Two-frequency radiative transfer and asymptotic solution

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Two-frequency radiative transfer (2f-RT) theory is developed for classical waves in random media. Depending on the ratio of the wavelength to the scale of medium fluctuation, the 2f-RT equation is either a Boltzmann-like integral equation with a complex-valued kernel or a Fokker–Planck-like differential equation with complex-valued coefficients in the phase space. The 2f-RT equation is used to estimate three physical parameters: the spatial spread, the coherence length, and the coherence bandwidth (Thouless frequency). A closed-form solution is given for the boundary layer behavior of geometrical radiative transfer and shows highly nontrivial dependence of mutual coherence on the spatial displacement and frequency difference. It is shown that the paraxial form of 2f-RT arises naturally in anisotropic media that fluctuate slowly in the longitudinal direction.

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1. INTRODUCTION

Let $U_j, j=1,2$, be the random, scalar wave field of wavenumber $k_j, j=1,2$. The mutual coherence function and its cross-spectral version, known as the two-frequency mutual coherence function, defined by

$$\Gamma_{12}(\mathbf{x},\mathbf{y}) = \left\langle U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\right\rangle, \quad (1)$$

where $\langle\cdot\rangle$ stands for ensemble averaging, is the central quantity of optical coherence theory, from which the two-space, two-time correlation function can be obtained via the Fourier transform in frequency and therefore plays a fundamental role in analyzing propagation of random pulses [1–5]. The motivation for the scaling factors in Eq. (1) will be given below; cf. Eq. (3).

In this paper I set out to analyze the two-frequency mutual coherence as function of the spatial displacement and frequency difference for classical waves in multiply scattering media. This problem has been extensively studied in the physics literature (see [3,6–8] and references therein). Here I derive from the multiscale expansion (MSE) the two-frequency version of the radiative transfer equation, which is then used to estimate qualitatively the three physical parameters: the spatial and spatial frequency spreads and the coherence bandwidth, also known as the Thouless frequency in condensed matter physics. Moreover, I show that the boundary layer behavior of the two-frequency radiative transfer (2f-RT) equation is analytically solvable in geometrical optics. The closed-form solution (43) provides detailed information of the two-frequency mutual coherence beyond the current physical picture [7–9] [see the discussion about expression (44)].

To this end, I introduce the two-frequency Wigner distribution whose ensemble average is equivalent to the two-frequency mutual coherence and is a natural extension of the standard Wigner distribution widely used in optics [10,11]. A different version of two-frequency Wigner distribution for parabolic waves was introduced earlier [12], and with it the corresponding radiative transfer equation has been derived with full mathematical rigor [13,14]. In the case of anisotropic media fluctuating slowly in the longitudinal direction the 2f-RT equation developed here reduces to that of the paraxial waves in similar media, which lends support to the validity of MSE. The other regime where the two frequency radiative transfer equation has been obtained with full mathematical rigor is geometrical optics [15].

The main difference between the 2f-RT and the standard theory is that the former retains the wave nature of the process and is not just about energy transport. Hence the governing equation cannot be derived simply based on the energy conservation law.

2. TWO-FREQUENCY WIGNER DISTRIBUTION

Let $U_j, j=1,2$ be governed by the reduced wave equation

$$\Delta U_j(\mathbf{r}) + k_j^2(\nu_j + V_j(\mathbf{r}))U_j(\mathbf{r}) = f_j(\mathbf{r}), \qquad \mathbf{r}\in\mathbb{R}^3, \quad j=1,2, \quad (2)$$

where ν_j and V_j are, respectively, the mean and fluctuation of the refractive index associated with the wavenumber k_j and are in general complex valued. The source terms f_j may result from the initial data or the external sources. Here and below the vacuum phase speed is set to be unity. To solve Eq. (2) one also needs some boundary condition that is assumed to be vanishing at the far field.

We define the two-frequency Wigner distribution as

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}} U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right) U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y}. \tag{3}$$

In view of the definition, we see that both $\mathbf{x}$ and $\mathbf{p}$ are dimensionless. Here the choice of the scaling factors is crucial; namely, the spatial dependence of the wave field should be measured with respect to the probing wavelength. The benefit is that this choice leads to a closed-form equation for W. It is easy to see that the ensemble average $\langle W\rangle$ is just the (partial) Fourier transform of the mutual coherence function (1). The two-frequency Wigner distribution defined here has a different scaling factor from the one introduced for the parabolic waves [12].

The purpose of introducing the two-frequency Wigner distribution is to develop a two-frequency theory in analogy to the well-studied standard theory of radiative transfer. Although definition (3) requires the domain to be $\mathbb{R}^3$, the governing radiative transfer equation, once obtained, can be (inverse) Fourier transformed back to get the governing equation for the two-point function $U_1(\mathbf{r}_1)U_2^*(\mathbf{r}_2)$ or Γ_{12}, as their boundary conditions are usually easier to describe [cf. Eq. (42)].

The Wigner distribution has the following easy-to-check properties:

$$\int |W|^2(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p} = \left(\frac{\sqrt{k_1k_2}}{2\pi}\right)^3 \int |U_1|^2(\mathbf{x})\mathrm{d}\mathbf{x}\int |U_2|^2(\mathbf{x})\mathrm{d}\mathbf{x},$$

$$\int W(\mathbf{x},\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{y}}\mathrm{d}\mathbf{p} = U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right), \tag{4}$$

$$\int W(\mathbf{x},\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{q}}\mathrm{d}\mathbf{x} = (\pi^2k_1k_2)^3\hat{U}_1\left(\frac{k_1\mathbf{p}}{4}+\frac{k_1\mathbf{q}}{2}\right) \times\hat{U}_2^*\left(\frac{k_2\mathbf{p}}{4}-\frac{k_2\mathbf{q}}{2}\right), \tag{5}$$

where $\hat{\cdot}$ stands for the Fourier transform and hence contains all the information in the two-point two-frequency function. In particular,

$$\int \mathbf{p}W(\mathbf{x},\mathbf{p})\mathrm{d}\mathbf{p} = -i\left[\frac{1}{2k_1}\nabla U_1\left(\frac{\mathbf{x}}{k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}\right) - \frac{1}{2k_2}U_1\left(\frac{\mathbf{x}}{k_1}\right)\nabla U_2^*\left(\frac{\mathbf{x}}{k_2}\right)\right],$$

which, in the case of $k_1=k_2$, is proportional to the energy flux density.

Let us now derive the equation for the two-frequency Wigner distribution. After taking the derivative $\mathbf{p}\cdot\nabla$ and some calculation we have

$$\begin{aligned}\mathbf{p}\cdot\nabla W = {}& \frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right) \\ &\times V_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\mathrm{d}\mathbf{y} - \frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}-\frac{\mathbf{y}}{2k_1}\right) \\ &\times U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)V_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} \\ &+\frac{i}{2}(\nu_1-\nu_2^*)W+F,\end{aligned} \tag{6}$$

where the function F depends linearly on U_j and f_j:

$$\begin{aligned}F = {}& -\frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}f_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} \\ &+\frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)f_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y}.\end{aligned} \tag{7}$$

Substituting the spectral representation of V_j,

$$V_j(\mathbf{x}) = \int e^{i\mathbf{q}\cdot\mathbf{x}}\hat{V}_j(\mathrm{d}\mathbf{q}), \tag{8}$$

into the expression and using the definition of W, we then obtain the exact equation

$$\begin{aligned}&\mathbf{p}\cdot\nabla W - \frac{i}{2}(\nu_1-\nu_2^*)W - F \\ &= \frac{i}{2}\int \hat{V}_1(\mathrm{d}\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}/k_1}W\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1}\right) \\ &\quad -\frac{i}{2}\int \hat{V}_2^*(\mathrm{d}\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}/k_2}W\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_2}\right).\end{aligned} \tag{9}$$

Here and below $\hat{V}_2^*$ is the complex conjugate of the Fourier spectral measure $\hat{V}_2$. The full derivation of Eq. (9) is given in Appendix A.

Let us pause to compare the classical wave with the quantum wave function in the context of two-frequency formulation. The quantum wave functions Ψ_j at two different frequencies ω_1,ω_2 satisfy the stationary Schrödinger equation

$$\frac{\hbar^2}{2}\Delta\Psi_j + [\nu_j + V_j(\mathbf{x})]\Psi_j = -\omega_j\hbar\Psi_j + f_j, \qquad j=1,2, \tag{10}$$

where ν_j+V_j are hypothetical, energy-dependent real-valued potentials. Here the source terms f_j equal the initial data f of the time-dependent problem. Usually, in the quantum mechanical context, the potential function does not explicitly depend on the energy level (i.e., it is dispersionless).

The natural definition of the two-frequency Wigner distribution for the quantum wave functions is

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi_1\left(\mathbf{x}+\frac{\hbar\mathbf{y}}{2}\right)\Psi_2^*\left(\mathbf{x}-\frac{\hbar\mathbf{y}}{2}\right)\mathrm{d}\mathbf{y}, \tag{11}$$

which satisfies the Wigner–Moyal equation

$$\begin{aligned}\mathbf{p}\cdot\nabla W + i(\omega_2-\omega_1)W + \frac{i}{\hbar}(\nu_2^*-\nu_1)W \\ = \frac{i}{\hbar}\int \hat{V}_1(\mathrm{d}\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}W\left(\mathbf{x},\mathbf{p}-\frac{\hbar\mathbf{q}}{2}\right) \\ -\frac{i}{\hbar}\int \hat{V}_2^*(\mathrm{d}\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}}W\left(\mathbf{x},\mathbf{p}-\frac{\hbar\mathbf{q}}{2}\right)+F,\end{aligned} \tag{12}$$

where F has an expression similar to Eq. (7). The main difference between the quantum and classical waves in the Wigner formulation is that the derivation of a closed-form equation does not require rescaling each energy component with respect to its de Broglie wavelength. The implication in radiative transfer will be further discussed [see the remark following Eq. (27)].

3. TWO-FREQUENCY RADIATIVE TRANSFER SCALING

Let us assume that $V_j(\mathbf{x}), j=1,2$ are real-valued, centered, random stationary (i.e., statistically homogeneous) ergodic fields admitting spectral representation (8) with the spectral measures $\hat{V}_j(\mathrm{d}\mathbf{p}), j=1,2$ such that

$$\langle \hat{V}_j(\mathrm{d}\mathbf{p})\hat{V}_j^*(\mathrm{d}\mathbf{q})\rangle = \delta(\mathbf{p}-\mathbf{q})\Phi_j(\mathbf{p})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q},$$

where Φ_j are the (nonnegative-valued) power spectral densities of the random fields $V_j, j=1,2$. The above δ function is a consequence of the statistical homogeneity of the random field V_j. As V_j, $j=1,2$ are real valued, $\hat{V}_j^*(\mathrm{d}\mathbf{p}) = \hat{V}_j(-\mathrm{d}\mathbf{p})$, and hence the power spectral densities $\Phi_j(\mathbf{p})$ satisfy the symmetry property $\Phi_j(\mathbf{p})=\Phi_j(-\mathbf{p})$, $\forall \mathbf{p}$.

We will also need the cross-frequency correlation, and we postulate the existence of the cross-frequency spectrum Φ_{12} such that

$$\langle \hat{V}_1(\mathrm{d}\mathbf{p})\hat{V}_2^*(\mathrm{d}\mathbf{q})\rangle = \delta(\mathbf{p}-\mathbf{q})\Phi_{12}(\mathbf{p})\mathrm{d}\mathbf{p}\mathrm{d}\mathbf{q}.$$

Here Φ_{12} need not be real valued.

An important regime of multiple scattering of classical waves takes place when the scale of medium fluctuation is much smaller than the propagation distance but is comparable with or much larger than the wavelength [3,16]. The radiative transfer regime can be characterized by the scaling limit, which replaces ν_j+V_j in Eq. (2) with

$$\frac{1}{\gamma^2\epsilon^2}\left(\nu_j+\sqrt{\epsilon}V_j\left(\frac{\mathbf{r}}{\epsilon}\right)\right), \qquad \gamma>0, \qquad \epsilon\ll 1, \tag{13}$$

where ϵ is the ratio of the scale of medium fluctuation to the $O(1)$ propagation distance and γ the ratio of the wavelength to the scale of medium fluctuation. Hence $\gamma\epsilon$ is the ratio of the wavelength to the propagation distance, and the prefactor $(\gamma\epsilon)^{-2}$ arises from rescaling the wavenumber $k\to k/(\epsilon\gamma)$. This is the so-called weak coupling (or disorder) limit in kinetic theory, which prohibits the Anderson localization from happening [17]. Note that the resulting medium fluctuation $\epsilon^{-3/2}V_j(\mathbf{r}/\epsilon)$ converges to a spatial white noise in three dimensions.

Physically speaking, the radiative transfer scaling belongs to the diffusive wave regime under the condition of a large dimensionless conductance $g=N\ell_t/L$, where ℓ_t is the transport mean free path, L is the sample size in the direction of propagation, and $N=2\pi A/\lambda^2$ is the number of transverse modes, limited by the illuminated area A and the wavelength of radiation λ [6,7]. The dimensionless conductance g can be expressed as $g=k\ell_t\theta$ with the inverse Fresnel number $\theta=A/(\lambda L)$. With the scaling of Eq. (13), $k\ell_t\sim\theta\sim\gamma^{-1}\epsilon^{-1}$, and hence $g\sim\gamma^{-2}\epsilon^{-2}\gg 1$ for any finite γ as $\epsilon\to 0$.

Anticipating small-scale fluctuation due to Eq. (13), we modify the definition of the two-frequency Wigner distribution in the following way:

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\gamma\epsilon\mathbf{y}}{2k_1}\right)U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\gamma\epsilon\mathbf{y}}{2k_2}\right)\mathrm{d}y.$$

Equation (9) now becomes

$$\mathbf{p}\cdot\nabla W - F = \frac{i}{2\epsilon\gamma}(\nu_1-\nu_2^*)W + \frac{1}{\sqrt{\epsilon}}\mathcal{L}W, \tag{14}$$

where the operator $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L}W(\mathbf{x},\mathbf{p}) = \frac{i}{2\gamma}\int \hat{V}_1(\mathrm{d}\mathbf{q})\exp\left(i\frac{\mathbf{q}\cdot\mathbf{x}}{\epsilon k_1}\right)W\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}\right) \\ -\frac{i}{2\gamma}\int \hat{V}_2^*(\mathrm{d}\mathbf{q})\exp\left(-i\frac{\mathbf{q}\cdot\mathbf{x}}{\epsilon k_2}\right)W\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2}\right).\end{aligned}$$

To capture the cross-frequency correlation in the radiative transfer regime, we also need to restrict the frequency difference range

$$\lim_{\epsilon\to 0} k_1 = \lim_{\epsilon\to 0} k_2 = k, \qquad \frac{k_2-k_1}{\epsilon\gamma k} = \beta, \tag{15}$$

where k, $\beta>0$ are independent of ϵ and γ. Assuming the differentiability of the mean refractive index's dependence on the wavenumber, we write

$$\frac{\nu_2^*-\nu_1}{2\epsilon\gamma} = \nu', \tag{16}$$

where ν' is independent of ϵ, γ.

4. MULTISCALE EXPANSION

To derive the radiative transfer equation for the two-frequency Wigner distribution let us employ MSE [18,19], which begins with introducing the fast variable

$$\tilde{\mathbf{x}} = \mathbf{x}/\epsilon$$

and treating $\tilde{\mathbf{x}}$ as independent from the slow variable $\mathbf{x}$. Consequently the derivative $\mathbf{p}\cdot\nabla$ consists of two terms,

$$\mathbf{p}\cdot\nabla = \mathbf{p}\cdot\nabla_{\mathbf{x}} + \epsilon^{-1}\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}. \tag{17}$$

Then MSE posits the following asymptotic expansion:

$$W(\mathbf{x},\mathbf{p}) = \bar{W}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + \sqrt{\epsilon}W_1(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + \epsilon W_2(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + O(\epsilon^{3/2}), \qquad \tilde{\mathbf{x}} = \mathbf{x}\epsilon^{-1}, \tag{18}$$

whose proper sense will be explained below.

Substituting the ansatz into Eq. (14) and using Eq. (17), we determine each term of Eq. (18) by equating terms of the same order of magnitude, starting with the highest order, ϵ^{-1}.

The ϵ^{-1}-order equation has one term,

$$\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}\bar{W} = 0,$$

which can be solved by setting $\bar{W}=\bar{W}(\mathbf{x},\mathbf{p})$. That is, to the leading order W is independent of the fast variable. Since the fast variable is due to medium fluctuation, this suggests that $\bar{W}$ is deterministic.

The next is the $\epsilon^{-1/2}$-order equation:

$$\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_1 = \mathcal{L}\bar{W}. \tag{19}$$

We seek a solution that is stationary in $\tilde{\mathbf{x}}$, square integrable in $\mathbf{p}$, and has finite second moment. The solvability condition (Fredholm alternative) is that the right-hand side, $\mathcal{L}\bar{W}$, satisfies $\int\mathbb{E}[\Psi^*\mathcal{L}\bar{W}]\mathrm{d}\mathbf{p}=0$ for any $\tilde{\mathbf{x}}$-stationary, square-integrable field $\Psi(\tilde{\mathbf{x}},\mathbf{p})$ satisfying $\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}\Psi=0$. The solvability condition is, however, not easy to enforce. Alternatively, we can consider the regularized equation

$$\epsilon W_1^{\epsilon} + \mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_1^{\epsilon} = \mathcal{L}\bar{W}, \tag{20}$$

which is always solvable for $\epsilon>0$ and admits of the solution

$$W_1^{\epsilon}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) = \frac{i}{2\gamma}\int \hat{V}_1(\mathrm{d}\mathbf{q})\frac{\exp\left(i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_1}\right)}{\epsilon + i\mathbf{q}\cdot\mathbf{p}/k_1}\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}\right) - \frac{i}{2\gamma}\int \hat{V}_2^*(\mathrm{d}\mathbf{q})\frac{\exp\left(-i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_2}\right)}{\epsilon - i\mathbf{q}\cdot\mathbf{p}/k_2}\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2}\right). \tag{21}$$

In the jargon of asymptotic analysis [18], $\sqrt{\epsilon}W_1^{\epsilon}$ is called the first *corrector*. To control the first corrector, let us choose $\bar{W}$ such that $\mathcal{L}\bar{W}$ has zero mean. This is a necessary condition, as we seek an $\tilde{\mathbf{x}}$-stationary solution and consequently $\langle\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_1\rangle=\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}\langle W_1\rangle=0$. Needless to say, this condition is weaker than the solvability condition stated above and is satisfied for any deterministic $\bar{W}$, since both V_1 and V_2 have zero mean.

Indeed, under the assumption of deterministic $\bar{W}$, the resulting equation will be much simplified; so let us impose this property on $\bar{W}$ from now on. The fact that in the limit $\bar{W}$ is deterministic can be proved rigorously in the paraxial regime [14].

Finally, the $O(1)$ equation is

$$\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_2(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) = -\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}(\mathbf{x},\mathbf{p}) - i\nu'\bar{W} + F + \frac{i}{2\gamma}\int \hat{V}_1(\mathrm{d}\mathbf{q})\exp\left(i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_1}\right)\times W_1^{\epsilon}\left(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}\right) - \frac{i}{2\gamma}\int \hat{V}_2^*(\mathrm{d}\mathbf{q})\times\exp\left(-i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_2}\right)W_1^{\epsilon}\left(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2}\right), \tag{22}$$

which can be solved with regularization as in Eq. (20) and yields the second corrector ϵW_2^{ϵ}. Again we impose on the right-hand side of Eq. (22) the weaker condition of zero mean. Using Eq. (21) in Eq. (22), taking the ensemble average, and passing to the limit $\epsilon\to 0$, we obtain the governing equation for $\bar{W}$:

$$\begin{aligned}
&\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}(\mathbf{x},\mathbf{p}) + i\nu'\bar{W} - \langle F\rangle \\
&= -\frac{k_1^3}{2\gamma^4}\int \mathrm{d}\mathbf{q}\Phi_1\left(\frac{k_1}{\gamma}(\mathbf{p}-\mathbf{q})\right)\pi\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\bar{W}(\mathbf{x},\mathbf{p}) \\
&+ \frac{ik_1^3}{2\gamma^4}\fint \mathrm{d}\mathbf{q}\frac{\Phi_1\left(\frac{k_1}{\gamma}(\mathbf{p}-\mathbf{q})\right)}{|\mathbf{p}|^2-|\mathbf{q}|^2}\bar{W}(\mathbf{x},\mathbf{p}) \\
&- \frac{k_2^3}{2\gamma^4}\int \mathrm{d}\mathbf{q}\Phi_2\left(\frac{k_2}{\gamma}(\mathbf{p}-\mathbf{q})\right)\pi\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\bar{W}(\mathbf{x},\mathbf{p}) \\
&- \frac{ik_2^3}{2\gamma^4}\fint \mathrm{d}\mathbf{q}\frac{\Phi_2\left(\frac{k_2}{\gamma}(\mathbf{p}-\mathbf{q})\right)}{|\mathbf{p}|^2-|\mathbf{q}|^2}\bar{W}(\mathbf{x},\mathbf{p}) \\
&+ \frac{1}{4\gamma^2}\int \mathrm{d}\mathbf{q}\Phi_{12}(\mathbf{q})e^{i\tilde{\mathbf{x}}\cdot\mathbf{q}(k_1^{-1}-k_2^{-1})}\pi\delta\left(\frac{\mathbf{q}}{k_2}\cdot\left(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}\right)\right) \\
&\times\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2}\right) \\
&+ \frac{1}{4\gamma^2}\int \mathrm{d}\mathbf{q}\Phi_{12}(\mathbf{q})e^{i\tilde{\mathbf{x}}\cdot\mathbf{q}(k_1^{-1}-k_2^{-1})}\pi\delta\left(\frac{\mathbf{q}}{k_1}\cdot\left(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2}\right)\right) \\
&\times\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2}\right) + \frac{i}{4\gamma^2}\int \mathrm{d}\mathbf{q}\left[\frac{1}{\frac{\mathbf{q}}{k_2}\cdot\left(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}\right)}\right. \\
&\left.-\frac{1}{\frac{\mathbf{q}}{k_1}\cdot\left(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2}\right)}\right]\Phi_{12}(\mathbf{q})e^{i\tilde{\mathbf{x}}\cdot\mathbf{q}(k_1^{-1}-k_2^{-1})} \\
&\times\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2}\right),
\end{aligned}$$

where we have used the fact that in the sense of generalized function

$$\lim_{\eta\to 0}\frac{1}{\eta+i\xi}=\pi\delta(\xi)-\frac{i}{\xi},$$

with the second term giving rise to the Cauchy principal value integral denoted $⨍$. From Eq. (7) we have the expression for $\langle F\rangle$,

$$\langle F\rangle=-\frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}f_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\left\langle U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\right\rangle \mathrm{d}\mathbf{y}+\frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\left\langle U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_2}\right)\right\rangle f_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y},$$

which depends only on the mean fields $\langle U_1\rangle$, $\langle U_2\rangle$, both assumed known throughout the paper.

Putting all the terms together with the regularization, we arrive at the following MSE:

$$W(\mathbf{x},\mathbf{p})=\bar{W}(\mathbf{x},\mathbf{p})+\sqrt{\epsilon}W_1^{\epsilon}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})+\epsilon W_2^{\epsilon}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}), \quad (23)$$

which satisfies

$$\left(\mathbf{p}\cdot\nabla-\frac{1}{\sqrt{\epsilon}}\mathcal{L}\right)W+i\nu' W-F=(i\nu'-1)\sqrt{\epsilon}W_1^{\epsilon}+\sqrt{\epsilon}\mathbf{p}\cdot\nabla_{\mathbf{x}}W_1^{\epsilon}-\sqrt{\epsilon}\mathcal{L}W_2^{\epsilon}+(i\nu'-1)\epsilon W_2^{\epsilon}+\epsilon\mathbf{p}\cdot\nabla_{\mathbf{x}}W_2^{\epsilon}. \quad (24)$$

Unfortunately, the right-hand side of Eq. (24) does not vanish in the strong L^2 topology, but only in the weak topology, as in

$$\lim_{\epsilon\to 0}\epsilon\int \mathrm{d}\mathbf{x}\left\langle\left|\int \mathrm{d}\mathbf{p}W_1^{\epsilon}\left(\mathbf{x},\frac{\mathbf{x}}{\epsilon},\mathbf{p}\right)\psi(\mathbf{p})\right|^2\right\rangle=0, \qquad \forall\,\psi\in L^2 \quad (25)$$

(see Appendix B). It is not clear at this point how to justify the preceding argument and construction of asymptotic solution with full mathematical rigor. Fortunately, in the regime of geometrical optics, the rigorous asymptotic result can be obtained by a probabilistic method [15] and is the same as derived by MSE (see Section 6). Another regime for which the asymptotic result can be fully justified is paraxial waves, which we will turn to in Section 5.

Owing to assumption (15) and the assumed continuous dependence of the medium fluctuation on the frequency, we have $\lim\Phi_1=\lim\Phi_2=\lim\Phi_{12}=\Phi$. As a consequence, all the Cauchy principal value integrals cancel out. With some changes of variables the governing equation for $\bar{W}$ takes the much simplified form

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}+i\nu'\bar{W}-\langle F\rangle=\frac{\pi k^3}{\gamma^4}\int \mathrm{d}\mathbf{q}\Phi\left(\frac{k}{\gamma}(\mathbf{p}-\mathbf{q})\right)\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\times[e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})\beta}\bar{W}(\mathbf{x},\mathbf{q})-\bar{W}(\mathbf{x},\mathbf{p})]. \quad (26)$$

The δ function in the scattering kernel is due to elastic scattering, which preserves the wavenumber. When $\beta=0$ (then $\nu_1=\nu_2$ and $i\nu'\sim$ the imaginary part of ν), Eq. (26) reduces to the standard form of the radiative transfer equation for the phase space energy density [16,20–22]. For $\beta>0$, the wave feature is retained in Eq. (26). When $\beta\to\infty$, the first term in the bracket on the right-hand side of Eq. (26) drops out because of rapid phase fluctuation, so the random scattering effect is pure damping:

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}+i\nu'\bar{W}-\langle F\rangle=-\frac{\pi k^3}{\gamma^4}\int \mathrm{d}\mathbf{q}\Phi\left(\frac{k}{\gamma}(\mathbf{p}-\mathbf{q})\right)\times\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\bar{W}(\mathbf{x},\mathbf{p}).$$

As a comparison, for Schrödinger equation (10) in the frequency domain, we modify the Wigner distribution as

$$W(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi_1\left(\mathbf{x}+\frac{\epsilon\hbar\mathbf{y}}{2}\right)\Psi_2^*\left(\mathbf{x}-\frac{\epsilon\hbar\mathbf{y}}{2}\right)\mathrm{d}\mathbf{y}$$

and in the limit $\epsilon\to 0$ obtain the radiative transfer equation by following the same procedure:

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}+i(\omega_2-\omega_1)\bar{W}+\frac{2i}{\hbar}\nu'\bar{W}-\langle F\rangle=\frac{4\pi}{\hbar^4}\int \mathrm{d}\mathbf{q}\Phi\left(\frac{\mathbf{p}-\mathbf{q}}{\hbar}\right)\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)[\bar{W}(\mathbf{x},\mathbf{q})-\bar{W}(\mathbf{x},\mathbf{p})]. \quad (27)$$

The absence of the factor $e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})\beta}$ in Eq. (27), and therefore the cross-frequency interference, is the main characteristic of 2f-RT for quantum waves.

5. PARAXIAL 2f-RT: ANISOTROPIC MEDIUM

The forward-scattering approximation, also called the paraxial approximation, is valid when backscattering is negligible, and, as we shall see now, this is the case for anisotropic media fluctuating slowly in the (longitudinal) direction of propagation. Let z denote the longitudinal coordinate and $\mathbf{x}_\perp$ the transverse coordinates. Let p and $\mathbf{p}_\perp$ denote the longitudinal and transverse components of $\mathbf{p}\in\mathbb{R}^3$, respectively. Let $\mathbf{q}=(q,\mathbf{q}_\perp)\in\mathbb{R}^3$ be likewise defined.

Consider now a highly anisotropic spectral density for a medium fluctuating much more slowly in the longitudinal direction, i.e., replacing $\Phi((\mathbf{p}-\mathbf{q})k/\gamma)$ in Eq. (26) with

$$\frac{1}{\eta}\Phi\left(\frac{k}{\eta\gamma}(p-q),\frac{k}{\gamma}(\mathbf{p}_\perp-\mathbf{q}_\perp)\right), \qquad \eta\ll 1,$$

which, in the limit $\eta\to 0$, tends to

$$\frac{\gamma}{k}\delta(p-q)\int \mathrm{d}w\Phi\left(w,\frac{k}{\gamma}(\mathbf{p}_\perp-\mathbf{q}_\perp)\right). \quad (28)$$

Writing $\bar{W}=\bar{W}(z,\mathbf{x}_\perp,p,\mathbf{p}_\perp)$, we can approximate Eq. (26) by

$$p\partial_z\bar{W}+\mathbf{p}_\perp\cdot\nabla_{\mathbf{x}_\perp}\bar{W}+i\nu'\bar{W}-\langle F\rangle=\frac{\pi k^2}{\gamma^3}\int \mathrm{d}\mathbf{q}_\perp\int \mathrm{d}w\Phi\left(w,\frac{k}{\gamma}(\mathbf{p}_\perp-\mathbf{q}_\perp)\right)\delta(|\mathbf{p}_\perp|^2-|\mathbf{q}_\perp|^2)\times[e^{i\mathbf{x}_\perp\cdot(\mathbf{p}_\perp-\mathbf{q}_\perp)\beta}\bar{W}(z,\mathbf{x}_\perp,p,\mathbf{q}_\perp)-\bar{W}(z,\mathbf{x}_\perp,p,\mathbf{p}_\perp)]. \quad (29)$$

Equation (29) is identical to the 2f-RT equation rigorously derived directly from the paraxial wave equation for similar anisotropic media [13,14]. This is somewhat surprising in view of the different scaling factors in the definition of two-frequency Wigner distributions in the two cases.

Note that in Eq. (29) the longitudinal momentum p plays the role of a parameter and does not change during propagation and scattering. An important implication of this observation is that Eq. (29) can be solved as an evolution equation in the direction of increasing z with the one-sided boundary condition (e.g., at $z=\text{constant}$). In other words, the influence from the other boundary vanishes as the longitudinal direction is infinitely long. The initial value problem of Eq. (29) is much easier to solve than the boundary value problem of Eq. (26).

6. TWO-FREQUENCY GEOMETRICAL RADIATIVE TRANSFER (2f-GRT)

Let us consider the further limit $\gamma \ll 1$ when the wavelength is much shorter than the correlation length of the medium fluctuation. To this end, the following form is more convenient to work with:

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W} + i\nu'\bar{W} - \langle F\rangle = \frac{\pi k}{2\gamma^2}\int \mathrm{d}\mathbf{q}\Phi(\mathbf{q})\delta\left(\mathbf{q}\cdot\left(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k}\right)\right) \times\left[e^{i\mathbf{x}\cdot\mathbf{q}\beta\gamma/k}\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{k}\right)-\bar{W}(\mathbf{x},\mathbf{p})\right], \tag{30}$$

which is obtained from Eq. (26) after a change of variables. We expand the right-hand side of Eq. (30) in γ and pass to the limit $\gamma\to 0$ to obtain

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W} + i\nu'\bar{W} - \langle F\rangle = \frac{1}{4k}(\nabla_{\mathbf{p}} - i\beta\mathbf{x})\cdot\mathbf{D}\cdot(\nabla_{\mathbf{p}} - i\beta\mathbf{x})\bar{W} \tag{31}$$

with the (momentum) diffusion coefficient

$$\mathbf{D}(\mathbf{p}) = \pi\int \Phi(\mathbf{q})\delta(\mathbf{p}\cdot\mathbf{q})\mathbf{q}\otimes\mathbf{q}\mathrm{d}\mathbf{q}. \tag{32}$$

The symmetry $\Phi(\mathbf{p})=\Phi(-\mathbf{p})$ plays an explicit role here in rendering the right-hand side of Eq. (30) a second-order operator in the limit $\gamma\to 0$. Equation (31) can be rigorously derived from geometrical optics by a probabilistic method [15].

A. Spatial (Frequency) Spread and Coherence Bandwidth

Through dimensional analysis, Eq. (31) yields qualitative information about important physical parameters of the stochastic medium. To show this, let us assume for simplicity the isotropy of the medium, i.e., $\Phi(\mathbf{p})=\Phi(|\mathbf{p}|)$, so that $\mathbf{D}=C|\mathbf{p}|^{-1}P(\mathbf{p})$, where

$$C = \frac{\pi}{3}\int \delta\left(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\right)\Phi(|\mathbf{q}|)|\mathbf{q}|\mathrm{d}\mathbf{q} \tag{33}$$

is a constant and $P(\mathbf{p})$ the orthogonal projection onto the plane perpendicular to $\mathbf{p}$. In view of Eq. (31), C (and $\mathbf{D}$) has the dimension of inverse length, while the variables $\mathbf{x}$ and $\mathbf{p}$ are dimensionless.

Now consider the following change of variables:

$$\mathbf{x} = \sigma_x k\tilde{\mathbf{x}}, \qquad \mathbf{p} = \sigma_p\tilde{\mathbf{p}}/k, \qquad \beta = \beta_c\tilde{\beta}, \tag{34}$$

where σ_x and σ_p are, respectively, the spreads in position and spatial frequency, and β_c is the coherence bandwidth. Let us substitute Eq. (34) into Eq. (31) and aim for the standard form

$$\tilde{\mathbf{p}}\cdot\nabla_{\tilde{\mathbf{x}}}\bar{W} + i\nu'\bar{W} - \langle F\rangle = (\nabla_{\tilde{\mathbf{p}}} - i\beta\tilde{\mathbf{x}})\cdot|\tilde{\mathbf{p}}|^{-1}P_{(\tilde{\mathbf{p}})}(\nabla_{\tilde{\mathbf{p}}} - i\beta\tilde{\mathbf{x}})\bar{W}. \tag{35}$$

The first term on the left-hand side yields the first duality relation

$$\sigma_x/\sigma_p \sim 1/k^2. \tag{36}$$

The balance of terms in each pair of parentheses yields the second duality relation

$$\sigma_x\sigma_p \sim 1/\beta_c, \tag{37}$$

whose left-hand side is the *space-spread–bandwidth product*. Finally, removal of the constant C determines

$$\sigma_p \sim k^{2/3}C^{1/3}, \tag{38}$$

from which σ_x and β_c can be determined by using relations (36) and (37):

$$\sigma_x \sim k^{-4/3}C^{1/3}, \qquad \beta_c \sim k^{2/3}C^{-2/3}.$$

We do not know if, as it stands, Eq. (35) is analytically solvable, but we can solve analytically for its boundary layer behavior.

B. Boundary Layer Asymptotics: Paraxial 2f-GRT

Consider the half-space $z\geq 0$ occupied by an isotropic random medium characterized by the power spectral density $\Phi=\Phi(|\mathbf{p}|)$ and a collimated narrowband beam propagating in the z direction and incident normal to the boundary of the medium. Near the point of incidence on the boundary the corresponding two-frequency Wigner distribution would be highly concentrated at the longitudinal momentum, say, $p=1$. Hence we can assume that the projection $P(\mathbf{p})$ in Eq. (35) is effectively just the projection onto the transverse plane coordinated by $\mathbf{x}_\perp$, and we can approximate Eq. (31) by

$$(\partial_z + \mathbf{p}_\perp\cdot\nabla_{\mathbf{x}_\perp})\bar{W} + i\nu'\bar{W} - \langle F\rangle = \frac{C_\perp}{4k}(\nabla_{\mathbf{p}_\perp} - i\beta\mathbf{x}_\perp)^2\bar{W}, \tag{39}$$

where the constant $C_\perp$ is given by

$$C_\perp = \frac{\pi}{2}\int_{\mathbf{R}^2}\Phi(0,\mathbf{q}_\perp)|\mathbf{q}_\perp|^2\mathrm{d}\mathbf{q}_\perp.$$

Note again that the longitudinal momentum p plays the role of a parameter in Eq. (39), which then can be solved in the direction of increasing z as an evolution equation with initial data given at a fixed z. This is another instance of paraxial approximation.

Let σ_* be the spatial spread in the transverse coordinates , ℓ_c the coherence length in the transverse dimensions and β_c the coherence bandwidth. Let L be the scale of the boundary layer. We then seek the following change of variables:

$$\tilde{\mathbf{x}}_\perp = \frac{\mathbf{x}_\perp}{\sigma_* k}, \quad \tilde{\mathbf{p}}_\perp = \mathbf{p}_\perp k \ell_c, \quad \tilde{z} = \frac{z}{Lk}, \quad \tilde{\beta} = \frac{\beta}{\beta_c}, \quad (40)$$

to remove all the physical parameters from Eq. (39) and to aim for the form

$$\partial_{\tilde{z}} \bar{W} + \tilde{\mathbf{p}}_\perp \cdot \nabla_{\tilde{\mathbf{x}}_\perp} \bar{W} + Lki\nu' \bar{W} - Lk\langle F\rangle = (\nabla_{\tilde{\mathbf{p}}_\perp} - i\tilde{\beta}\tilde{\mathbf{x}}_\perp)^2 \bar{W}. \quad (41)$$

The same reasoning as above now leads to

$$\ell_c \sigma_* \sim L/k, \qquad \sigma_*/\ell_c \sim 1/\beta_c, \qquad \ell_c \sim k^{-1} L^{-1/2} C_\perp^{-1/2}$$

and hence

$$\sigma_* \sim L^{3/2} C_\perp^{-1/2}, \qquad \beta_c \sim k^{-1} C_\perp^{-1} L^{-2}.$$

The layer thickness L may be determined by $\ell_c \sim 1$ or equivalently $L \sim C_\perp^{-1} k^{-2}$. After the inverse Fourier transform Eq. (41) becomes

$$\partial_{\tilde{z}} \Gamma - i \nabla_{\tilde{\mathbf{y}}_\perp} \cdot \nabla_{\tilde{\mathbf{x}}_\perp} \Gamma + Lki\nu' \Gamma - Lk\langle F\rangle = -|\tilde{\mathbf{y}}_\perp + \tilde{\beta}\tilde{\mathbf{x}}_\perp|^2 \Gamma, \quad (42)$$

which is the governing equation for the two-frequency mutual coherence in the normalized variables. With data given on $\tilde{z}=0$ and vanishing far-field boundary condition in the transverse directions, Eq. (42) can be solved analytically, and its Green's function is given by

$$\frac{e^{-iLk\nu'}(i4\tilde{\beta})^{1/2}}{(2\pi)^2 \tilde{z} \sinh[(i4\tilde{\beta})^{1/2}\tilde{z}]} \exp\left(\frac{1}{i4\tilde{\beta}\tilde{z}} |\tilde{\mathbf{y}}_\perp - \tilde{\beta}\tilde{\mathbf{x}}_\perp - \mathbf{y}'_\perp + \tilde{\beta}\mathbf{x}'_\perp|^2 \right)$$
$$\times \exp\left\{ - \frac{\coth[(i4\tilde{\beta})^{1/2}\tilde{z}]}{(i4\tilde{\beta})^{1/2}} \left| \tilde{\mathbf{y}}_\perp + \tilde{\beta}\tilde{\mathbf{x}}_\perp - \frac{\mathbf{y}'_\perp + \tilde{\beta}\mathbf{x}'_\perp}{\cosh[(i4\tilde{\beta})^{1/2}\tilde{z}]} \right|^2 \right\}$$
$$\times \exp\left\{ - \frac{\tanh[(i4\tilde{\beta})^{1/2}\tilde{z}]}{(i4\tilde{\beta})^{1/2}} |\mathbf{y}'_\perp + \tilde{\beta}\mathbf{x}'_\perp|^2 \right\}. \quad (43)$$

Formula (43) is consistent with the asymptotic result in the literature, which mainly concerns the cross-frequency correlation of intensity. In the radiative transfer regime considered here, the cross-spectral correlation of intensity is the square of the two-frequency mutual coherence and has the commonly accepted form [8,9,23]

$$\exp(-2\sqrt{2\tilde{\beta}}), \quad (44)$$

which is just the large $\tilde{\beta}$ asymptotic of the squared factor $|\sinh[(i4\tilde{\beta})^{1/2}\tilde{z}]|^{-2}$ in formula (43) for $\tilde{z}=1$(see Ref. [15] for detailed comparison). Moreover, formula (43) provides detailed information about the simultaneous dependence of the mutual coherence on the frequency difference and spatial displacement [7,8].

Surprisingly, a closely related equation arises in the two-frequency formulation of the Markovian approximation of the paraxial waves [12]. The closed-form solution is crucial for analyzing the performance of time-reversal communication with broadband signals [24]. The solution procedure for formula (43) is similar to that given elsewhere [24] and is omitted here.

C. Paraxial 2f-GRT in Anisotropic Media

We use here the setting and notation defined in Section 5 for anisotropic media. For simplicity we will set $p=1$ and omit writing it out in $\bar{W}$. In view of Eq. (28) we replace $\Phi(\mathbf{q})$ in Eq. (32) with

$$\delta(q) \int \mathrm{d}w \Phi(w, \mathbf{q}_\perp)$$

and obtain the transverse diffusion coefficient

$$\mathbf{D}_\perp(\mathbf{p}_\perp) = \pi \int \mathrm{d}\mathbf{q}_\perp \int \mathrm{d}w \Phi(w, \mathbf{q}_\perp) \delta(\mathbf{p}_\perp \cdot \mathbf{q}_\perp) \mathbf{q}_\perp \otimes \mathbf{q}_\perp,$$

whereas the longitudinal diffusion coefficient is zero.

For simplicity we assume the isotropy in the transverse dimensions, $\Phi(w, \mathbf{p}_\perp) = \Phi(w, |\mathbf{p}_\perp|)$, so that $\mathbf{D}_\perp = C_\perp |\mathbf{p}_\perp|^{-1} P_\perp(\mathbf{p}_\perp)$, where

$$C_\perp = \frac{\pi}{2} \int \delta\left(\frac{\mathbf{p}_\perp}{|\mathbf{p}_\perp|} \cdot \frac{\mathbf{q}_\perp}{|\mathbf{q}_\perp|} \right) \Phi(w, |\mathbf{q}_\perp|) |\mathbf{q}_\perp| \mathrm{d}w \mathrm{d}\mathbf{q}_\perp$$

is a constant and $P_\perp(\mathbf{p}_\perp)$ is the orthogonal projection onto the transverse line perpendicular to $\mathbf{p}_\perp$. Hence Eq. (31) reduces to

$$(\partial_z + \mathbf{p}_\perp \cdot \nabla_{\mathbf{x}_\perp}) \bar{W} + i\nu' \bar{W} - \langle F\rangle$$
$$= \frac{C_\perp}{4k} (\nabla_{\mathbf{p}_\perp} - i\beta \mathbf{x}_\perp) \cdot |\mathbf{p}_\perp|^{-1} P_\perp(\mathbf{p}_\perp) (\nabla_{\mathbf{p}_\perp} - i\beta \mathbf{x}_\perp) \bar{W}. \quad (45)$$

Alternatively, Eq. (45) can also be derived from Eq. (29) by taking the geometrical optics limit as described at the beginning of Section 6.

Consider change of variables (40) to remove all the physical parameters from Eq. (45) to aim for the form

$$[\partial_{\tilde{z}} + \tilde{\mathbf{p}}_\perp \cdot \nabla_{\tilde{\mathbf{x}}_\perp}] \bar{W} + Lki\nu' \bar{W} - Lk\langle F\rangle$$
$$= (\nabla_{\tilde{\mathbf{p}}_\perp} - i\tilde{\beta}\tilde{\mathbf{x}}_\perp) \cdot |\tilde{\mathbf{p}}_\perp|^{-1} P_\perp(\tilde{\mathbf{p}}_\perp) (\nabla_{\tilde{p}_\perp} - i\tilde{\beta}\tilde{\mathbf{x}}_\perp) \bar{W}, \quad (46)$$

where L is interpreted as the distance of propagation. Following the same line of reasoning, we obtain that

$$\ell_c \sigma_* \sim L/k, \qquad \sigma_*/\ell_c \sim 1/\beta_c, \qquad \ell_c \sim C_\perp^{-1/3} L^{-1/3} k^{-1},$$

and hence

$$\sigma_* \sim C_\perp^{1/3} L^{4/3}, \qquad \beta_c \sim C_\perp^{-2/3} L^{-5/3} k^{-1}.$$

Unlike Eq. (39) it is unclear whether a closed-form solution to Eq. (45) exists.

7. DISCUSSION AND CONCLUSION

The standard (one-frequency) RT can be formally derived from the wave equation in at least two ways: the diagrammatic expansion method, as the ladder approximation of the Bethe–Salpeter equation [8,16] and the multiscale expansion [MSE] method advocated here [18]. The latter is considerably simpler than the former in terms of the amount of calculation involved. Both approaches have been developed with full mathematical rigor in some special cases (see [25,26] and the references therein). There are two regimes for which the 2f-RT equation has been derived with full mathematical rigor: first, for the paraxial wave equation by using the so-called martingale method in probability theory [13,14]; second, for spherical waves in geometrical optics by the path-integration method [15]. These rigorous results coincide with those derived here for the respective regimes and hence support the validity of MSE.

Within the framework of 2f-RT, a paraxial form arises naturally in anisotropic media that fluctuate slowly in the longitudinal direction. Another form of paraxial 2f-RT takes place in the boundary layer asymptotics of isotropic media. The latter equation turns out to be exactly solvable, and the boundary layer behavior is given in a closed form, revealing highly nontrivial structure of the two-frequency mutual coherence. In any case, dimensional analysis with the 2f-GRT equations yields qualitative scaling behavior of the spatial spread, the spatial frequency spread, and the coherent bandwidth in various regimes.

From the point of view of computation, especially Monte Carlo simulation, it appears to be natural to introduce the new quantity

$$\widetilde{W}(\mathbf{x},\mathbf{p}) = e^{-i\beta\mathbf{x}\cdot\mathbf{p}}\bar{W}(\mathbf{x},\mathbf{p})$$

and rewrite Eq. (26) in the following form:

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}\widetilde{W} + i\beta|\mathbf{p}|^2\widetilde{W} + i\nu'\widetilde{W} - e^{-i\beta\mathbf{x}\cdot\mathbf{p}}\langle F\rangle = \frac{\pi k^3}{\gamma^4}\int \mathrm{d}\mathbf{q}\Phi\left(\frac{k}{\gamma}(\mathbf{p}-\mathbf{q})\right)\times\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)[\widetilde{W}(\mathbf{x},\mathbf{q})-\widetilde{W}(\mathbf{x},\mathbf{p})].$$

The solution $\widetilde{W}$ can then be expressed as a path integration over the Markov process generated by the operator $\mathcal{A}$ defined by

$$\mathcal{A}W = -\mathbf{p}\cdot\nabla_{\mathbf{x}}W + \frac{\pi k^3}{\gamma^4}\int \mathrm{d}\mathbf{q}\Phi\left(\frac{k}{\gamma}(\mathbf{p}-\mathbf{q})\right)\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\times[W(\mathbf{x},\mathbf{q})-W(\mathbf{x},\mathbf{p})]$$

when V is real valued and Φ is nonnegative. I will pursue this observation in a separate publication [15].

APPENDIX A: DERIVATION OF EQ. (6)

Applying the operator $\mathbf{p}\cdot\nabla$ to definition (3), we obtain

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}W = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}2\mathbf{p}\cdot\nabla_{\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\times U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} - \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\times U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)2\mathbf{p}\cdot\nabla_{\mathbf{y}}U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y}$$

$$= \frac{2i}{(2\pi)^3}\int(\nabla_{\mathbf{y}}e^{-i\mathbf{p}\cdot\mathbf{y}})\cdot\nabla_{\mathbf{y}}U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\times U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} - \frac{2i}{(2\pi)^3}\int(\nabla_{\mathbf{y}}e^{-i\mathbf{p}\cdot\mathbf{y}})\times U_1\left(\frac{\mathbf{x}}{k_1}-\frac{\mathbf{y}}{2k_1}\right)\cdot\nabla_{\mathbf{y}}U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y}.$$

Integrating by parts with the first $\nabla_{\mathbf{y}}$ in the above integrals, we have

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}W = -\frac{2i}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\nabla_{\mathbf{y}}^2U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\times U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y} + \frac{2i}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\times U_1\left(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1}\right)\nabla_{\mathbf{y}}^2U_2^*\left(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}\right)\mathrm{d}\mathbf{y}, \tag{A1}$$

where the other resulting terms canceled each other. From Eq. (2),

$$\nabla_{\mathbf{y}}^2U_j\left(\frac{\mathbf{x}}{k_j}+\frac{\mathbf{y}}{2k_j}\right) = -\frac{1}{4}\left[\nu_j + V_j\left(\frac{\mathbf{x}}{k_j}+\frac{\mathbf{y}}{2k_j}\right)\right]U_j\left(\frac{\mathbf{x}}{k_j}+\frac{\mathbf{y}}{2k_j}\right) + \frac{1}{4}f_j\left(\frac{\mathbf{x}}{k_j}+\frac{\mathbf{y}}{2k_j}\right). \tag{A2}$$

Using Eq. (A2) in Eq. (A1), we arrive at Eq. (6).

APPENDIX B: WEAK CONVERGENCE OF CORRECTOR

First we shall see that the corrector does not vanish in the mean-square norm in any dimension, i.e., $\lim_{\epsilon\to 0}\epsilon\int\langle|W_1^\epsilon|^2\rangle\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{p}>0$ in general. For simplicity, we set $\gamma=1$ and consider the term involving $\hat{V}_1$ only. This can be seen in the following calculation:

$$\lim_{\epsilon\to 0}\frac{1}{4}\int \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{q}\Phi_1(\mathbf{q})\frac{\epsilon}{\epsilon^2+(\mathbf{p}\cdot\mathbf{q}/k_1)^2}\left|\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1}\right)\right|^2 = \frac{\pi}{4}\int \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{q}\Phi(\mathbf{q})\delta(\mathbf{p}\cdot\mathbf{q}/k)\left|\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k}\right)\right|^2,$$

which is positive in general.

Next we shall see that the corrector vanishes in the weak topology

$$\lim_{\epsilon\to 0}\epsilon\int \mathrm{d}\mathbf{x}\left\langle\left|\int \mathrm{d}\mathbf{p}W_1^\epsilon\left(\mathbf{x},\frac{\mathbf{x}}{\epsilon},\mathbf{p}\right)\psi(\mathbf{p})\right|^2\right\rangle = 0, \quad \forall\,\psi\in L^2. \tag{B1}$$

It suffices to prove Eq. (B1) for any smooth, compactly supported function ψ, and we have

$$\lim_{\epsilon\to 0}\frac{\epsilon}{4}\int \mathrm{d}\mathbf{p}\mathrm{d}\mathbf{p}'\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{q}\frac{\Phi_1(\mathbf{q})\psi(\mathbf{p})\psi^*(\mathbf{p}')}{(\epsilon+i\mathbf{p}\cdot\mathbf{q}/k_1)(\epsilon-i\mathbf{p}'\cdot\mathbf{q}/k_1)}$$
$$\times\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1}\right)\bar{W}^*\left(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1}\right)$$
$$=\lim_{\epsilon\to 0}\frac{\epsilon}{4}\int \mathrm{d}x\mathrm{d}\mathbf{q}\frac{\Phi_1(\mathbf{q})}{|\mathbf{q}|^2}\left[\pi\int \mathrm{d}\mathbf{p}\,\delta(\mathbf{p}\cdot\hat{\mathbf{q}}/k_1)\psi(\mathbf{p})\right.$$
$$\left.\times\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1}\right)-\fint \mathrm{d}\mathbf{p}\frac{i\psi(\mathbf{p})}{\mathbf{p}\cdot\hat{\mathbf{q}}/k_1}\bar{W}\left(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1}\right)\right]$$
$$\times\left[\pi\int \mathrm{d}\mathbf{p}'\,\delta(\mathbf{p}'\cdot\hat{\mathbf{q}}/k_1)\psi^*(\mathbf{p}')\bar{W}^*\left(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1}\right)\right.$$
$$\left.+\fint \mathrm{d}\mathbf{p}'\frac{i\psi^*(\mathbf{p}')}{\mathbf{p}'\cdot\hat{\mathbf{q}}/k_1}\bar{W}^*\left(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1}\right)\right],$$

where $\hat{\mathbf{q}}=\mathbf{q}/|\mathbf{q}|$ for sufficiently smooth $\bar{W}$, Φ, and rapidly decaying Φ. The essential point now is that $|\mathbf{q}|^{-2}$ is an integrable singularity in three dimensions, and hence the above expression vanishes in the limit.

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