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FROZEN PATH APPROXIMATION FOR TURBULENT DIFFUSION AND FRACTIONAL BROWNIAN MOTION IN RANDOM FLOWS*

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Abstract. We establish the conditions for the frozen path approximation for turbulent transport in a class of nonmixing Gaussian flows with long-range correlation. We identify the regimes of fractional Brownian motion limit as well as the Brownian motion limit.

Key words. turbulent transport, fractional Brownian motion, Taylor–Kubo formula

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1. Introduction. The study of turbulent transport is fundamental to understanding of temperature fields as well as pollutant or tracer particles movement in the atmosphere and oceans and solute transport in groundwater flows [1]. For a long time, the Brownian motion (BM) and the heat equation have been the paradigm for describing large-scale turbulent transport since Taylor's works in the 1920s. The wide applicability of the Brownian motion and the related Gaussian processes have much to do with the central limit theorem which is often assumed to be valid over large scales if there is no memory or intermittency effect.

To account for the memory or intermittency effect, anomalous diffusions have been introduced in recent years as phenomenological models within the framework of fractional kinetic equations or continuous-time random walks (see [21], [12], [19], and the references therein). The mechanisms for anomalous behaviors are generally attributed to long waiting times (subdiffusion) or long flights (superdiffusion) or both. The former results in fractional-in-time (hence non-Markovian) differential operators while the latter results in fractional-in-space differential operators. In both cases the underlying processes are non-Gaussian.

In this paper we derive rigorously the fractional Brownian motions (FBMs) as limiting processes of large-scale motions of particles being advected by a family of random flows that are decorrelated both in space and time but in a manner depending on the wave modes of the velocity. This dependence is described in terms of two crucial parameters (α and β) of the flows. Our limit theorem also characterizes the multiple-particle motions in the FBM regime. FBMs are Gaussian but non-Markovian processes and are different from the phenomenological models mentioned above. The FBMs we find in this paper are invariably superdiffusive due to the positive memory effect, while the FBMs we found elsewhere [7] for a different type of flows can be subdiffusive as well as superdiffusive. By varying the parameters we see that the

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limiting processes can switch from FBMs to the Brownian motion, and we characterize the boundary of transition precisely.

The FBM regime indicates the breakdown of the central limit theorem, but the Gaussianity persists in the limit and is inherited from that of the velocity field. It is an open problem if one would obtain non-Gaussian limits for non-Gaussian velocity fields, which are beyond the methodology of the paper.

For the particle displacement $\boldsymbol{x}(t)$ in a given random velocity $\mathbf{V}(t,\mathbf{x})$ we consider the general large-scale limit

$$\boldsymbol{x}^{\varepsilon}(t) = \varepsilon \boldsymbol{x}\left(\frac{t}{\varepsilon^{2q}}\right)$$

satisfying the equation

$$d\boldsymbol{x}^{\varepsilon}(t) = \varepsilon^{1-2q}\mathbf{V}(\varepsilon^{-2q}t, \varepsilon^{-p}\boldsymbol{x}^{\varepsilon}(t))dt + \varepsilon^{1-q}\sqrt{2\kappa}d\mathbf{B}(t), \quad p \geq 0, \tag{1.1}$$

for some $q > 0$ (to be determined) as ϵ tends to zero. Here $\mathbf{B}(t)$ is the Brownian motion and κ is the molecular diffusivity. The special case of $p = 0$ and $q = 1$ is the white-noise-in-time limit. The scaling limit with $p = 1$ is the homogenization limit.

We assume that, in addition to incompressibility, the velocity $\mathbf{V}(t,\mathbf{x}), (t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$, is a zero mean, time-stationary, space-homogeneous, isotropic, Ornstein–Uhlenbeck (thus, Gaussian and Markovian) process with *long-range correlations* (see below). Here the scaling exponent q depends on the correlation functions of the velocity. The scaling limit (1.1) has been studied by Kesten and Papanicolaou [11] in the case of $p = 0$ and Komorowski [13], [14] in the general case of $0 \leq p < 1$ for velocities sufficiently strongly mixing in time, and in this situation the scaling exponent is always $q = 1$, i.e., the *diffusive* scaling, and the limiting process is a Brownian motion with the diffusion coefficients given by the Taylor–Kubo formula [22]

$$D_{ij} = \int_0^{\infty} \left\{\mathbb{E}[V_i(t,\mathbf{0})V_j(0,\mathbf{0})] + \mathbb{E}[V_j(t,\mathbf{0})V_i(0,\mathbf{0})]\right\} dt. \tag{1.2}$$

To understand how the long-range correlation in velocity fields may change the diffusive scaling, we study the weak coupling limit for Ornstein–Uhlenbeck velocities with long-range correlations in both space and time (thus, nonmixing) defined as follows. We define the family of velocity fields with power-law spectra as follows. Let $(\Omega, \mathcal{V}, P)$ be a probability space of which each element is a velocity field $\mathbf{V}(t,\mathbf{x})$, $(t,\mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ satisfying the following properties.

- (H1) $\mathbf{V}(t,\mathbf{x})$ is time-stationary, space-homogeneous, and centered, i.e., $\mathbb{E}\mathbf{V}(0,\mathbf{0}) = \mathbf{0}$, and Gaussian. Here $\mathbb{E}$ stands for the expectation with respect to the probability measure P.
- (H2) The two-point correlation tensor $\mathbf{R} = [R_{ij}]$ is given by

$$\begin{aligned} R_{ij}(t,\mathbf{x}) &= \mathbb{E}\left[V_i(t,\mathbf{x})V_j(0,\mathbf{0})\right] \\ &= \int_{\mathbb{R}^d} \cos(\mathbf{k}\cdot\mathbf{x}) \exp\left(-|\mathbf{k}|^{2\beta}t\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \\ &\quad \times \left(\mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^{d-1}}\right) d\mathbf{k}, \quad \beta \geq 0, \quad d \geq 2, \end{aligned} \tag{1.3}$$

with the spatial spectral density

$$\mathcal{E}(k) = \frac{a(k)}{k^{2\alpha-1}}, \quad \alpha < 1, \tag{1.4}$$

where $a : [0, +\infty) \to \mathbb{R}_+$ is a compactly supported, continuous, nonnegative function. The factor $\mathbf{I} - \mathbf{k} \otimes \mathbf{k}/|\mathbf{k}|^2$ in (1.3) ensures the incompressibility. Note that for $\alpha < 1$ the instantaneous two-point correlation functions $R_{ij}(0, \mathbf{x})$ decays to zero as $|\mathbf{x}|$ tends to infinity. The velocity is strongly temporally mixing if and only if $\beta = 0$ (see [20]).

We show that the scaling limit is either a Brownian motion or a persistent (i.e., superdiffusive) FBM as stated in the following theorem.

THEOREM 1. *Let the velocity field satisfy properties* (H1)–(H2) *with* $p < 1$.

Case 1. *For* $\alpha + \beta < 1$ *and the scaling exponent*

$$q = 1,$$

the solution $\boldsymbol{x}^\varepsilon(t)$ *converges in distribution, as* ε *tends to zero, to the Brownian motion with the covariance matrix given by the Kubo formula* (1.2) *plus* $\kappa\mathbf{I}$.

Case 2. *For* $1 < \alpha + \beta$, $\alpha + 2\beta < 1 + 1/p$, *and the scaling exponent*

$$q := \frac{\beta}{\alpha + 2\beta - 1}, \tag{1.5}$$

the solution $\boldsymbol{x}^\varepsilon(t)$ *converges in probability, as* ε *tends to zero, to a fractional Brownian motion* $\mathbf{B}_H(t)$ *with covariance given by*

$$Cov(\mathbf{B}_H(t_1), \mathbf{B}_H(t_2)) = \frac{1}{2}\mathbf{D}\left\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\right\} \tag{1.6}$$

with the coefficients $\mathbf{D}$

$$\mathbf{D} = \int_{\mathbb{R}^d} \frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}\right) \frac{a(0)}{|\mathbf{k}|^{d-1}} d\mathbf{k} \tag{1.7}$$

and the Hurst exponent H,

$$1/2 < H = 1/2 + \frac{\alpha + \beta - 1}{2\beta} < 1. \tag{1.8}$$

The homogenization scaling with $p = 1$ has been considered in [2], [3], [8], [15] and the corresponding scaling exponent q is the same. But the eddy diffusion matrix is no longer given by the Kubo formula.

We also establish the following results, which are very useful for understanding the simultaneous limit of the motion of multiple particles.

THEOREM 2. *Under the same assumptions of Theorem* 1, *the following approximations are valid in the respective regimes in the mean square sense for sufficiently small* ε:

Case 1.

$$\boldsymbol{x}^\varepsilon(t) = \mathbf{W}_\varepsilon(t) + o(1)$$

with

$$\begin{aligned}\mathbf{W}_\varepsilon(t) := \int_0^t \int_{\mathbb{R}^d} &\frac{|\mathbf{k}|^\beta \mathcal{E}^{\frac{1}{2}}(|\mathbf{k}|)}{(|\mathbf{k}|^{2\beta} + \varepsilon^2)|\mathbf{k}|^{\frac{(d-1)}{2}}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}\right)^{\frac{1}{2}} \\ &[\cos(\varepsilon^{-p}\boldsymbol{x}^\varepsilon(s) \cdot \mathbf{k})\mathbf{W}_0(ds, d\mathbf{k}) + \sin(\varepsilon^{-p}\boldsymbol{x}^\varepsilon(s) \cdot \mathbf{k})\mathbf{W}_1(ds, d\mathbf{k})],\end{aligned} \tag{1.9}$$

where $\mathbf{W}_i(dt, d\mathbf{k})$, $i = 0, 1$, *are two independent copies of a d-dimensional space-time white-noise field (see* [16] *for a thorough discussion).*

Case 2.

$$\boldsymbol{x}^\varepsilon(t) = \boldsymbol{x}^\varepsilon(0) + \int_0^t \varepsilon^{1-2q}\mathbf{V}(\varepsilon^{-2q}s, \varepsilon^{-p}\boldsymbol{x}^\varepsilon(0))ds + o(1). \tag{1.10}$$

The surprising feature about the approximation (1.10) is that the "frozen path" approximation is asymptotically exact on the time scale of observation. Thus the multiple-point motion can be easily derived.

The process $\mathbf{W}_\varepsilon(t)$ defined by (1.9) is a continuous martingale with the quadratic variation

$$\langle \mathbf{W}_\varepsilon \rangle_t = t \int_{\mathbb{R}^d} \frac{|\mathbf{k}|^{2\beta}\mathcal{E}(|\mathbf{k}|)}{(|\mathbf{k}|^{2\beta} + \varepsilon^2)^2}\left(\mathbf{I} - \frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\frac{d\mathbf{k}}{|\mathbf{k}|^{d-1}}.$$

Thus we know that $\mathbf{W}_\varepsilon(t)$, $t \geq 0$, is a Brownian motion. It is easy to check that the ratio $\langle \mathbf{W}_\varepsilon \rangle_t / t$ converges to the Kubo formula as ε tends to zero.

Theorem 1 characterizes the limit of one-point motion whereas Theorem 2 enables us to calculate the limit of multiple-point motion with each particle starting from a different point. It is straightforward to check from the corresponding approximations (1.9) and (1.10) that any two particles with a *fixed* initial separation in $\boldsymbol{x}^\varepsilon(0)$ become, in the limit $\varepsilon \to 0$, *independent* Brownian or FBMs for $p > 0$. However, if the initial separation of particles is of order ε^p, then the resulting limit processes are correlated as in the case of $p = 0$ which has been studied in [6]. The proofs of Theorem 1 and 2 use (finite) diagrammatic expansion and are given in sections 4–6.1. In the main text we present the physical explanation of the theorems in terms of the frozen path approximation. The results are shown schematically in Figure 1. In section 7 we provide a scaling argument for the case of $p > 1$ for the fractional Brownian regime.

When an additional infrared cutoff of the size ε^γ is introduced in the velocity spectrum, the results depend on whether the cutoff is *subcritical*, $\gamma < (\alpha + 2\beta - 1)^{-1}$, or *supercritical*, $\gamma > (\alpha + 2\beta - 1)^{-1}$. A supercritical cutoff does not affect the diagram, but a subcritical cutoff does. In particular, the regime of FBM limit disappears, and the limit is always a Brownian motion when the infrared cutoff is subcritical (see [2], [3]). We will not further discuss the effect of infrared cutoff in this paper.

The effect of molecular diffusion on transport may be subtle (see [18], [7]). However, for isotropic flows with monotonically decaying temporal correlation, small molecular diffusivity is negligible and will only affect results perturbatively. So we set $\kappa = 0$ from now on to simplify the presentation.

2. Brownian motion limit. Let us first consider the case of the Brownian motion limit. We express the displacement in the integral form

$$\boldsymbol{x}_\varepsilon(t) = \boldsymbol{x}_\varepsilon(0) + \frac{1}{\varepsilon}\int_0^t \mathbf{V}\left(\frac{t_1}{\varepsilon^2}, \frac{\boldsymbol{x}_\varepsilon(t_1)}{\varepsilon^p}\right) dt_1.$$

Assuming for simplicity that the spatial derivative of the velocity field is uniformly bounded, we know that the frozen path approximation

$$\boldsymbol{x}_\varepsilon(t) \approx \tilde{\boldsymbol{x}}_\varepsilon(t) = \boldsymbol{x}_\varepsilon(0) + \frac{1}{\varepsilon}\int_0^t \mathbf{V}\left(\frac{s}{\varepsilon^2}, \frac{\boldsymbol{x}_\varepsilon(0)}{\varepsilon^p}\right) ds, \quad 0 < t < \tau,$$

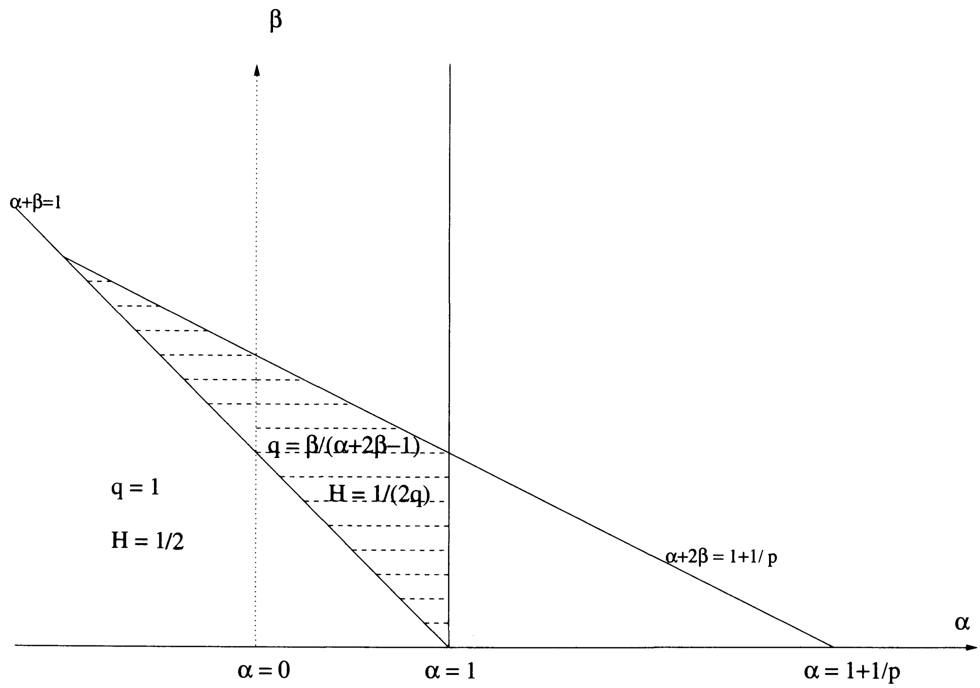

FIG. 1. *Phase diagram with supercritical infrared cutoff.*

is accurate pathwise with an error of $O(\tau^{3/2}\varepsilon^{-1-p})$ on the time scale

$$\varepsilon^2 \ll \tau \ll \varepsilon^{p+1} \tag{2.1}$$

(cf. (3.5)). One then expects that, for small ε, the displacement $\boldsymbol{x}_\varepsilon(t)$ is approximately the sum, $\tilde{\boldsymbol{x}}_\varepsilon(t)$, of t/τ random variables in the form

$$\Delta\tilde{\boldsymbol{x}}_\varepsilon^n(\tau) = \tilde{\boldsymbol{x}}_\varepsilon((n+1)\tau) - \tilde{\boldsymbol{x}}_\varepsilon(n\tau) = \varepsilon\int_{n\tau/\varepsilon^2}^{(n+1)\tau/\varepsilon^2} \mathbf{V}\left(s, \frac{\tilde{\boldsymbol{x}}_\varepsilon(n\tau)}{\varepsilon^p}\right) ds, \quad n = 0, 1, 2, \ldots.$$

Since $\tau \gg \varepsilon^2$, by the central limit theorem for processes with mixing, stationary increments (cf. [20]), the process $\Delta\tilde{\boldsymbol{x}}_\varepsilon^n(t)$,

$$\Delta\tilde{\boldsymbol{x}}_\varepsilon^n(t) = \varepsilon\int_{n\tau/\varepsilon^2}^{(n\tau+t)/\varepsilon^2} \mathbf{V}\left(s, \frac{\tilde{\boldsymbol{x}}_\varepsilon(n\tau)}{\varepsilon^p}\right) ds, \quad 0 < t \le \tau,$$

conditioned on $\tilde{\boldsymbol{x}}_\varepsilon(n\tau)$, is approximately a Brownian motion, starting at 0, with diffusion coefficient given by the Taylor–Kubo formula (1.2). Since $\tau \gg \varepsilon^2$ and the Taylor–Kubo formula converges, $\Delta\tilde{\boldsymbol{x}}_\varepsilon^n$ are nearly uncorrelated for different n and the total error made by the frozen path approximation is $O(\tau\varepsilon^{-1-p})$, which is negligible for $\tau \ll \varepsilon^{1+p}$.

The question is, What is the region in the (α, β) plane where the classical turbulent diffusion theorem, with the Taylor–Kubo formula (1.2), holds? It is easy to find the necessary condition by imposing the convergence of the Taylor–Kubo formula

(1.2). A straightforward calculation

$$
\begin{aligned}
D^*_{ij} &= \int_0^\infty R_{ij}(t,\mathbf{0})dt \\
&= \int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}} \int_0^\infty \exp\left(-|\mathbf{k}|^{2\beta} t\right) dt d\mathbf{k} \\
&= \int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} d\mathbf{k} \qquad (2.2)
\end{aligned}
$$

leads to the condition

$$\alpha + \beta < 1. \qquad (2.3)$$

It turns out that (2.3) is also sufficient. In other words, the classical turbulent diffusion theorem holds for this family of Gaussian velocity fields if and only if (2.3) is true (see section 6.1).

Let us see what the frozen path approximation tells us. The covariance of the Gaussian increment $\Delta\tilde{\boldsymbol{x}}^n_\varepsilon(t),\ 0 < t < \tau$ (given by (2.1)), stationary with respect to n, can be expressed as

$$
\begin{aligned}
&2\varepsilon^2 \int_0^{t/\varepsilon^2} \int_0^{s_1} R_{ij}(s_1 - s_2, \mathbf{0})\, ds_2\, ds_1 \\
&= 2\varepsilon^2 \int_0^{t/\varepsilon^2} \int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} (1 - e^{-s_1|\mathbf{k}|^{2\beta}}) d\mathbf{k} ds_1 \\
&= 2\int_0^t \int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} (1 - e^{-t_1|\mathbf{k}|^{2\beta}/\varepsilon^2}) d\mathbf{k} dt_1 \\
&= 2D^*_{ij} t - 2\int_0^t \int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} e^{-t_1|\mathbf{k}|^{2\beta}/\varepsilon^2} d\mathbf{k} dt_1 \\
&= 2D^*_{ij} t - 2\int_{\mathbb{R}^d} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} \times \frac{\varepsilon^2}{|\mathbf{k}|^{2\beta}} (1 - e^{-t|\mathbf{k}|^{2\beta}/\varepsilon^2})\, d\mathbf{k} \qquad (2.4)
\end{aligned}
$$

with D^* given by the Taylor–Kubo formula (2.2). The last integral can be estimated by breaking it into two parts: $|\mathbf{k}|^{2\beta} < \varepsilon^2/t$ and $|\mathbf{k}|^{2\beta} \geq \varepsilon^2/t$. The first part has the asymptotic

$$
\begin{aligned}
&2t \int_{|\mathbf{k}|^{2\beta} < \varepsilon^2/t} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}}\, d\mathbf{k} \\
&\sim |\mathbf{k}|^{2-2\alpha-2\beta} \Big|_0^{(\frac{\varepsilon^2}{t})^{1/(2\beta)}} t \\
&= \varepsilon^{2(1-\alpha-\beta)/\beta} t^{(\alpha+2\beta-1)/\beta}, \qquad (2.5)
\end{aligned}
$$

which, if $\alpha + 2\beta > 1$, gives rise to the *subdiffusive* FBM with the Hurst exponent

$$H = \frac{\alpha + 2\beta - 1}{2\beta} \ < \ 1/2$$

and vanishing coefficient since $\alpha + \beta < 1$. The second part can be estimated by

$$2\int_{|\mathbf{k}|^{2\beta} \geq \varepsilon^2/t} \frac{\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d+2\beta-1}} \times \frac{\varepsilon^2}{|\mathbf{k}|^{2\beta}}\, d\mathbf{k}$$

$$\sim 2\varepsilon^2 \left[\left(\frac{\varepsilon^2}{t} \right)^{(1-\alpha-2\beta)/\beta} - K^{2(1-\alpha-2\beta)} \right]$$

$$= 2\varepsilon^{2(1-\alpha-\beta)/\beta} t^{(\alpha+2\beta-1)/\beta} - 2\varepsilon^2 K^{2(1-\alpha-2\beta)}. \tag{2.6}$$

Thus, if $\alpha + 2\beta < 1$, the second term in (2.6) dominates the first and, if $\alpha + 2\beta > 1$, the first dominates. But both (2.5) and (2.6) are negligible relative to the leading term $2D^*_{ij}t$.

Therefore, for $\alpha + \beta < 1$, the displacement $\boldsymbol{x}_\varepsilon(t)$ behaves like the Brownian motion, with the diffusion coefficient given by the Taylor–Kubo formula (2.2), plus a correction term. When $\alpha + 2\beta > 1$ the correction term is like a *subdiffusive* FBM.

3. FBM limit. What happens if (2.3) is violated? The divergence of the Taylor–Kubo formula (1.2) suggests a superdiffusive behavior and, consequently, a different scaling limit.

Consider the superdiffusive scaling on the displacement

$$\boldsymbol{x}_\varepsilon(t) = \varepsilon \boldsymbol{x}\left(\frac{t}{\varepsilon^{2q}} \right), \quad q < 1. \tag{3.1}$$

The equation of motion becomes

$$\frac{d\boldsymbol{x}_\varepsilon(t)}{dt} = \frac{1}{\varepsilon^{2q-1}} \mathbf{V}\left(\frac{t}{\varepsilon^{2q}}, \frac{\boldsymbol{x}_\varepsilon(t)}{\varepsilon^p} \right), \quad p < 1. \tag{3.2}$$

The frozen path argument will show that for

$$\alpha + \beta > 1, \quad \alpha < 1, \tag{3.3}$$

and

$$q = \frac{\beta}{\alpha + 2\beta - 1}, \tag{3.4}$$

the solution $\boldsymbol{x}_\varepsilon(t)$ of (3.2) converges to an FBM.

First we note that the frozen path approximation

$$\boldsymbol{x}_\varepsilon(t) \approx \tilde{\boldsymbol{x}}_\varepsilon(t) = \boldsymbol{x}_\varepsilon(0) + \frac{1}{\varepsilon^{2q-1}} \int_0^t \mathbf{V}\left(\frac{t_1}{\varepsilon^{2q}}, \frac{\boldsymbol{x}_\varepsilon(0)}{\varepsilon^p} \right) dt_1$$

is accurate with the error $O(\tau^{1+1/(2q)} \varepsilon^{1-p-2q})$ on the (rescaled) time scale τ,

$$\varepsilon^{2q} \ll \tau \ll \varepsilon^{p+2q-1}, \tag{3.5}$$

provided that the scaling exponent q is the right one (i.e., $\boldsymbol{x}_\varepsilon(t), t > 0$ is $O(1)$). The upper limit on τ is imposed in (3.5) because the total error made by the frozen path approximation is then $O(\tau \varepsilon^{1-p-2q})$, which is negligible.

Let us calculate the covariance of the Gaussian increment

$$\Delta \tilde{\boldsymbol{x}}^n_\varepsilon(t) = \varepsilon \int_{n\tau/\varepsilon^{2q}}^{(n\tau+t)/\varepsilon^{2q}} \mathbf{V}\left(s, \frac{\tilde{\boldsymbol{x}}_\varepsilon(n\tau)}{\varepsilon^p} \right) ds, \quad 0 < t \leq \tau,$$

which is stationary in n. Denoting by $\mathbf{R}_s$ the symmetric part of the covariance matrix $\mathbf{R}$, we have

$$
\begin{aligned}
&\mathbb{E}[\Delta\tilde{\mathbf{x}}_\varepsilon^n(t)\otimes\Delta\tilde{\mathbf{x}}_\varepsilon^n(t)]\\
&=2\varepsilon^2\int_0^{t/\varepsilon^{2q}}\int_0^{s_1}\mathbf{R}_s(s_1-s_2,0)\,ds_1\,ds_2\\
&=2\varepsilon^2\int_0^{t/\varepsilon^{2q}}\int_{\mathbb{R}^d}\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d+2\beta-1}}(1-e^{-s_1|\mathbf{k}|^{2\beta}})\,ds_1\,d\mathbf{k}\\
&=2\varepsilon^{2(1-q)}\int_0^t\left(\int_{|\mathbf{k}|^{2\beta}<\varepsilon^{2q}/t_1}+\int_{|\mathbf{k}|^{2\beta}\geq\varepsilon^{2q}/t_1}\right)\\
&\qquad\times\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d+2\beta-1}}(1-e^{-t_1|\mathbf{k}|^{2\beta}/\varepsilon^{2q}})\,d\mathbf{k}\,dt_1.
\end{aligned}
$$

The first integral has the order of magnitude

$$
\varepsilon^{2(1-q)}\int_0^t\int_{|\mathbf{k}|^{2\beta}<\varepsilon^{2q}/t_1}\frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d+2\beta-1}}\times\frac{t_1|\mathbf{k}|^{2\beta}}{\varepsilon^{2q}}\,d\mathbf{k}\,dt_1\sim\varepsilon^{2[1-q(\alpha+2\beta-1)/\beta]}t^{(\alpha+2\beta-1)/\beta}.
$$

The second integral has the order of magnitude

$$
\varepsilon^{2(1-q)}\int_0^t\int_{|\mathbf{k}|^{2\beta}\geq\varepsilon^{2q}/t_1}\frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d+2\beta-1}}\,d\mathbf{k}\,dt_1\sim\varepsilon^{2[1-q(\alpha+2\beta-1)/\beta]}t^{(\alpha+2\beta-1)/\beta}.
$$

They are of the same sign so they do not cancel with each other. With (3.4) both terms behave like the FBM of finite, constant coefficients with the Hurst exponent $H=1/(2q)$ on the (rescaled) time scales in the range given by (3.5). In particular, for $p=0$, the FBM limit holds up to order one time as is rigorously proved in [6]. The scaling with (3.4) is *superdiffusive* since $q<1$ for $\alpha+\beta>1$. This is the result of *positive* correlation between successive increments. For the FBM-like behavior to persist up to order one times for $p>0$ the stationary increments at different times must have the right positive correlation. This is proved in section 6.

4. Estimation by diagrammatic method. We now turn to the proof of Theorem 1. We shall only calculate the mean square displacement of the particle. We make use of a spectral representation of the velocity field as follows. Let $\hat{\mathbf{V}}_0(t,d\mathbf{k}),\hat{\mathbf{V}}_1(t,d\mathbf{k})$ be two independent copies of real $\mathbb{R}^d$-valued, Gaussian, random spectral measures with the structure matrix

$$
\mathbb{E}[\hat{\mathbf{V}}_i(t,d\mathbf{k})\otimes\hat{\mathbf{V}}_i(0,d\mathbf{k})]=\frac{\mathrm{e}^{-|\mathbf{k}|^{2\beta}t}\mathcal{E}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)d\mathbf{k},\quad i=0,1. \tag{4.1}
$$

The modes of the random measure can be intuitively thought of as mutually independent "infinitesimal" Ornstein–Uhlenbeck processes, that is, a stationary solution of a properly understood (e.g., in the sense of generalized functions) stochastic differential equation

$$
\begin{aligned}
&d_t\hat{\mathbf{V}}_i(t,d\mathbf{k})\\
&=-|\mathbf{k}|^{2\beta}\hat{\mathbf{V}}_i(t,d\mathbf{k})dt+|\mathbf{k}|^{(2\beta+1-d)/2}\mathcal{E}^{\frac{1}{2}}(|\mathbf{k}|)\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\mathbf{W}_i(dt,d\mathbf{k}),\quad i=0,1.
\end{aligned} \tag{4.2}
$$

Here $\mathbf{W}_i(dt, d\mathbf{k})$, $i = 0, 1$, are independent $\mathbb{R}^d$-valued, uncorrelated space-time white-noise random measures.

We can write then that

$$\mathbf{V}(t, \mathbf{x}) = \int \hat{\mathbf{V}}(t, \mathbf{x}, d\mathbf{k}), \tag{4.3}$$

with

$$\hat{\mathbf{V}}(t, \mathbf{x}, d\mathbf{k}) := \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{V}}(t, d\mathbf{k}) \tag{4.4}$$

and $\hat{\mathbf{V}}(t, \cdot)$ a $\mathbb{C}^d$-valued, componentwise Gaussian random measure given by

$$\hat{\mathbf{V}}(t, A) := \frac{1}{2}[\hat{\mathbf{V}}_0(t, A) + \hat{\mathbf{V}}_0(t, -A)] + \frac{\mathrm{i}}{2}[\hat{\mathbf{V}}_1(t, A) - \hat{\mathbf{V}}_1(t, -A)]. \tag{4.5}$$

The velocity field is temporally Markovian because for any Borel set A and $s \le t$

$$\mathbb{E}_s\hat{\mathbf{V}}(t, d\mathbf{k}) = \mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}\hat{\mathbf{V}}(s, d\mathbf{k}). \tag{4.6}$$

Here $\mathbb{E}_s$ denotes the conditional expectation with respect to the history of the random field determined up to time s. Another property of temporal dynamics of the field is its *reversibility*, which can be expressed in the following form. For any $s_1 \ge s_2 \ge \cdots s_n \ge 0$ and functions F, G of appropriate arguments, we have

$$\begin{aligned} &\mathbb{E}\left\{\mathbb{E}_0\left[F(\hat{\mathbf{V}}(s_1, d\mathbf{k}_1), \ldots, \hat{\mathbf{V}}(s_n, d\mathbf{k}_n))\right] G(\hat{\mathbf{V}}(0, d\mathbf{k}_{n+1}))\right\} \\ &\quad = \mathbb{E}\left[F(\hat{\mathbf{V}}(s_1 - s_n, d\mathbf{k}_1), \ldots, \hat{\mathbf{V}}(0, d\mathbf{k}_n))\mathbb{E}_0 G(\hat{\mathbf{V}}(s_n, d\mathbf{k}_{n+1}))\right]. \end{aligned} \tag{4.7}$$

As explained in the introduction, the molecular diffusion has only a perturbative effect and will be set to zero to simplify the calculation. The motion of the tracer is then described by

$$\frac{d\boldsymbol{x}(t)}{dt} = \mathbf{V}(t, \varepsilon^{1-p}\boldsymbol{x}(t)). \tag{4.8}$$

Let us set

$$\boldsymbol{x}^{\varepsilon}(t) = \varepsilon \int_0^{t\varepsilon^{-2q}} \mathbf{V}(s, \varepsilon^{1-p}\boldsymbol{x}(s))ds, \tag{4.9}$$

where $p < 1$ and $\mathbf{x}(t)$ is given by (4.8) and q is to be specified later.

For any $t \ge s$ define $\Delta_n(t, s) := [(s_1, \ldots, s_{n+1}) : t \ge s_1 \ge \cdots \ge s_{n+1} \ge s]$. To compute the mean square displacement of the particle we write

$$\begin{aligned} \mathbb{E}\left[\mathbf{x}_\varepsilon(t) \otimes \mathbf{x}_\varepsilon(t)\right] &= \varepsilon^2 \int_0^{t\varepsilon^{-2q}} ds \int_0^s \left\{\mathbb{E}\left[\mathbf{V}(s_1, \varepsilon^{1-p}\mathbf{x}(s_1)) \otimes \mathbf{V}(0, \mathbf{0})\right]\right. \\ &\qquad \left. + \mathbb{E}\left[\mathbf{V}(0, \mathbf{0}) \otimes \mathbf{V}(s_1, \varepsilon^{1-p}\mathbf{x}(s_1))\right]\right\} ds_1 \\ &= \sum_{n=0}^{N-1} \mathcal{I}_{n,\varepsilon}(t) + \mathcal{R}_{N,\varepsilon}(t), \end{aligned} \tag{4.10}$$

with $\mathcal{I}_{n,\varepsilon}(t)$ the symmetric part of the matrix

$$\mathcal{I}^0_{n,\varepsilon}(t) := 2\varepsilon^{n(1-p)+2}\int_0^{t\varepsilon^{-2q}} ds \int\cdots\int_{\Delta_n(s,0)} \mathbb{E}\left\{\mathbb{E}_0\left[\mathbf{W}_n(s_1,\ldots,s_{n+1},\mathbf{0})\right] \otimes \mathbf{V}(0,\mathbf{0})\right\} ds_1\cdots ds_{n+1} \tag{4.11}$$

and $\mathcal{R}_{N,\varepsilon}(t)$ the symmetric part of the matrix

$$\mathcal{R}^0_{N,\varepsilon}(t) = 2\varepsilon^{N(1-p)+2}\int_0^{t\varepsilon^{-2q}} ds \int\cdots\int_{\Delta_N(s,0)} \mathbb{E}\left\{\mathbb{E}_{s_{N+1}}\left[\mathbf{W}_N(s_1,\ldots,s_{N+1},\varepsilon^{1-p}\mathbf{x}(s_{N+1}))\right] \otimes \mathbf{V}(0,\mathbf{0})\right\} ds_1\cdots ds_{N+1},$$

where $\mathbf{W}_n(\cdot)$ is defined inductively by

$$\mathbf{W}_0(s_1,\mathbf{x}) := \mathbf{V}(s_1,\mathbf{x}), \tag{4.12}$$

$$\mathbf{W}_n(s_1,\ldots,s_{n+1},\mathbf{x}) := \mathbf{V}(s_{n+1},\mathbf{x}) \cdot\nabla\mathbf{W}_{n-1}(s_1,\ldots,s_n,\mathbf{x}) \qquad \text{for } n=1,2,\ldots. \tag{4.13}$$

To estimate both $\mathcal{I}_n$ and the remainder term $\mathcal{R}_{N,\varepsilon}(t)$ we shall deal with expectations of polynomial-like expressions in a Gaussian variable. To calculate the expectation of multiple product of Gaussian random variables, we use Feynman diagrams borrowed from quantum field theory (see, e.g., [9] and [10]). A *Feynman diagram* $\mathcal{F}$ (of order n = number of vertexes and rank r = number of edges) is a graph consisting of a set $B(\mathcal{F})$ of n vertexes and a set $E(\mathcal{F})$ of r edges without common endpoints. So there are r pairs of vertexes, each joined by an edge, and $n-2r$ unpaired vertexes, called *free vertexes*. Let $B(\mathcal{F})$ be a subset of positive integers. An edge whose endpoints are $m,n\in B$ is represented by $\widehat{mn}$ (unless otherwise specified, we always assume $m<n$); an edge includes its endpoints. A diagram $\mathcal{F}$ is said to be *based on* $B(\mathcal{F})$. Denote the set of free vertexes by $A(\mathcal{F})$, so $A(\mathcal{F}) = \mathcal{F}\setminus E(\mathcal{F})$. The diagram is *complete* if $A(\mathcal{F})$ is empty and *incomplete* otherwise. Denote by $\mathcal{G}(B)$ the set of all diagrams based on B, by $\mathcal{G}_c(B)$ the set of all complete diagrams based on B, and by $\mathcal{G}_{\text{in}}(B)$ the set of all incomplete diagrams based on B. A diagram $\mathcal{F}'\in\mathcal{G}_c(B)$ is called a *completion* of $\mathcal{F}\in\mathcal{G}_i(B)$ if $E(\mathcal{F})\subseteq E(\mathcal{F}')$.

Let $\mathbb{Z}_n := \{1,2,3,\ldots,n\}$. For $n\geq 1$ we define inductively a class $\mathfrak{S}_n$ of certain Feynman diagrams based on $\mathbb{Z}_n$ as follows. For $n=1$, $\mathfrak{S}_1$ consists of the trivial diagram $\mathcal{F}$ with vertex 1. Given $\mathfrak{S}_{n-1}$, $\mathfrak{S}_n$ consists of all the *descendants* of $\mathfrak{S}_{n-1}$. A descendant $\mathcal{F}'$ of $\mathcal{F}\in\mathfrak{S}_{n-1}$ is a graph based on $\mathbb{Z}_n$ such that $A(\mathcal{F}')\neq\emptyset$ and

$$\mathcal{F}'_{|n-1} = \mathcal{F}, \tag{4.14}$$

where $\mathcal{F}'_{|n-1}$ is the restriction of $\mathcal{F}'$ to $\mathbb{Z}_{n-1}$ with edges of the type $\widehat{mn}$, $m=1,2,\ldots,n-1$, deleted. We call $\mathcal{F}$ the *predecessor* of $\mathcal{F}'$. The predecessor of any $\mathcal{F}'\in\mathfrak{S}_n$ is clearly unique. For $\mathcal{F}\in\mathfrak{S}_n$ set $A_k(\mathcal{F}) = A(\mathcal{F}_{|k})$, $k=1,2,\ldots,n$.

Adopting the multi-index notation for any $N\in\mathbb{Z}^+$, $\mathbf{n}=(n_1,\ldots,n_{N+1})$, $n_j\in\{1,2,3,\ldots,d\}$, and $|\mathbf{n}| := n_1+n_2+\ldots+n_{N+1}$, we have the following formula.

LEMMA 1. *Let $N\geq 1$ and $s_1\geq s_2\geq\cdots\geq s_{N+1}\geq 0$, $i\in\{1,\ldots,d\}$, $\mathbf{x}\in\mathbb{R}^d$. We have then that*

(4.15)

$$\mathbb{E}_{s_{N+1}} W_{N,i}(s_1,\ldots,s_{N+1},\mathbf{x}) = \sum_{\mathbf{n},\mathcal{F}} \int\cdots\int \exp\left\{ \mathrm{i} \sum_{m\in A_N(\mathcal{F})\cup\{N+1\}} \mathbf{k}_m\cdot\mathbf{x} \right\}$$

$$\times\, \mathrm{i}^N \prod_{j=1}^{N} \left(\sum_{m\in A_j(\mathcal{F})} \mathbf{k}_m \right) \exp\left\{ - \sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_j - s_{j+1}) \right\}$$

$$\times\, C_{\mathbf{n},N} \prod_{\widehat{mm'}\in E(\mathcal{F})} \mathbb{E}\left[\widehat{V}_{n_m}(0,d\mathbf{k}_m)\widehat{V}_{n_{m'}}(0,d\mathbf{k}_{m'}) \right] \prod_{m\in A_N(\mathcal{F})\cup\{N+1\}} \widehat{V}_{n_m}(s_{N+1},d\mathbf{k}_m),$$

where $|C_{\mathbf{n},N}|\le 1$. *The summation extends over all integer valued multi-indices* $\mathbf{n} = (n_1,\ldots,n_{N+1})$, $n_1 = i$, *and all Feynman diagrams* $\mathcal{F}\in\mathfrak{S}_N$.

The proof of Lemma 1 is a straightforward moment calculation with jointly Gaussian random variables using spectral representation (4.3)–(4.4). The free vertexes arise from centering and the edges from covariance of each pair. The condition $A(\mathcal{F}')\neq\emptyset$ is due to the gradient operation. The term $C_{\mathbf{n},N}$ contains an $O(1)$ factor like

$$\prod_{\widehat{mm'}\in E(\mathcal{F})} \left[1 - e^{-2|\mathbf{k}_{m'}|^{2\beta}(s_{m'}-s_{N+1})} \right]$$

resulting from replacing the conditional covariance by the covariance of the pairing (cf. [6]).

Using Lemma 1 we can write that

$$\text{(4.16)}\quad \int_0^{t\varepsilon^{-2q}} ds \int\cdots\int_{\Delta_N(s,0)} \mathbb{E}_{s_{N+1}} W_{N,i}(s_1,\ldots,s_{N+1},\mathbf{x})ds_1\cdots ds_{N+1}$$

$$= \sum \int_0^{t\varepsilon^{-2q}} ds \int_0^{s} ds' \int\cdots\int_{\Delta_{N-1}(s,s')} \varphi_N(\mathbf{k}_1,\ldots,\mathbf{k}_N) P_N(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_N;\mathcal{F})$$

$$\times \prod_{\widehat{mm'}\in E(\mathcal{F})} \mathbb{E}\left[\widehat{V}_{n_m}(0,d\mathbf{k}_m)\widehat{V}_{n_{m'}}(0,d\mathbf{k}_{m'}) \right] \prod_{m\in A_N(\mathcal{F})\cup\{N+1\}} \widehat{V}_{n_m}(s',d\mathbf{k}_m)$$

for $i = 1,\ldots,d$. Here,

(4.17)

$$\varphi_N(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_N) := \mathrm{i}^N C_{\mathbf{n},N} \prod_{j=1}^{N} \frac{\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}}{1-\exp\left\{ -\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} t\varepsilon^{-2q} \right\}} \times \frac{\sum_{m\in A_j(\mathcal{F})} \mathbf{k}_m}{\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|}$$

$$\times \exp\left\{ \mathrm{i} \sum_{m\in A_N(\mathcal{F})\cup\{N+1\}} \mathbf{k}_m\cdot\mathbf{x} \right\}$$

$$\times \int\cdots\int_{\Delta_{N-1}(s,s')} \prod_{j=1}^{N} \exp\left\{ - \sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_j - s_{j+1}) \right\} ds_1\cdots ds_N$$

and

(4.18)

$$P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N;\mathcal{F}) = \prod_{j=1}^{N}\left\{\left(\sum_{m\in A_j(\mathcal{F})}|\mathbf{k}_m|\right)\times\frac{1-\exp\left\{-\sum_{m\in A_j(\mathcal{F})}|\mathbf{k}_m|^{2\beta}t\varepsilon^{-2q}\right\}}{\sum_{m\in A_j(\mathcal{F})}|\mathbf{k}_m|^{2\beta}}\right\}.$$

It is elementary to check that

$$|\varphi_N(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_N)|\le 1. \tag{4.19}$$

To further estimate the expression (4.16) we need to have more refined analysis of the graphs $\mathcal{F}\in\mathfrak{S}_N$. Define

$$r_N(\mathcal{F}) := \max\{m : m\in A_N(\mathcal{F})\}, \tag{4.20}$$
$$a_N(\mathcal{F}) := \min\{m : m\in A_N(\mathcal{F})\}. \tag{4.21}$$

We define $a_k(\mathcal{F}),\ k<N$, as

$$a_{k-1}(\mathcal{F}) = \min\{m : m\in A_{a_k}(\mathcal{F})\}$$

successively unless $a_k=1$. In other words, a_{k-1} is the smallest integer which is the left endpoint of an edge with its right endpoint greater than a_k; cf. (4.14). Below we will use the short-hand notation $a_k := a_k(\mathcal{F})$. Note that $A_N(\mathcal{F})$ and $A_{a_k-1}(\mathcal{F}),\ a_k>1$, are mutually disjoint. Let

$$\mathcal{A}(\mathcal{F}) := A_N(\mathcal{F}) \bigcup_{k:\,a_k>1} A_{a_k-1}(\mathcal{F}). \tag{4.22}$$

Observe that any vertex $m\in\mathcal{A}(\mathcal{F})$ cannot be a right endpoint of any edge in $E(\mathcal{F})$. For any $m\in\mathcal{A}(\mathcal{F})$ let m^* be the nearest vertex in $A(\mathcal{F})$ to the right of m, i.e.,

$$m^* := \min[k:\ k\in\mathcal{A}(\mathcal{F}),\ k>m] \tag{4.23}$$

if the relevant set is nonempty; otherwise, set $m^*:=N$. Let

$$q_m := \#\{\widehat{pp'}\in E(\mathcal{F}) : m<p'<m^*\} \tag{4.24}$$

and let $e(\mathcal{F})$, $c(\mathcal{F})$ be the cardinalities of $E(\mathcal{F})$ and $A_N(\mathcal{F})$, respectively. It is easy to see that

$$\sum_{m\in\mathcal{A}} q_m = e(\mathcal{F}), \tag{4.25}$$

and thus

$$\sum_{m\in\mathcal{A}(\mathcal{F})} q_m + e(\mathcal{F}) + c(\mathcal{F}) = N. \tag{4.26}$$

4.1. Estimates for the remainder terms $\mathcal{R}_{N,\varepsilon}(t)$. By the Cauchy–Schwartz inequality we get that

$$|\mathcal{R}_{N,\varepsilon}(t)|^2 \le 4\varepsilon^{2N(1-p)}\mathbb{E}|\mathbf{V}(0,\mathbf{0})|^2 \times \max_{0\le s\le t\varepsilon^{-2}} \mathbb{E}\left|\int_0^s ds' \int\cdots\int_{\Delta_{N-1}(s,s')} \mathbb{E}_{s'}\mathbf{W}_N(s_1,\ldots,s_N,s',\varepsilon\mathbf{x}(s'))ds_1\cdots ds_N\right|^2. \tag{4.27}$$

The stationarity of the Lagrangian velocity field implies that the right-hand side of (4.27) is equal to

$$4\varepsilon^{2N(1-p)}\mathbb{E}|\mathbf{V}(0,\mathbf{0})|^2 \max_{0\le s\le t\varepsilon^{-2}} \mathbb{E}\left|\int_0^s ds' \int\cdots\int_{\Delta_N(s',0)} \mathbb{E}_0\mathbf{W}_N(s_1,\ldots,s_N,0,\mathbf{0})ds_1\cdots ds_N\right|^2. \tag{4.28}$$

Subsequently using (4.16) for the multiple time integration of the conditional expectations in (4.28), we deduce that the above expression is less than or equal to

$$4C\varepsilon^{2N(1-p)}t^2\varepsilon^{4(1-2q)}\mathbb{E}\left|\sum_{\mathcal{F},\mathbf{n}}\int\cdots\int \varphi_N(\mathbf{x},\mathbf{k}_1,\ldots,\mathbf{k}_N)P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N;\mathcal{F}) \times \prod_{\widehat{mm'}\in E(\mathcal{F})} \mathbb{E}\left[\widehat{V}_{n_m}(0,d\mathbf{k}_m)\widehat{V}_{n_{m'}}(0,d\mathbf{k}_{m'})\right] \prod_{m\in A_N(\mathcal{F})\cup\{N+1\}} \widehat{V}_{n_m}(0,d\mathbf{k}_m)\right|^2. \tag{4.29}$$

The summation above extends over all Feynman diagrams $\mathcal{F}\in\mathfrak{S}_N$ and multi-indices $\mathbf{n}$.

By introducing an identical copy of the diagram which is supported on $\{N+2,N+2,\ldots,2N+2\}$, the expression in (4.29) can be written in the form

$$\begin{aligned}4C\varepsilon^{2N(1-p)}t^2\varepsilon^{4(1-2q)}\sum\int\cdots\int &\varphi_N(\mathbf{0},\mathbf{k}_1,\ldots,\mathbf{k}_N)\varphi_N(\mathbf{0},\mathbf{k}'_1,\ldots,\mathbf{k}'_N)\\ &P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N;\mathcal{F})P_N(\mathbf{k}'_1,\ldots,\mathbf{k}'_N;\mathcal{F})\\ \times\Bigg|\prod_{\widehat{mm'}\in E(\mathcal{F})} &\mathbb{E}\left[\widehat{V}_{n_m}(0,d\mathbf{k}_m)\widehat{V}_{n_{m'}}(0,d\mathbf{k}_{m'})\right]\\ \times\prod_{\widehat{mm'}\in E(\mathcal{F})} &\mathbb{E}\left[\widehat{V}_{n_m}(0,d\mathbf{k}'_m)\widehat{V}_{n_{m'}}(0,d\mathbf{k}'_{m'})\right]\mathbb{E}\left[\prod_{m\in A^{(2)}(\mathcal{F})}\widehat{V}_{n_m}(0,d\mathbf{k}_m)\right]\Bigg|,\end{aligned}$$

where $\mathbf{k}_{N+1+j} := \mathbf{k}'_j$ and

$$A^{(2)}(\mathcal{F}) = A_N(\mathcal{F})\cup\{N+1\}\cup\{j+N+1 : j\in A_N(\mathcal{F})\cup\{N+1\}\}. \tag{4.30}$$

Using the elementary inequality

$$\frac{1-\mathrm{e}^{-xt/\varepsilon^2}}{x} \le \frac{C}{x+\varepsilon^{2q}/t}, \quad t,x>0,$$

for a constant C independent of ε, x, we conclude that (see (4.18))

$$|P_N(\mathbf{k}_1,\ldots,\mathbf{k}_N;\mathcal{F})| \le \prod_{j=1}^{N} \frac{\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|}{\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} + \varepsilon^{2q}/t}.$$

The expression (4.29) can be now estimated by

$$\begin{aligned}(4.31)\qquad & Ct^2\varepsilon^{4(1-2q)}\sum\int_0^K\cdots\int_0^K Q_N(k_1,\ldots,k_N;\mathcal{F})\,Q_N(k_1',\ldots,k_N';\mathcal{F})\\ &\times \prod_{\widehat{mm'}\in E(\mathcal{F})}\left[\frac{\delta(k_m-k_{m'})dk_m dk_{m'}}{k_m^{2\alpha-1}}\times\frac{\delta(k_m'-k_{m'}')dk_m dk_{m'}}{{k_m'}^{2\alpha-1}}\right]\\ &\times \prod_{\widehat{mm'}\in E(\mathcal{F}')}\frac{\delta(k_m-k_{m'})dk_m dk_{m'}}{k_m^{2\alpha-1}},\end{aligned}$$

with $k_m = |\mathbf{k}_m|$ and

$$Q_N(k_1,\ldots,k_N;\mathcal{F}) := \prod_{j=1}^{N}\frac{\sum_{m\in A_j(\mathcal{F})} k_m}{\sum_{m\in A_j(\mathcal{F})} k_m^{2\beta} + \varepsilon^{2q}/t}.$$

The summation extends over all Feynman diagrams $\mathcal{F}\in\mathfrak{S}_N$ and all complete diagrams $\mathcal{F}'$ made of the vertexes of $A^{(2)}(\mathcal{F})$.

When $2\beta \le 1$, $Q_N(k_1,\ldots,k_N;\mathcal{F})$ is bounded. This in turn implies that the expression (4.31) diverges at most at the rate $\varepsilon^{4(1-2q)}$. The estimate (4.27) implies then that $\mathcal{R}_{N,\varepsilon}(t)$ vanishes with $\varepsilon \downarrow 0$ and $N > 2/(1-p)$.

Let us assume therefore that $2\beta > 1$. There exists then a constant C, depending only on t, N, β, and K, such that

$$(4.32)\qquad \frac{\sum_{m\in A_j(\mathcal{F})} k_m}{\sum_{m\in A_j(\mathcal{F})} k_m^{2\beta} + \varepsilon^{2q}/t} \le C\frac{k_{m_j} + \varepsilon^{q/\beta}}{k_{m_j}^{2\beta}+\varepsilon^{2q}}\quad \forall m_j\in A_j(\mathcal{F})$$

and thus

$$(4.33)\qquad Q_N(k_1,\ldots,k_N;\mathcal{F}) \le C\prod_{j=1}^{N}\frac{k_{m_j}+\varepsilon^{q/\beta}}{k_{m_j}^{2\beta}+\varepsilon^{2q}}$$

for all $m_j \in A_j(\mathcal{F})$. Hereby we make the following definite choice of m_j: let $m_j := j$ if j is *not* the right endpoint of an edge of the diagram $\mathcal{F}$. Otherwise, let m_j be the closest vertex from $\mathcal{A}(\mathcal{F})$ to the left of j.

Denote by $E'(\mathcal{F})$ the set of the edges of the diagram $\mathcal{F}$ with neither endpoint belonging to $\mathcal{A}(\mathcal{F})$ (see (4.22)) by cardinality of e'. In view of (4.25), (4.26), and the identity

$$(4.34)\qquad e'(\mathcal{F}) + \#[A(\mathcal{F})\setminus A_N(\mathcal{F})] = e(\mathcal{F}),$$

the expression on the right-hand side of (4.33) can be written as

$$C \prod_{\widehat{mm'}\in E'(\mathcal{F})} \frac{k_m+\varepsilon^{q/\beta}}{k_m^{2\beta}+\varepsilon^{2q}} \times \prod_{m\in\mathcal{A}(\mathcal{F})\setminus A_N(\mathcal{F})} \left(\frac{k_m+\varepsilon^{q/\beta}}{k_m^{2\beta}+\varepsilon^{2q}}\right)^{q_m+1} \times \prod_{m\in A_N(\mathcal{F})} \left(\frac{k_m+\varepsilon^{q/\beta}}{k_m^{2\beta}+\varepsilon^{2q}}\right)^{q_m+1}. \tag{4.35}$$

From (4.31), (4.33), and (4.35) we conclude that

$$|\mathcal{R}_{N,\varepsilon}(t)|^2 \le C\varepsilon^{2N(1-p)+4(1-2q)} \sum \prod_{m\in\mathcal{A}(\mathcal{F})\setminus A_N(\mathcal{F})} \left[\int_0^K \left(\frac{k+\varepsilon^{q/\beta}}{k^{2\beta}+\varepsilon^{2q}}\right)^{q_m+1} \frac{dk}{k^{2\alpha-1}}\right]^2 \times \left[\int_0^K \frac{(k+\varepsilon^{q/\beta})dk}{(k^{2\beta}+\varepsilon^{2q})k^{2\alpha-1}}\right]^{2e'} \prod_{\widehat{mm'}\in E(\mathcal{F}')} \int_0^K \left(\frac{k+\varepsilon^{q/\beta}}{k^{2\beta}+\varepsilon^{2q}}\right)^{2+q_m+q_{m'}} \frac{dk}{k^{2\alpha-1}}. \tag{4.36}$$

Here the summation extends over all possible diagrams $\mathcal{F}$, $\mathcal{F}'$ as in (4.31). The meanings of q_m's related to the diagram $\mathcal{F}$ are the same as introduced in the previous section. We adopt also the convention that $q_{N+1} = q_{2N+2} = -1$ and $q_{N+1+m} := q_m$.

4.2. Estimates for $\mathcal{I}_{n,\varepsilon}(t)$ for $n \ge 1$. The calculation is similar to that for the remainder term carried out in the previous section, so we shall sketch only the main points.

From (4.16) we infer that the i,jth entry of the matrix $\mathcal{I}_{n,\varepsilon}(t)$, given by (4.11), equals

$$2\varepsilon^{n(1-p)+2} \sum \int_0^{t\varepsilon^{-2q}} ds \int\cdots\int \varphi_{n+1}(\mathbf{k}_1,\dots,\mathbf{k}_{n+1}) \times P_n(\mathbf{k}_1,\dots,\mathbf{k}_n;\mathcal{F}) \left(\sum_{m\in A_{n+1}(\mathcal{F})} |\mathbf{k}_m|^{2\beta}+\varepsilon^{2q}/t\right)^{-1} \times \left|\prod_{\widehat{mm'}\in E(\mathcal{F})} \mathbb{E}\left[\widehat{V}_{l_m}(0,d\mathbf{k}_m)\widehat{V}_{l_{m'}}(0,d\mathbf{k}_{m'})\right] \mathbb{E}\left[\prod_{m\in A_{n+1}(\mathcal{F})\cup\{n+2\}} \widehat{V}_{l_m}(0,d\mathbf{k}_m)\right]\right|. \tag{4.37}$$

Here the summation extends over all multi-indices $\mathbf{l} = (l_1,\dots,l_{n+2})$ such that $l_1 = i$, $l_{n+2} = j$, and all Feynman diagrams $\mathcal{F} \in \mathfrak{S}_{n+1}$. Proceeding with the same type of estimates as in the case of the remainder term we conclude that

$$|\mathcal{I}_{n,\varepsilon}(t)| \le Ct\varepsilon^{n(1-p)} \sum \prod_{m\in\mathcal{A}(\mathcal{F})\setminus A_{n+1}(\mathcal{F})} \int_0^K \left(\frac{\varepsilon^{\frac{q}{\beta}}+k}{\varepsilon^{2q}+k^{2\beta}}\right)^{q_m+1} \frac{dk}{k^{2\alpha-1}} \times \left[\int_0^K \frac{(\varepsilon^{\frac{q}{\beta}}+k)dk}{(\varepsilon^{2q}+k^{2\beta})k^{2\alpha-1}}\right]^{e'} \prod_{\widehat{mm'}\in\mathcal{F}'} \int_0^K \frac{(\varepsilon^{\frac{q}{\beta}}+k)^{2+q_m+q_{m'}-r_{m,m'}}}{(\varepsilon^{2q}+k^{2\beta})^{2+q_m+q_{m'}}} \times \frac{dk}{k^{2\alpha-1}}. \tag{4.38}$$

Here the summation extends over all Feynman diagrams $\mathcal{F} \in \mathfrak{S}_{n+1}$ and all complete diagrams $\mathcal{F}'$ made of the vertexes of $A_{n+1}(\mathcal{F}) \cup \{n+2\}$; $r_{m,m'} := \delta_{m,m_{n+1}} + \delta_{m',m_{n+1}}$. Also, we adopt convention $q_{n+2} = -1$.

5. Case 1. $\alpha+\beta<1$, $2p\beta<1$, $0\le p<1$—Brownian motion ($q=1$). We shall give the proof only in the case $2p\beta<1$. Also, for clarity we shall calculate only the asymptotic of the mean square displacement of $\mathbf{x}_\varepsilon(t)$, referring an interested reader to our paper [5], where the proof of the martingale version of our theorem has been laid out for $p=0$. A suitable adaptation of the proof to the case $p\in[0,1)$ and $2p\beta<1$ is possible along the lines of the argument we present below.

After an elementary calculation we deduce that under the assumption $\alpha+\beta<1$

$$\lim_{\varepsilon\downarrow 0}\mathcal{I}_{0,\varepsilon}(t)=\mathbf{D}t$$

with

$$\mathbf{D}=\int_{\mathbb{R}^d}\left(\mathbf{I}-\frac{\mathbf{k}\otimes\mathbf{k}}{|\mathbf{k}|^2}\right)\frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+2\beta-1}}\frac{d\mathbf{k}}{|\mathbf{k}|^{d-1}},$$

provided that $q=1$.

Estimates for $\mathcal{R}_{N,\varepsilon}(t)$. We observe that

$$\int_0^K\frac{k+\varepsilon^{1/\beta}}{k^{2\beta}+\varepsilon^2}\frac{dk}{k^{2\alpha-1}}\le C, \tag{5.1}$$

$$\int_0^K\left(\frac{\varepsilon^{1/\beta}+k}{\varepsilon^2+k^{2\beta}}\right)^{q_m+1}\frac{dk}{k^{2\alpha-1}}\le C(1+\varepsilon^{\gamma(m)}), \tag{5.2}$$

$$\int_0^K\left(\frac{\varepsilon^{1/\beta}+k}{\varepsilon^2+k^{2\beta}}\right)^{2+q_m+q_{m'}}\frac{dk}{k^{2\alpha-1}}\le C(1+\varepsilon^{\gamma(\widehat{mm'})}), \tag{5.3}$$

with

$$\gamma(m):=\frac{1}{\beta}[2-2\alpha+(q_m+1)(1-2\beta)], \tag{5.4}$$

$$\gamma(\widehat{mm'}):=\frac{1}{\beta}[2-2\alpha+(q_m+q_{m'}+2)(1-2\beta)]. \tag{5.5}$$

We conclude therefore that

$$|\mathcal{R}_{N,\varepsilon}(t)|^2\le C\varepsilon^{\mu}, \tag{5.6}$$

with

$$\mu:=2N(1-p)-4+\kappa, \tag{5.7}$$

$$\kappa:=\frac{1}{\beta}\Big[2f'(2-\alpha-2\beta)+2f''(3-2\alpha-2\beta) \tag{5.8}$$
$$+(1-2\beta)\sum{}'(q_m+q_{m'})+2(1-2\beta)\sum{}''q_m\Big],$$

where the summation $\sum'$ extends over the edges $\widehat{mm'}$ of the diagram $\mathcal{F}'$ for which $\gamma(\widehat{mm'})<0$ and $\sum''$ extends over the vertexes m of $\mathcal{A}(\mathcal{F})\setminus A_N(\mathcal{F})$, for which $\gamma(m)<0$ (see (5.4), (5.5)) and f',f'' denote the cardinalities of the respective sets of edges and vertexes. Obviously,

$$f'\le c,\quad f''\le e-e', \tag{5.9}$$

with c the cardinality of $A_N(\mathcal{F})$ and e the number of edges of $\mathcal{F}$ (cf. (4.34)). Note that $c + 2e = N$.

Using $\sum'(q_m + q_{m'}) + 2\sum'' q_m \le 2e$ and $2\beta > 1$, we can write that

$$\kappa \ge \frac{2}{\beta}\left[f'(2-\alpha-2\beta) + f''(3-2\alpha-2\beta) + e(1-2\beta)\right]. \tag{5.10}$$

Since $N = c + 2e$ we conclude from (5.9) and (5.10) that

$$\begin{aligned}\mu &\ge -4 + 2N(1-p) + \frac{2}{\beta}\left[f'(2-\alpha-2\beta) + f''(3-2\alpha-2\beta) + e(1-2\beta)\right] \\ &\ge -4 + 2(c-f')(1-p) \\ &\quad + \frac{2}{\beta}\left\{f'[2-\alpha-(1+p)\beta] + f''(3-2\alpha-2\beta) + e(1-2p\beta)\right\} > 0,\end{aligned} \tag{5.11}$$

provided that $2p\beta < 1$ (note that then necessarily $2-\alpha-(1+p)\beta > 0$) and N is sufficiently large.

Estimates for $\mathcal{I}_{n,\varepsilon}(t)$ for $n \ge 1$. Using (5.1)–(5.2) and

$$\int_0^K \frac{(k+\varepsilon^{\frac{1}{\beta}})^{2+q_m+q_{m'}-r_{m,m'}}}{(k^{2\beta}+\varepsilon^2)^{2+q_m+q_{m'}}}\,\frac{dk}{k^{2\alpha-1}} \le C(1+\varepsilon^{\tilde\gamma(\widehat{mm'})}),$$

with

$$\tilde\gamma(\widehat{mm'}) := \frac{1}{\beta}[2-2\alpha+(q_m+q_{m'}+2)(1-2\beta) - r_{m,m'}], \tag{5.12}$$

we conclude that

$$|\mathcal{I}_{n,\varepsilon}(t)| \le C\varepsilon^{\mu}, \tag{5.13}$$

where $\mu = n(1-p) + \kappa$ and

$$\begin{aligned}\kappa := \frac{1}{\beta}\Big[&2f'(2-\alpha-2\beta) + f''(3-2\alpha-2\beta) \\ &+(1-2\beta)\sum{}'(q_m+q_{m'}) + (1-2\beta)\sum{}'' q_m - 1\Big].\end{aligned}$$

The summation $\sum'$ extends over the edges $\widehat{mm'}$ of the diagram $\mathcal{F}'$ for which $\tilde\gamma(\widehat{mm'}) < 0$ and $\sum''$ extends over the vertexes m of $\mathcal{A}(\mathcal{F}) \setminus A_{n+1}(\mathcal{F})$ for which $\gamma(m) < 0$. f', f'' denote the cardinalities of the respective sets of edges and vertexes. Finally, obtain that

$$\begin{aligned}\mu &\ge (c+1-2f'+e)(1-p) \\ &\quad + \frac{1}{\beta}\left[2p\beta + 2f'(2-\alpha-(1+p)\beta) + f''(3-2\alpha-2\beta) + e(1-2p\beta)\right] > 0.\end{aligned}$$

In conclusion, we proved that the utmost left-hand side of (4.10) tends to $\mathbf{D}t$ as $\varepsilon \downarrow 0$, provided that $\alpha + \beta < 1$.

6. Case 2. $1 < \alpha + \beta < 1 + 1/p$, $0 \le p < 1$—FBM. For $\alpha + \beta > 1$, it is straightforward to check that

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}_{0,\varepsilon} = \mathbf{D} t^{2H},$$

provided that $q = \beta/(\alpha + 2\beta - 1)$. Here

$$\mathbf{D} = \int_{\mathbb{R}^d} \frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}\right) \frac{a(0)}{|\mathbf{k}|^{d-1}} d\mathbf{k}$$

and the Hurst exponent H is given by

$$1/2 < H = 1/2 + \frac{\alpha + \beta - 1}{2\beta} < 1.$$

6.1. Case 2a. We assume that

$$3/2 < \alpha + \beta, \quad \alpha + 2\beta < 1 + 1/p, \quad 0 \le p < 1.$$

We shall only carry out the estimates of $\mathcal{R}_{N,\varepsilon}(t)$. One can easily obtain the respective estimates of $\mathcal{I}_{n,\varepsilon}(t)$. These estimates are very similar to the corresponding part of section 5. We use the notation introduced there.

As before we need only to consider the case $2\beta > 1$ (cf. (4.36)). Note that

$$\int_0^K \frac{(\varepsilon^{q/\beta} + k) dk}{(\varepsilon^{2q} + k^{2\beta}) k^{2\alpha - 1}} \le C(1 + \varepsilon^{\gamma}),$$

$$\int_0^K \left(\frac{\varepsilon^{q/\beta} + k}{\varepsilon^{2q} + k^{2\beta}}\right)^{q_m + 1} \frac{dk}{k^{2\alpha - 1}} \le C(1 + \varepsilon^{\gamma(m)}),$$

and

$$\int_0^K \left(\frac{\varepsilon^{q/\beta} + k}{\varepsilon^{2q} + k^{2\beta}}\right)^{2 + q_m + q_{m'}} \times \frac{dk}{k^{2\alpha - 1}} \le C(1 + \varepsilon^{\gamma(\widehat{mm'})}),$$

with

$$\gamma := \frac{3 - 2\alpha - 2\beta}{\alpha + 2\beta - 1}, \tag{6.1}$$

$$\gamma(m) := \frac{3 - 2\alpha - 2\beta + q_m(1 - 2\beta)}{\alpha + 2\beta - 1}, \tag{6.2}$$

$$\gamma(\widehat{mm'}) := \frac{4 - 2\alpha - 4\beta + (q_m + q_{m'})(1 - 2\beta)}{\alpha + 2\beta - 1} \tag{6.3}$$

(cf. (5.4)–(5.5)).

Estimating the same way as in (5.6)–(5.11) we obtain

$$|\mathcal{R}_{N,\varepsilon}(t)|^2 \le C t^4 \varepsilon^{\mu},$$

with

$$\mu := 2N(1-p) + 4(1-2q) + \kappa, \tag{6.4}$$

$$\kappa := 2(e' + f'')\frac{3-2\alpha-2\beta}{\alpha+2\beta-1}$$

$$+\frac{1}{\alpha+2\beta-1}\left[2f'(2-\alpha-2\beta) + (1-2\beta)\sum{}'(q_m + q_{m'}) + 2(1-2\beta)\sum{}'' q_m\right]$$

(cf. (5.9)). We have

$$\begin{aligned}\mu &\geq 4(1-2q) + 2N(1-p) + 2(e'+f'')\frac{3-2\alpha-2\beta}{\alpha+2\beta-1}\\ &\quad + \frac{2}{\alpha+2\beta-1}\left[f'(2-\alpha-2\beta) + e(1-2\beta)\right]\\ &\geq 4(1-2q) + 2(c+2e)(1-p) + 2e\frac{3-2\alpha-2\beta}{\alpha+2\beta-1}\\ &\quad + \frac{2}{\alpha+2\beta-1}\left[c(2-\alpha-2\beta) + e(1-2\beta)\right]\\ &\geq 4(1-2q) + 2Np\,\frac{1+1/p-\alpha-2\beta}{\alpha+2\beta-1} > 0,\end{aligned}$$

provided that N is sufficiently large. This in turn implies that $|\mathcal{R}_{N,\varepsilon}(t)|^2$ vanishes as $\varepsilon \downarrow 0$ for such a choice of N.

6.2. Case 2b. Here we assume that

$$1 < \alpha + \beta < 3/2, \quad \alpha + 2\beta < 1 + 1/(2p) + (\alpha+\beta-1)/p, \quad 0 \leq p < 1.$$

In this case one can write κ in (6.4) as

$$\begin{aligned}\kappa &= 2f''\,\frac{3-2\alpha-2\beta}{\alpha+2\beta-1} + \frac{1}{\alpha+2\beta-1}\\ &\quad\times\left[2f'(2-\alpha-2\beta) + (1-2\beta)\sum{}'(q_m+q_{m'}) + 2(1-2\beta)\sum{}'' q_m\right]\end{aligned}$$

and hence

$$\begin{aligned}\mu &\geq 4(1-2q) + 2(c+2e)(1-p)\\ &\quad + 2f''\,\frac{3-2\alpha-2\beta}{\alpha+2\beta-1} + \frac{2}{\alpha+2\beta-1}\left[f'(2-\alpha-2\beta) + e(1-2\beta)\right]\\ &\geq 4(1-2q) + 2ep\,\frac{(\alpha+\beta-1)/p + 1/(2p) - \alpha - 2\beta}{\alpha+2\beta-1} + 2f''\,\frac{3-2\alpha-2\beta}{\alpha+2\beta-1}\\ &\quad + 2(c-f')(1-p) + 2f'p\,\frac{1+1/p-\alpha-2\beta}{\alpha+2\beta-1} > 0,\end{aligned}$$

provided that N is sufficiently large. This in turn implies that $|\mathcal{R}_{N,\varepsilon}(t)|^2$ vanishes as $\varepsilon \downarrow 0$ for such a choice of N.

7. FBM limit with $p \geq 1$: Heuristics. In this section we give an argument indicating that the FBM limit holds for $p > 1$. The argument is similar to the one given in [8].

Let $\mathbf{U}^\varepsilon(t, \mathbf{x})$ be the Gaussian velocity with energy spectrum given by

$$\mathcal{E}_\varepsilon(k) = \frac{a(\varepsilon^p k)}{k^{2\alpha-1}}. \tag{7.1}$$

Then it follows from the spectral representation of the velocity correlation function that $\mathbf{U}^\varepsilon$ is related to $\mathbf{V}$ via

$$\mathbf{V}\left(\frac{t}{\varepsilon^{2q}}, \frac{\mathbf{x}}{\varepsilon^p}\right) = \varepsilon^{p(1-\alpha)}\mathbf{U}^\varepsilon\left(\frac{t}{\varepsilon^{2(q-p\beta)}}, \mathbf{x}\right).$$

With a unique pair of parameters q, η_ε,

$$q = \beta/(\alpha + 2\beta - 1), \quad \eta_\varepsilon = \varepsilon^{1+p-p(\alpha+2\beta)}, \tag{7.2}$$

the equation of motion can be written as

$$\frac{d\mathbf{x}^\varepsilon(t)}{dt} = \frac{1}{\eta_\varepsilon^{2q-1}}\mathbf{U}^\varepsilon\left(\frac{t}{\eta_\varepsilon^{2q}}, \mathbf{x}^\varepsilon(t)\right). \tag{7.3}$$

Since η_ε must tend to zero we require that

$$\alpha + 2\beta < 1 + \frac{1}{p}. \tag{7.4}$$

Condition (7.4) is also related to the fact that the velocity $\mathbf{U}^\varepsilon$ has increasingly smaller scales as ε tends to zero.

The following physical argument shows that, under the conditions (7.4) and

$$\alpha + \beta > 1,$$

the ultraviolet divergence in $\mathbf{U}^\varepsilon$ has no physical significance. The small-scale velocity associated with high wave number $|\mathbf{k}|$ has the amplitude

$$\left(\int_{c_1|\mathbf{k}|\leq|\mathbf{k}'|\leq c_2|\mathbf{k}|} \mathcal{E}(\mathbf{k}')d|\mathbf{k}'|\right)^{1/2} \sim |\mathbf{k}|^{1-\alpha}, \quad |\mathbf{k}| \gg 1,$$

and the correlation time is of the order $|\mathbf{k}|^{-2\beta}$. Then particles transported by small-scale velocity travel a distance less than or equal to the sum of $t|\mathbf{k}|^{2\beta}$ number uncorrelated random variables of magnitude $|\mathbf{k}|^{1-\alpha}|\mathbf{k}|^{-2\beta}$. Thus, on the time scale $t \sim \eta_\varepsilon^{-2q}$, the displacement caused by high wave number $\mathbf{k}$ is of the order less than or equal to $\sqrt{\eta_\varepsilon^{-2q}|\mathbf{k}|^{2\beta}|\mathbf{k}|^{1-\alpha-2\beta}}$, as suggested by the turbulent diffusion limit theorem for mixing flows [4], which equals $\eta_\varepsilon^{-q}|\mathbf{k}|^{1-\alpha-\beta}$ and is always smaller than η_ε^{-1} (the spatial scale of observation) if $\alpha + \beta > 1$ and $q < 1$ (superdiffusive scaling). With (7.2) the two conditions ($\alpha + \beta > 1$ and $q < 1$) are equivalent. It is clear that for $|\mathbf{k}| = O(1)$ the previous argument is still valid.

Now, if we neglect the high wave numbers in (7.3) the equation becomes

$$d\boldsymbol{x}^\varepsilon(t)/dt = \eta_\varepsilon^{1-2q}\mathbf{V}(t/\eta_\varepsilon^{2q}, \boldsymbol{x}^\varepsilon(t)), \tag{7.5}$$

which has the asymptotic solution

$$\boldsymbol{x}^{\varepsilon}(0) + \eta_{\varepsilon} \int_{0}^{t/\eta_{\varepsilon}^{2q}} \mathbf{V}(\boldsymbol{x}^{\varepsilon}(0), s) ds \tag{7.6}$$

converging to an FBM (Theorems 1 and 2).

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