(s) = I - C(sI - A_P)^{-1}PC^*,
\]

where \( G(s) = \hat{M}(s)^{-1}\hat{N}(s) \) and the coprime property is

\[
\hat{M}(s)X(s) - \hat{N}(s)Y(s) = I \quad \text{for } \text{Re}(s) \geq 0,
\]

with \( X(s) = I - C(sI - A_Q)^{-1}PC^* \), \( Y(s) = B^*Q(sI - A_Q)^{-1}PC^* \).

The normalization property is

\[
\hat{M}(i\omega)^*\hat{M}(i\omega) + \hat{N}(i\omega)\hat{N}(i\omega)^* = I, \quad \omega \in \mathbb{R}.
\]
For background theory of coprime factorizations for state linear systems (bounded $B,C$) consult Curtain and Zwart [7] – in particular, Section 7.3, Example 7.29 and Lemma 9.4.10; for the PS-case see Curtain [4]. Since we shall repeatedly appeal to known properties of the normalized left-coprime factor system and its Hankel singular values, we collect them here.

**Lemma 4.6.** If $\Sigma(A,B,C)$ is an exponentially stabilizable and detectable state linear system with finite-rank input and output operators, its normalized left-coprime factor system $\Sigma(A_P, [B,-PC^*], C, [0,I])$ with transfer function $[\tilde{N}, \tilde{M}]$ is an exponentially stable state linear system with a nuclear Hankel operator. Its singular values $\{\sigma_i; i = 1, \ldots, \infty\}$, ordered according to decreasing magnitude, satisfy

$$\sigma_1 < 1, \sum_{i=1}^{\infty} \sigma_i < \infty.$$  \hspace{1cm} (4.13)

The singular values are system invariants; i.e., they comprise a property of the transfer function only and are independent of the realization.

Furthermore, the controllability and observability gramians $L_{B_{nlc}}$ and $L_{C_{nlc}}$, respectively, of $\Sigma(A_P, [B,-PC^*], C, [0,I])$ are nuclear and satisfy

$$L_{B_{nlc}} = P, \quad L_{C_{nlc}} = Q(I + PQ)^{-1}, \quad Q = (I - L_{C_{nlc}}L_{B_{nlc}})^{-1}L_{C_{nlc}},$$  \hspace{1cm} (4.14)

where $Q,P$ are the stabilizing solutions of the Riccati equations (4.8), (4.9), respectively. The nonzero eigenvalues of $L_{B_{nlc}}^{-1}L_{C_{nlc}}$, $(L_{C_{nlc}}L_{B_{nlc}})^{-1}$ are system invariants.

**Proof.** Hankel operators have been defined in Section 4.2. The nuclearity of $L_{B_{nlc}}$, $L_{C_{nlc}}$ and the Hankel operator $\Gamma$ is shown in Curtain and Sasane [39]. This implies the second part of (4.13). The first part follows from Lemma 9.4.7 in Curtain and Zwart [7], since the coprime factors are normalized. The connection between the controllability and observability gramians of $\Sigma(A_P, [B,-PC^*], C, [0,I])$ in (4.14) and $P,Q$, was proven in Lemma 9.4.10 of [7]. The relationship between the eigenvalues of $L_{C_{nlc}}^{-1}L_{B_{nlc}}$ and those of the Hankel operator is shown in Lemma 8.2.9 of [7].

Lemma 4.6 allows us to deduce some interesting properties of the solutions $Q,P$ of (4.8), (4.9).

**Corollary 4.7.** Let $\Sigma(A,B,C)$ be an exponentially stabilizable and detectable state linear system on the Hilbert space $Z$. There exist unique solutions $Q,P$ to (4.8), (4.9) in the class of self-adjoint operators on $Z$ such that $A_Q$ and $A_P$ generate exponentially stable semigroups $T_Q(t), T_P(t)$, respectively, on $Z$. $P,Q$ and $PQ$ are nuclear operators. $-1 \in \rho(PQ) = \rho(QP)$ and the eigenvalues of $PQ$ (QP) are positive and are system invariants.

**Proof.** The first statements are well known (see Curtain and Zwart [7], Chapter 6). That $-1 \in \rho(PQ) = \rho(QP)$ was noted in [7], Corollary 9.4.11. The nuclearity
of \(L_{B_{nlc}}, L_{C_{nlc}}\) was proven in Lemma 4.6 and the nuclearity of \(P, Q\) follows from (4.14). The relationship (4.14) and the one-one relationship between \(\Sigma(A, B, C)\) and \(\Sigma(A_P, [B, -PC^*], C, [0, I])\) show that the eigenvalues of \(QP\) are positive and depend only on the transfer function.□

Following the finite-dimensional terminology we call the square roots of the eigenvalues of \(PQ\) the \textit{LQG-characteristic values} of \(G\) or of \(\Sigma(A, B, C)\). From (4.14) we deduce the following simple relationship between the Hankel singular values of \([\tilde{N}, \tilde{M}]\), \(\{\sigma_i, i = 1, \ldots, \infty\}\), and the LQG-characteristic values of \(G\), \(\{\mu_i, i = 1, \ldots, \infty\}\), where they are ordered according to decreasing magnitude:

\[
\begin{align*}
\mu_i^2 &= \frac{\sigma_i^2}{(1 - \sigma_i^2)} , \quad \sigma_i^2 = \frac{\mu_i^2}{1 + \mu_i^2} . \\
& (4.15)
\end{align*}
\]

The diagram depicted in Figure 4.1 may help us to understand the one-one relationship between the exponentially stabilizable and detectable state linear system \(\Sigma(A, B, C)\) and its normalized left-coprime factor system \(\Sigma(A_P, [B, -PC^*], C, [0, I])\).

Notice that there are two ways of connecting the two systems: one with state-space descriptions using the one-one relationship between the solutions of the LQG-Riccati equations of \(\Sigma(A, B, C)\) and the solutions of the Lyapunov equations of \(\Sigma(A_P, [B, -PC^*], C, [0, I])\). The other is via the transfer function relationship \(G = \tilde{M}^{-1}\tilde{N}\), although we note that normalized left-coprime factorizations are only unique up to left multiplication by a constant unitary matrix. There is a complete generalization of these one-one relationships to the exponentially stabilizable and detectable PS-class; i.e., the unique solutions to the LQG-Riccati equations (4.8), (4.9) are related to the observability and controllability gramians of the normalized left-coprime factor system via (4.14); see Curtain [4]. In fact, not only the stabilizing solutions but any pair of solutions \(Q, P\) to (4.8), (4.9) correspond to a pair of solutions to the Lyapunov equations of the normalized left-coprime factor system in a one-one way.

In Meyer [25] the next step was to find a balanced realization of the stable system \([\tilde{N}, \tilde{M}]\) and, using (4.14), prove that \(PQ\) can be diagonalized. This approach
Theorem 4.8. Let $\Sigma(A, B, C)$ be exponentially stabilizable and detectable with bounded, finite-rank inputs and outputs. The transfer function possesses an LQG-balanced realization on the state space $l_2$ which is a PS-system with respect to the spaces $W, V$, where $W = \Sigma^{-\frac{1}{2}}l_2 \hookrightarrow l_2 \hookrightarrow \Sigma^{\frac{1}{2}}l_2 = V$. The Riccati equations (4.8), (4.9) both have the solution

$$
\Lambda = \text{diag} \frac{\sigma_i}{\sqrt{1 - \sigma_i^2}} = \text{diag}(\mu_i).
$$

Proof. We apply Lemma 4.4 to obtain a balanced realization of $\Sigma(A, [B, -PC^*], C, [0, I])$ on $l_2$. This is a PS-system with controllability and observability gramians both equal to $\Sigma$ and its transfer function is $[\tilde{N}, \tilde{M}]$. We now apply a new coordinate transformation

$$
S = (I - \Sigma^2)^{-\frac{1}{4}} : l_2 \rightarrow l_2
$$

to obtain yet another realization of $[\tilde{N}, \tilde{M}]$ on $l_2$, the system $\text{PS}\Sigma(A^l, B^l, C^l, [0, I])$, where

$$
T^l(t) = \text{ST}_P(t)S^{-1}, \quad A^lS = SA_P, \quad B^l = SB_P, \quad C^l = C_P S^{-1}.
$$

$S$ is boundedly invertible. Thus it is easy to see that $\text{PS}\Sigma(A^l, B^l, C^l, [0, I])$ is also a PS-system on $l_2$, its controllability gramian is $\Sigma S^2$ and its observability gramian is $\Sigma S^{-2}$. Considering Figure 4.1, we wish to identify $\text{PS}\Sigma(A^l, B^l, C^l, [0, I])$ as a normalized left-coprime factor system of a system on the left. Our candidate is the PS-system $\text{PS}\Sigma(A^{lqg}, B^{lqg}, C^{lqg})$ on $l_2$ given by

$$
A^{lqg} = A^l + \Lambda(C^l)^* C^l, \quad B^{lqg} = B^l, \quad C^{lqg} = C^l.
$$

To see that this is a well-defined PS-system, we recall from Curtain et al. [6], Theorem 4.1, that it is sufficient that $\Lambda(C^l)^* \in \mathcal{L}(\mathbb{C}^k, W)$. Now $\Lambda < 1/\sqrt{1 - \sigma_i^2 \Sigma}$ and in [5], it is shown that $\Sigma(C^l)^*$ is bounded from $\mathbb{C}^k$ to $\Sigma^{\frac{1}{2}}l_2 = W$. To connect these two systems we define $\tilde{P}, \tilde{Q} \in \mathcal{L}(l_2)$ via the relationship (4.14):

$$
\tilde{P} = \Sigma S^2 = \Lambda; \quad \tilde{Q} = \Sigma S^{-2}(I - \Sigma^2)^{-1} = \Lambda.
$$

The obvious next step is to verify that $\tilde{Q}$ and $\tilde{P}$ are solutions to the LQG-Riccati equations (4.8), (4.9), respectively, of $\text{PS}\Sigma(A^{lqg}, B^{lqg}, C^{lqg})$. For bounded input and output operators, this was done in Lemma 9.4.10 of [7] by direct verification. It is tempting to appeal to the generalization of the smooth PS-case in [4]. However, the smooth property is used in an essential way in the proof and it is unlikely that our PS-realization will be smooth. Fortunately, using a different approach, the result can be shown to hold for nonsmooth PS-systems (see [35]). Note that we do not require that the solutions be unique, since there is a one-one relationship between the pairs $\{L_B, L_C\}$ and $\{\tilde{P}, \tilde{Q}\}$. Next we verify that $\text{PS}\Sigma(A^{lqg}, B^{lqg}, C^{lqg})$ has the transfer function $\tilde{M}^{-1}\tilde{N} = G$ as in Exercise 7.29 of [7], noting that this also extends
to the PS-class (see [4]). So $PS\Sigma(A_{lqg}^{t}, B_{lqg}^{t}, C_{lqg}^{t})$ is an LQG-balanced realization of the transfer function $G$.

We remark that an alternative route would be to use normalized right-coprime factorizations.

We continue with the approach of Meyer [25] and introduce a sequence of truncations of the LQG-balanced realization which are themselves LQG-balanced. Henceforth we use the notation $PS\Sigma(A, B, C)$ for the LQG-balanced realization constructed in Theorem 4.8. We use the infinite matrix representation for operators on $l_{2}$ with respect to the usual orthonormal basis. Thus $PS\Sigma(A, B, C)$ has the observability gramian $\Sigma(I - \Sigma)^{-\frac{1}{2}}$ and the controllability gramian $\Sigma(I - \Sigma)^{\frac{1}{2}}$ and the corresponding Riccati equation solutions for $PS\Sigma(A, B, C)$ are given by $\hat{P} = \hat{Q} = \Lambda = \Sigma(I - \Sigma)^{-\frac{1}{2}}$. Choose a positive integer $r$ such that $\mu_{r} > \mu_{r+1}$ and partition $PS\Sigma(A, B, C)$ and $PS\Sigma(A_{r}, B_{r}, C_{r}, [0, I])$ compatibly with the following partition of $\hat{P} = \hat{Q} = \Lambda = \text{diag}(\Lambda_{r}, *)$, where $\hat{P}_{r} = \hat{Q}_{r} = \Lambda_{r} = \text{diag}(\mu_{1}, \ldots, \mu_{r})$.

$PS\Sigma(A, B, C)$ and $PS\Sigma(A_{r}, B_{r}, C_{r}, [0, I])$ become

\[
\hat{A} = \begin{bmatrix} \hat{A}_{11}(r) & * \\ * & * \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} \hat{B}_{1}(r) \\ * \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} \hat{C}_{1}(r) & * \end{bmatrix},
\]

\[
A_{r} = \begin{bmatrix} A_{11}(r) & * \\ * & * \end{bmatrix}, \quad B_{r} = \begin{bmatrix} B_{11}(r) & B_{12}(r) \\ * & * \end{bmatrix},
\]

\[
C_{r} = \begin{bmatrix} C_{1}(r) & * \end{bmatrix}.
\]

For simplicity of notation we henceforth assume that the $\mu_{i}$ are all distinct.

The $r$th order LQG-balanced truncation of $PS\Sigma(A, B, C)$ is defined to be the finite-dimensional system $\Sigma(\hat{A}_{11}(r), \hat{B}_{1}(r), \hat{C}_{1}(r))$ and the $r$th order truncation of $PS\Sigma(A_{r}, B_{r}, C_{r}, [0, I])$ is $\Sigma(A_{11}(r), [B_{11}(r), B_{12}(r)], C_{1}(r), [0, I_{r}])$.

The following result shows that $\Sigma(\hat{A}_{11}(r), \hat{B}_{1}(r), \hat{C}_{1}(r))$ is a good candidate as a reduced-order model for $PS\Sigma(A, B, C)$ and hence for $G$. The approximations converge in the gap topology.

**Theorem 4.9.** The transfer functions $G_{r}$ of the LQG-balanced truncations $\Sigma(\hat{A}_{11}(r), \hat{B}_{1}(r), \hat{C}_{1}(r))$ converge in the gap topology as $r \to \infty$ to $G$, the transfer function of $PS\Sigma(A, B, C)$ and of $\Sigma(A, B, C)$.

**Proof.** (a) First we show that $\Sigma(A_{11}(r), [B_{11}(r), B_{12}(r)], C_{1}(r), [0, I_{r}])$ is the normalized left-coprime factor system of $\Sigma(\hat{A}_{11}(r), \hat{B}_{1}(r), \hat{C}_{1}(r))$. The LQG-Riccati equations have solutions $\text{diag}(\mu_{1}, \ldots, \mu_{r})$ which correspond to the observability and controllability gramians of $\Sigma(A_{11}(r), [B_{11}(r), B_{12}(r)], C_{1}(r), [0, I_{r}])$ via (4.14). Moreover, from the block structure it is clear that

\[
A_{11}(r) = \hat{A}_{11}(r) - \Lambda_{r} \hat{C}_{1}(r)^{*} \hat{C}_{1}(r), \quad [B_{11}(r), B_{12}(r)] = [\hat{B}_{1}(r), -\Lambda_{r} \hat{C}_{1}(r)^{*}],
\]

\[
C_{1}(r) = \hat{C}_{1}(r).
\]
So recalling Figure 4.1 and arguing as in the proof of Theorem 4.8, it is easy to see that \( \Sigma(A_{11}(r), B_{11}(r), C_{1}(r), [0, I]) \) is the normalized left-coprime factor system of \( \Sigma(A_{11}(r), B_{1}(r), C_{1}(r)) \) on \( l_2 \) and \( G_r = \tilde{M}_r^{-1}\tilde{N}_r \) for each \( r \leq n \).

(b) Note that \([\tilde{N}_r, \tilde{M}_r]\) is also the transfer function of the \( r \)th order truncations of the balanced and output normal realizations of \([\tilde{N}, \tilde{M}]\). So appealing to the theory in [10], Theorem 5.1, we conclude that

\[
\|[\tilde{N}_r, \tilde{M}_r] - [\tilde{N}, \tilde{M}]\|_\infty \leq 2 \sum_{i=r+1}^{\infty} \sigma_i. \tag{4.16}
\]

By (4.13) \( \sum_{i=1}^{\infty} \sigma_i < \infty \), and so the normalized left-coprime factors \([\tilde{M}_r, \tilde{M}_r]\) of \( G_r \) converge to those of \( G \) in the \( H_\infty \)-norm, i.e., we have convergence in the gap topology.

### 4.4 Numerical Computation of LQG-Balanced Truncations

Here we address the question of computing the balanced and LQG-balanced truncations from a given state-space description \( \Sigma(A, B, C) \). Since the balanced case can be seen as a special case of the LQG-case, we consider the latter. In the finite-dimensional case studied in Mustafa and Glover [30], it is very easy to calculate \( P \) and \( Q \) and then diagonalize \( PQ \). In infinite dimensions, the best you can hope for is to obtain a good numerical approximation to the operator solutions to the LQG-Riccati equations. As pointed out in Morris [29], if we intend to use the truncations for control design, we shall require not only sufficient conditions on a sequence of systems \( \Sigma(A^n, B^n, C^n) \) approximating \( \Sigma(A, B, C) \) to ensure convergence of the solutions to the LQG-Riccati equations, but additional ones to ensure convergence of the transfer functions in the gap topology.

**Theorem 4.10.** Suppose that the exponentially stabilizable and detectable state linear system \( \Sigma(A, B, C) \) with finite-rank and bounded input and output operators has transfer function \( G \) and denote the solutions to the control and filter Riccati equations (4.8) and (4.9) by \( Q \) and \( P \), respectively. Let \( \Sigma(A^n, B^n, C^n) \) be a sequence of finite-dimensional linear systems which satisfies the following assumptions:

(A1) \( Z^n \) is a sequence of finite-dimensional subspaces of \( Z \) and \( \Pi^n \) is the orthogonal projection of \( Z \) into \( Z^n \) such that

\[ \Pi^n z \to z \text{ as } n \to \infty \text{ for all } z \in Z, \]

\[ B^n = \Pi^n B, \quad C^n = C|_{Z^n}; \]

(A2) \( A^n \in \mathcal{L}(Z^n) \) and for each \( z \in Z \) there holds

(i) \( e^{A^n t}\Pi^n z \to T(t)z \),

(ii) \( (e^{A^n t})^*\Pi^n z \to T(t)^*z \)

uniformly in \( t \) on bounded intervals as \( n \to \infty \).
(A3) \((A^n, B^n)\) is uniformly exponentially stabilizable; i.e., there exists a uniformly bounded sequence of operators \(F^n \in \mathcal{L}(\mathbb{R}^n, \mathbb{C}^m)\) such that

\[
\left\| e^{(A^n - B^n F^n) t} \Pi^n \right\|_{\mathbb{R}} \leq M_1 e^{-\alpha t} \|z\|_{\mathbb{R}}
\]

for some positive constants \(M_1 \geq 1\) and \(\alpha\);

(A4) \((A^n, C^n)\) is uniformly exponentially detectable; i.e., there exists a uniformly bounded sequence of operators \(L^n \in \mathcal{L}(\mathbb{C}^k, \mathbb{R}^n)\) such that

\[
\left\| e^{(A^n - C^n L^n) t} \Pi^n \right\|_{\mathbb{R}} \leq M_2 e^{-\beta t} \|z\|_{\mathbb{R}}
\]

for some positive constants \(M_2 \geq 1\) and \(\beta\).

Let \(Q^n, P^n\) be the unique stabilizing solutions to the LQG-Riccati equations (4.8) and (4.9) corresponding to \(\Sigma(A^n, B^n, C^n)\) with transfer function \(G^n\). Then \(Q^n\) and \(P^n\) converge in the nuclear norm to \(Q\), respectively, \(P\) as \(n \to \infty\). \(T^n_P(t) = e^{(A^n - P^n(C^n)^*) t}\) converges strongly to \(T_P(t)\) uniformly on compact intervals and there exist constants \(M_3 \geq 1\), \(\gamma > 0\) such that

\[
\|T^n_P(t)\| \leq M_3 e^{-\gamma t} \quad \text{for all } n \text{ and } t > 0. \tag{4.17}
\]

Similarly, \(T^n_P(t)^*\) converges strongly to \(T_P(t)^*\) as \(n \to \infty\). Let \([\hat{N}^n, \hat{M}^n]\) denote the transfer function of the normalized left-coprime factor system \(\Sigma(A^n, B^n, -P^n(C^n)^*)\), \(C^n\) of \(\Sigma(A^n, B^n, C^n)\). Then

\[
\|\hat{N}^n - \hat{M}^n\|_{\infty} \to 0 \quad \text{as } n \to \infty; \tag{4.18}
\]

i.e., \(G^n\) converges to \(G\) in the gap topology.

Let \(\Gamma, \Gamma^n\) denote the Hankel operators of \([\hat{N}, \hat{M}],[\hat{N}^n, \hat{M}^n]\), respectively, and their singular values, ordered according to decreasing magnitude, by \(\{\sigma_i; i = 1, \ldots, \infty\}\), \(\{\sigma^n_i; i = 1, \ldots, s(n)\}\), respectively. We have \(\sigma_1 < 1, \sigma^n_1 < 1\) for all \(n\) and the singular values and Schmidt vectors of \(\Gamma^n\) converge to those of \(\Gamma\) according to

\[
\sup_i |\sigma^n_i - \sigma_i| \to 0 \quad \text{as } n \to \infty \tag{4.19}
\]

\[
\sum_{i=1}^{s(n)} \sigma^n_i \to \sum_{i=1}^{\infty} \sigma_i \quad \text{as } n \to \infty. \tag{4.20}
\]

**Proof.** (a) The proof of the strong convergence of \(P^n, Q^n, T^n_P(t), T^n_P(t)^*\) and (4.17) can be found in Ito [14] and Kappel and Salamon [18]. Equation (4.17) and the fact that \(B, C\) have finite rank can be used to show uniform convergence of \(P^n\) and \(Q^n\) (see the remarks in [18] on p.1143). In part (d) we prove convergence in the nuclear norm.
(b) The convergence in the gap topology is given in Morris [29], but for completeness we give a simpler proof. We consider the impulse responses

\[ h^n(t) = C^n T^n_p(t) [B^n, -P^n(C^n)^*], \quad h(t) = CT_p(t) [B, -PC^*], \]

corresponding to \([\tilde{N}^n, \tilde{M}^n]\), \([\tilde{N}, \tilde{M}]\) minus the constant terms. Now since \(P^n, B^n, C^n, T^n_p(t)\) all converge strongly as \(n \to \infty\), they are uniformly bounded in norm in \(n\) and, moreover, \(h_n(t) \to h(t)\) pointwise as \(n \to \infty\). Using (4.17) we see that

\[ h^n(t) \leq \text{constant} \cdot e^{-\gamma t} \in L_1(0, \infty), \]

and from the Lebesgue dominated convergence theorem we have that \(h^n \to h\) in the \(L_1(0, \infty; \mathbb{C}^{m \times k})\) norm as \(n \to \infty\). This implies (4.18).

(c) Since \(\sigma_i\) and \(\sigma^n_i\) are singular values of a normalized left-coprime factor system, the largest is strictly less than 1 (Lemma 9.4.7 of [7]).

(d) To establish the convergence of the Schmidt vectors and of the singular values of \(\Gamma^n\) according to (4.19), (4.20), we show that \(\Gamma^n\) converges in the nuclear norm to \(\Gamma\).

Recall from Lemma 8.2.2 in [7] that \(\Gamma = \mathcal{C}_{nlc} B_{nlc}\), where \(\mathcal{C}_{nlc} \in \mathcal{L}(Z, L_2(0, \infty; \mathbb{C}^k))\), and \(B_{nlc} \in \mathcal{L}(L_2(0, \infty; \mathbb{C}^m), Z)\) are the observation and control maps, respectively, of \(\Sigma(A_P, [B, -PC^*], C, [0, I])\) defined by

\[ \mathcal{C}_{nlc} z = CT_p(t) z \quad \text{for} \quad z \in Z, \]

\[ B_{nlc} u = \int_0^\infty T_p(t) [B, -PC^*] u(t) \, dt \quad \text{for} \quad u \in L_2(0, \infty; \mathbb{C}^m). \]

Similarly, \(\Gamma^n = \mathcal{C}^n_{nlc} B^n_{nlc}\), where \(\mathcal{C}^n_{nlc}, B^n_{nlc}\) are the observability, respectively, controllability maps of \(\Sigma(A^n_P, [B^n, -P^n(C^n)^*], C^n)\). Recall from Partington [36], Corollary 1.4, that

\[ ||\Gamma^n||_N \leq ||\mathcal{C}^n_{nlc}||_{HS} ||B_{nlc}||_{HS}, \]

where \(N, HS\) denote the nuclear and Hilbert-Schmidt norms, respectively. So, if we show that

\[ ||\mathcal{C}^n_{nlc} - \mathcal{C}_{nlc}||_{HS} = ||(\mathcal{C}^n_{nlc})^* - (\mathcal{C}_{nlc})^*||_{HS} \to 0, \]

\[ ||B^n_{nlc} - B_{nlc}||_{HS} \to 0 \quad \text{as} \quad n \to \infty, \]

then

\[ ||\Gamma^n - \Gamma||_N \leq ||\mathcal{C}^n_{nlc}(B^n_{nlc} - B_{nlc})||_N + ||(\mathcal{C}^n_{nlc} - \mathcal{C}_{nlc})B_{nlc}||_N \]

\[ \leq ||\mathcal{C}^n_{nlc}||_{HS} ||B^n_{nlc} - B_{nlc}||_{HS} + ||\mathcal{C}^n_{nlc} - \mathcal{C}_{nlc}||_{HS} ||B_{nlc}||_{HS} \to 0 \quad \text{as} \quad n \to \infty. \]

From the duality between the observability and controllability maps, it suffices to prove that

\[ ||\mathcal{C}^n_{nlc} - \mathcal{C}_{nlc}||_{HS} \to 0 \quad \text{as} \quad n \to 0. \quad (4.21) \]
To do this we recall from Weidmann [43], Theorem 6.12 that
\[ \| C_{nlc} \|_H^2 \leq \int_0^\infty \| C T_P(t) \|^2 \, dt, \]
where the second norm is in \( L(Z, C^k) \). So we need to show that
\[ \| C_{nlc} - C_{nlc} \|_H^2 \leq \int_0^\infty \| C^n T_P^n(t) - C T_P(t) \|^2 \, dt \to 0 \quad \text{as} \quad n \to \infty. \]

Now \( C^n, T_P^n(t) \) and \( T_P^n(t)^* \) all converge strongly to \( C, T_P(t) \) and \( T_P(t)^* \), respectively, as \( n \to \infty \), and since \( C \) has finite-rank, we have
\[ \| C^n T_P^n(t) - C T_P(t) \| \to 0 \quad \text{as} \quad n \to \infty. \]

From (4.17) and (A1) we have
\[ \| C^n T_P^n(t) \| \leq \| C \| M_3 e^{-\gamma t} \in L_2(0, \infty). \]

So applying the Lebesgue dominated convergence theorem we obtain
\[ \int_0^\infty \| C^n T_P^n(t) - C T_P(t) \|^2 \, dt \to 0 \quad \text{as} \quad n \to \infty, \]
and \( \Gamma^n \) converges in the nuclear norm to \( \Gamma \).

We remark that the assumption that \( \Pi^n \) are orthogonal projections is not essential and it can be relaxed to allow for more general Galerkin approximations (see Ito and Kappel [16]). An earlier result in Gibson [9] on retarded systems showed that \( Q^n \) converges to \( Q \) in the nuclear norm for the special case that \( C^n = C \) for all \( n \). The convergence of the solutions to the Riccati equations in the nuclear norm in Theorem 4.10 appears to be a new result. We shall see in Section 4.30 that it is necessary for a successful controller design.

One can use the same approach to prove convergence for Lyapunov equations as promised at the end of Section 4.2.

**Corollary 4.11.** Suppose that the exponentially stable state linear system \( \Sigma(A, B, C) \) with finite-rank and bounded input and output operators has the transfer function \( G \) and Hankel operator \( \Gamma \). Denote the Hankel singular values (ordered according to decreasing magnitude) by \( \{ \sigma_i ; i = 1, \ldots, \infty \} \) and the solutions to the Lyapunov equations (4.3) and (4.4) by \( L_G, L_B \), respectively. Let \( \Sigma(A^n, B^n, C^n) \) be a sequence of finite-dimensional linear systems which satisfies the assumptions (A1), (A2), and \( A^n \) is uniformly exponentially stable; i.e., there exist positive constants \( \alpha, M \) such that
\[ (A5) \quad \left\| e^{A^n t} \Pi^n z \right\|_Z \leq M e^{-\alpha t} \| z \|_Z. \quad (4.22) \]
Let \( L^n_C, L^n_B \) denote the unique solutions to the Lyapunov equations (4.4), (4.3), respectively, corresponding to \( \Sigma(A^n, B^n, C^n) \) with transfer functions \( G^n \). Then \( L^n_C, L^n_B \) converge in the nuclear norm to \( L_C, L_B \), respectively, and \( G^n \) converges to \( G \) in the \( H_\infty \)-norm as \( n \to \infty \).

Let \( \Gamma^n \) denote the Hankel operators and \( \sigma_i^n \) the Hankel singular values of \( \Sigma(A^n, B^n, C^n) \) (ordered according to decreasing magnitude: \( \sigma_i^n \leq \sigma_{i+1}^n \)). Then \( \Gamma^n \) converges to \( \Gamma \) in the nuclear norm
\[
\sum_{i=1}^{s(n)} \sigma_i^n \to \sum_{i=1}^\infty \sigma_i \quad \text{as } n \to \infty.
\]

### 4.5 Robust Controller Design via LQG-Balanced Truncation

Let us first clarify that what we mean in this section by stability of a system is the concept of input-output stability defined in Definition 9.1.2 in [7]. At the same time, we recall that if \( \Sigma(A, B, C) \) is exponentially stabilizable and detectable and the input and output spaces are finite-dimensional and it is stabilized in the input-output sense by a controller with an exponentially stabilizable and detectable realization \( \Sigma(A^K, B^K, C^K, D^K) \), then the semigroup of the resulting closed-loop system is exponentially stable (Exercise 9.6.2 in [7]). Of course, it is always possible to find a stabilizable and detectable (actually controllable and observable) realization of a rational transfer function, and so we can achieve exponential stability of the closed-loop system by a suitable implementation of the controller.

Our aim is to design a finite-dimensional controller that stabilizes an exponentially stabilizable and detectable state linear system \( \Sigma(A, B, C) \) with bounded finite-rank input and output operators. As many people do, we shall start with a state-space description \( \Sigma(A, B, C) \) and approximate it by a sequence \( \Sigma(A^n, B^n, C^n) \) satisfying the conditions in Theorem 4.10. In Morris [28] it is shown that for sufficiently large \( n \) the popular LQG-controller with transfer function \( K^n \) designed to stabilize the reduced-order model \( \Sigma(A^n, B^n, C^n) \) will also exponentially stabilize \( \Sigma(A, B, C) \). Although this is a pleasing result, it is known that LQG-controllers do not have very good robustness properties, even in finite dimensions. So even if it stabilizes \( \Sigma(A, B, C) \), it may not stabilize the physical plant. We propose an alternative robust controller design based on reduced-order models of \( \Sigma(A^n, B^n, C^n) \) obtained using LQG-balanced truncations. Since the controller is designed to be robust, it will stabilize not only \( \Sigma(A, B, C) \) but other neighboring ones as well.

The first step will be to compute the numerical approximations \( P^n, Q^n \) to the LQG-Riccati equations corresponding to \( \Sigma(A^n, B^n, C^n) \) as outlined in Section 4.4 and then to compute the LQG-balanced truncations of this finite-dimensional system (a finite-dimensional computation in MATLAB). These two approximation steps are entirely different approximation procedures with different types of errors involved. Nonetheless, we shall show how, starting from our DPS \( \Sigma(A, B, C) \) we obtain a sequence of reduced-order models with transfer functions \( G^n \) which converge to the transfer function of \( G \) in the gap topology as \( q \to \infty \). We propose a
controller design based on a reduced-order model $G^q$ and prove that it is guaranteed to stabilize the original system. Moreover, we give a priori computable estimates for $q$ which determines the order of the controller.

4.5.1 Robust Controller Design via LQG-Balanced Truncation

**Step 1:** Using your favorite approximating sequence $\Sigma(A^n, B^n, C^n)$ to $\Sigma(A, B, C)$, which satisfies the assumptions in Theorem 4.10, find numerical approximations $P_n$ and $Q_n$ for sufficiently large $n$. Calculate and order the eigenvalues values of $P^nQ^n$, $\{(\mu_i^n)^2, i = 1, \ldots, s(n)\}$ in decreasing magnitude. Calculate the related Hankel singular values $(\sigma_i^n)^2 = \frac{(\mu_i^n)^2}{1 + (\mu_i^n)^2}$ and the sum $\sum_{i=1}^{s(n)} (\sigma_i^n)^2$. An indication of “large enough” is that this sum appears close to a limit.

**Step 2:** Obtain the LQG-balanced realization of $\Sigma(A^n, B^n, C^n)$ on $l_2$ as explained in Section 4.3.

**Step 3:** Choose $r$ so that $\sum_{i=r+1}^{s(n)} (\sigma_i^n)$ is small compared with $\sum_{i=1}^{s(n)} (\sigma_i^n)$. Form the $r$th order LQG-balanced truncation of $\Sigma(A^n, B^n, C^n)$ which we denote by $\Sigma(\tilde{A}^n(r), \tilde{B}^n(r), \tilde{C}^n(r))$ and its transfer function by $G^r_n$.

**Step 4:** Design a controller for $\Sigma(\tilde{A}^n(r), \tilde{B}^n(r), \tilde{C}^n(r))$ which is robustly stabilizing with respect to normalized coprime factor perturbations as outlined in Chapter 9.4 of [7]. The central controller (Theorem 9.4.16) has the transfer function

$$K^r_n(s) = \frac{1}{\varepsilon^2 - 1} \tilde{B}^n(r)^* \Lambda^n(r)(sI - \tilde{A}^n(r))^{-1}W^n(r)\tilde{C}^n(r)^*, \quad (4.23)$$

where

$$\Lambda^n(r) = \text{diag}\{\mu_i^n\}_{i=1}^r, \quad W^n(r) = \text{diag}\left\{\frac{(1 - \varepsilon^2)\mu_i^n}{1 - \varepsilon^2 - \varepsilon^2(\mu_i^n)^2}\right\}_{i=1}^r,$$

$$\tilde{A}^n(r) = \tilde{A}^n(r) - \tilde{B}^n(r)\tilde{B}^n(r)^*\Lambda^n(r) + \frac{1}{\varepsilon^2 - 1}W^n(r)\tilde{C}^n(r)^*\tilde{C}^n(r)$$

and $\varepsilon$ is chosen strictly less than the maximal robustness margin attainable for $G^r_n$ which is

$$\varepsilon^n_{\max} = \sqrt{1 - (\sigma_1^n)^2} = \frac{1}{\sqrt{1 + (\mu_1^n)^2}}. \quad (4.24)$$

Note that the maximal robustness margin is independent of $r$.

This controller is robust in the sense that it stabilizes $G^r_n$ and all perturbed systems $G_{\Delta}$ with a left-coprime factorization of the form $G_{\Delta} = (\tilde{M}^n(r) + \Delta_1)^{-1}(\tilde{N}^n(r) + \Delta_2)$ and such that

$$\|[\Delta_1, \Delta_2]\|_\infty < \varepsilon. \quad (4.25)$$
So it will stabilize $G$ provided that

$$\| [[\hat{N}, \hat{M}] - [\tilde{N}^n(r), \tilde{M}^n(r)]]_\infty < \varepsilon < \sqrt{1 - (\sigma_1^n)^2}. \quad (4.26)$$

**Step 5:** Tune $r$ in Step 4 to obtain a controller which achieves a satisfactory level of robustness and performance with respect to the original system $\Sigma(A, B, C)$. The idea is to choose $n$ large and $r$ as small as possible so as to obtain a low-order controller; the order of the controller is equal to the order of $G^n(r)$.

We now prove that we can always choose $n >> r$ so that this controller robustly stabilizes our original system.

**Theorem 4.12.** Under the assumptions of Theorem 4.10, given a positive $\varepsilon < \sqrt{1 - \sigma_1^2}$, we can always find two integers $n >> r$ such that the controller $K^n_r$ given by (4.23) stabilizes $G$ with a robustness margin with respect to left-coprime factor perturbations of $\varepsilon - \sqrt{1 - \sigma_1^2}$.

**Proof.** (a) Given $\delta > 0$, (4.18), (4.19) and (4.20) show that we can always find a sufficiently large $N = N(\delta)$ such that for all $n > N(\delta)$ there holds

$$\| [[\hat{N}, \hat{M}] - [\tilde{N}^n, \tilde{M}^n]]_\infty < \delta, \quad (4.27)$$

$$\left| \sqrt{1 - \sigma_1^2} - \sqrt{1 - (\sigma_1^n)^2} \right| < \delta, \quad (4.28)$$

$$\sum_{i=r+1}^{s(n)} \sigma_i^n < \delta + 2 \sum_{i=r+1}^\infty \sigma_i. \quad (4.29)$$

Now using the above inequalities, we estimate the “gap” between $G$ and $G^n(r)$:

$$\| [[\hat{N}, \hat{M}] - [\tilde{N}^n(r), \tilde{M}^n(r)]]_\infty \leq \| [[\hat{N}, \hat{M}] - [\tilde{N}^n, \tilde{M}^n]]_\infty + \| [[\tilde{N}^n, \tilde{M}^n] - [\tilde{N}^n(r), \tilde{M}^n(r)]]_\infty$$

$$\leq \delta + 2 \sum_{i=r+1}^{s(n)} \sigma_i^n \quad \text{using (4.27) and (4.16) applied to } G^n$$

$$< 2\delta + 2 \sum_{i=r+1}^\infty \sigma_i \quad \text{from (4.29).} \quad (4.30)$$

For a fixed $n > N(\delta)$ the maximum robustness margin is $\sqrt{1 - (\sigma_1^n)^2}$ and so we can design $K^n_r$ with a robustness margin of $\varepsilon = \sqrt{1 - \sigma_1^2} - \delta$ provided that

$$\varepsilon = \sqrt{1 - \sigma_1^2} - \delta < \sqrt{1 - (\sigma_1^n)^2}.$$

Equation (4.28) shows that this is satisfied and from (4.26) and (4.30), $K^n_r$ will stabilize $G$ if $2\delta + 2 \sum_{i=r+1}^\infty \sigma_i < \varepsilon$, i.e., if

$$2 \sum_{i=r+1}^\infty \sigma_i < \sqrt{1 - \sigma_1^2} - 3\delta. \quad (4.31)$$
We can always choose \( r \) to satisfy this (see (4.13)) and so ensure that \( \mathbf{K}_r^p \) stabilizes \( \mathbf{G} \).

(b) We now estimate the robustness margin of this controller with respect to \( \mathbf{G} \). Let \( \mathbf{G}_\Delta \) be a perturbation of \( \mathbf{G} \) with a left-coprime factorization \( \mathbf{G} = (\mathbf{M} + \Delta_1)^{-1}(\mathbf{N} + \Delta_2) \). We need to show that the gap with respect to \( \mathbf{G}_r^p \) satisfies (4.26). We estimate the gap

\[
\| \begin{bmatrix} \tilde{\mathbf{N}} + \Delta_1, \tilde{\mathbf{M}} + \Delta_2 \end{bmatrix} - [\mathbf{N}^p(r), \mathbf{M}^p(r)] \|_{\infty} \\
\leq \| [\tilde{\mathbf{N}}, \tilde{\mathbf{M}}] - [\mathbf{N}^p(r), \mathbf{M}^p(r)] \|_{\infty} + \| [\Delta_1, \Delta_2] \|_{\infty}
\]

\[
< 2\delta + 2 \sum_{i=r+1}^{\infty} \sigma_i + \| [\Delta_1, \Delta_2] \|_{\infty} \text{ by (4.30)}
\]

\[
< 2\delta + \sqrt{1 - \sigma_i^2} - 3\delta + \| [\Delta_1, \Delta_2] \|_{\infty} \text{ by (4.31)}
\]

\[
< \varepsilon
\]

provided that \( \| [\Delta_1, \Delta_2] \|_{\infty} < \varepsilon - \sqrt{1 - \sigma_i^2} + \delta \), and this establishes the robustness margin.

To discuss the performance we need to introduce the following index for the transfer function \( \mathbf{G} \) in closed-loop with a controller \( \mathbf{K} \):

\[
\mathcal{M}(\mathbf{G}, \mathbf{K}) = \left\| \begin{bmatrix} \mathbf{K} & \mathbf{I} \\ \mathbf{I} & \mathbf{G}_K \end{bmatrix} (\mathbf{I} - \mathbf{GK})^{-1} \mathbf{G} \|_{\infty}.
\]

(4.32)

The various components have the following interpretations:

- \( (I - \mathbf{GK})^{-1} \mathbf{G} \) corresponds to additive uncertainty in the controller,
- \( \mathbf{K}(I - \mathbf{GK})^{-1} \) corresponds to additive uncertainty on the plant,
- \( (I - \mathbf{GK})^{-1} \) is the sensitivity,
- \( \mathbf{K}(I - \mathbf{GK})^{-1} \mathbf{G} \) corresponds to input multiplicative uncertainty on the plant.

While we have no bounds on these components individually, they are each bounded by the bound on \( \mathcal{M}(\mathbf{G}, \mathbf{K}) \).

Applying the results from [7], Exercise 9.14 b, we deduce the following bounds on the performance of the controller \( \mathbf{K}_r^p \) on \( \mathbf{G}_r^p \) and \( \mathbf{G} \):

\[
\mathcal{M}(\mathbf{G}_r^p, \mathbf{K}_r^p) \leq \sqrt{1 + (\mu_r^p)^2} \leq \frac{1}{\varepsilon},
\]

\[
\mathcal{M}(\mathbf{G}, \mathbf{K}_r^p) \leq \frac{1}{\varepsilon - \sqrt{1 - \sigma_i^2}}.
\]

It is important to realize that the choice of the controller that is robustly stabilizing with respect to coprime factor perturbations was crucial. The above conclusions would not hold for other types of controllers which stabilize \( \mathbf{G}_r^p \). In particular, we can make no conclusions about the success of an LQG design based on \( \mathbf{G}_r^p \), as LQG designs have no guaranteed robustness margin.
4.6 Conclusions

In Section 4.2, I summarized the known theory on balanced realizations and truncations for the class of exponentially stable systems with bounded finite-rank input and output operators. This gives a theoretical basis for balanced truncations and, to a certain extent, POD approximations. However, the important point was made that nuclearity is an essential property if one intends to use balanced truncations for controller design. Moreover, for controller design, one should obtain balanced truncations (or POD bases) using the measured output and not the state. This has been observed in some numerical experiments (via private communication with H.V. Ly). Nuclearity is not sufficient to ensure that the finite-dimensional controller will achieve the desired performance on the original system; extra analysis is required.

Since balanced realizations only exist for stable systems, I introduced in Section 4.3 the concept of LQG-balancing and LQG-balanced truncations for the class of exponentially stabilizable and detectable state linear systems with bounded and finite-rank input and output operators. I synthesized known results on balanced realizations and on normalized left-coprime factorizations to obtain the main new result on the existence of LQG-balanced realizations (Theorem 4.8). An interesting feature was that the LQG-balanced realizations will not be in the original class but in the much larger Pritchard-Salamon class. The nice properties of this class played a key role in the proofs.

Finally, in Section 4.4 I addressed the question of the numerical computation of balanced and LQG-balanced truncations from a given DPS \( \Sigma(A, B, C) \). In Theorem 4.1, I extended known results on the convergence of numerical approximations of solutions to the LQG Riccati equations and on the convergence of the approximating transfer functions. Some proofs are new and the nuclear convergence of the approximating LQG solutions is a new result. In Section 4.5 I presented a low-order, robustly stabilizing controller design based on the LQG-balanced truncations. More important, in Theorem 4.12, I proved that this design always leads to a low-order controller which is guaranteed to stabilize the original DPS with a certain robustness margin. The surprising feature is that the previous theory followed fairly easily by weaving together existing results from the systems theory of Pritchard-Salamon systems.

Although the theory sounds promising, the real test is whether the LQG-balanced truncation design works well on real physical plants and on sophisticated simulations. In particular, comparisons with the standard LQG design should be made. Does this design lead to significantly lower-order controllers with a better robustness margin? There are many interesting possibilities for research in these directions.

Since the class considered here is restricted, it is interesting to ask whether the design can be shown to yield similar results for systems with unbounded inputs and outputs. It is striking how, in all the theorems, the nuclearity of the normalized coprime factor system played a key role. This suggests that the proposed controller design will only work with systems having this property. In this context, it is interesting to know that exponentially stable analytic systems are nuclear (see
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Curtain and Sasane [39]). Since such models occur frequently in the applications, an interesting open problem for future research is to investigate whether the theory extends to exponentially stabilizable and detectable regular analytic systems. It is also known that many delay systems are nuclear (see Glover, Lam and Partington [11] and Partington, Glover and Zwart [37]). So extensions in this direction may also be possible.

Since nuclearity appears to be so crucial, what can one do if the normalized coprime factor system is not nuclear? One could try precompensation, e.g., use a smoother control action \( \dot{u} = -u + v \), and then \( v(t) \) is the new control variable. This will have the effect of producing a normalized coprime factor system with a smoother Hankel operator.

Other closely related controller designs which have been developed for finite-dimensional systems are those based on \( H_\infty \)-balancing (see Mustafa and Glover [31, 30]). These theories can also be generalized to the class of exponentially stabilizable and detectable state linear systems with bounded and finite-rank input and output operators.

In some applications the performance requirements are frequency dependent. It should be possible to design controllers which take these into account using the loop-shaping design procedure in Chapter 6 of McFarlane and Glover [24].

Of course there are many other possibilities for future research motivated by the topics touched on in this chapter:

- LQG-balancing for well-posed linear systems,
- (LQG-) balancing for nonlinear infinite-dimensional systems,
- control design for specific performance objectives,
- robustness with respect to nonlinear perturbations,
- control design for nonlinear infinite-dimensional systems.

Some preliminary results on balancing of finite-dimensional nonlinear systems can be found in Scherpen [40] and Lall, Marsden and Glavaški [21]. Research into some of the above problems will lead to new theoretical results, but many will not be amenable to theory alone. The way to gain real insight into the control of physical DPS is to couple theory with well-designed numerical experiments on simulations and on real physical systems.

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