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TIME REVERSAL OF PARABOLIC WAVES AND TWO-FREQUENCY WIGNER DISTRIBUTION

Albert Fannjiang

Department of Mathematics
University of California at Davis
Davis, CA 95616

Knut Solna

Department of Mathematics
University of California at Irvine
Irvine, CA 92697

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Abstract. We consider propagation and time reversal of wave pulses in a random environment. The focus of our analysis is the development of an expression for the two frequency mutual coherence function for the harmonic wave field. This quantity plays a crucial role in the analysis of many wave propagation phenomena and we illustrate by explicitly considering time reversal in the context of time pulses with a high carrier frequency. In a time-reversal experiment the wave received by an active transducer or antenna (receiver-emitter) array, is recorded in a finite time window and then re-emitted into the medium time reversed, that is, the tails of the recorded signals are sent first. The re-emitted wave pulse will focus approximately on the original source location. We use explicit expressions for the mutual coherence functions and their asymptotic approximations in the regime of long or short propagation distance and a high carrier frequency to analyze the refocusing of the wave pulse in the time reversal experiment. A novel aspect of our analysis is that we are able to characterize precisely the decoherence length in temporal frequency. This allows us to analyze for instance the time reversal experiment when the mirror has a finite aperture in time.

1. **Introduction.** Wave propagation in a randomly layered medium is well understood and is analyzed in for instance [1]. The core theoretical result in [1] is a development of a family of equations that describe the hierarchy of moment equations for the harmonic wave field. In particular one obtains explicit expressions for the second order cross-moments in the case with a constant mean wave speed. These cross moments at nearby frequencies can be used to analyze for instance the energy spectrum of the incoherent waves reflected from a random half-space and also time reversal experiments in the context of a randomly layered half-space and high frequency waves. The paper [1] has led to a series of papers that analyze for

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instance various time reversal and imaging problems using the framework set forth. However, the analysis is restricted to randomly layered media.

In the present paper we consider a different regime corresponding to the paraxial wave equation [15], [20] where the longitudinal scattering, which is the main mode of scattering in the layered medium, is small and the main mode of scattering is lateral scattering. The paraxial approximation describes a situation with long range narrow angle propagation with relatively small fluctuations in the index of refraction.

The main focus of our paper is the development of equations governing the cross-moments of the harmonic wave field at nearby frequencies. As in the layered medium these are in fact explicitly solvable. Again these moment equations allow us to analyze a range of wave propagation phenomena and we consider in particular the time reversal experiment. This analysis generalizes the results in for instance [2, 7, 18] where a time reversal mirror that is constant in time is used which leads to phase conjugation at the mirror.

With a finite *temporal* aperture mirror we are then led to consider the coherence or wave field correlation at nearby frequencies. In the time reversal experiment that we analyze it is clear from time reversibility of the wave equation that if we capture, time reverse and re-emit a sufficient part of the wave field, the re-emitted wave will approximately refocus on the target [11]. The surprising and important fact is that the focusing resolution typically will be *enhanced* rather than hampered by heterogeneity or "randomness" in the medium. The effect has numerous applications. In the case of ultrasound, this process can be iterated to pinpoint the wave beam in order to destroy kidney stones, detect defects in materials and communicate with submarines. In the case of electromagnetic waves this effect holds the potential of increasing imaging resolution and channel capacity. The phenomenon has been studied in the literature, both from the experimental and theoretical points of view [2, 11, 14].

The outline of this paper is as follows. In Section 2 we formulate the propagation problem and introduce the paraxial approximation in the relevant scaling regime. Next, in Section 3, we introduce the two frequency Wigner-Moyal equation, the phase space formulation that will be useful for analyzing moments of the field. We review the white noise limit for this in Section 4. The white noise limit takes on a particular useful form in the high frequency or geometrical optics limit which we discuss in Section 5. Then we use the two frequency Wigner-Moyal framework that we have introduced to analyze respectively the time reversed wave field in Section 6 and the transmitted field in Section 7.

2. **Paraxial White Noise Regime.**

We start by considering the wave equation

$$\triangle' u - c^{-2} u_{t't'} \quad = \quad 0\,,$$

with the local speed c defined by

$$\frac{1}{c^2} \quad = \quad \frac{1+\eta}{\bar{c}^2}\,,$$

for $\bar{c}$ the constant effective medium speed and $\triangle'$ being the Laplacian in the space coordinates $(\mathbf{x}', z')$. The centered random medium fluctuations are denoted by η:

$$\eta = \eta\left(\frac{z'}{\ell_z}, \frac{\mathbf{x}'}{\ell_x}\right)\,,$$

which gives the fluctuation in the refractive index field and is a homogeneous, square-integrable random field. A relevant example is the generalized von Kármán spectral density with $H = 1/3$ [20].

We shall refer to the dimensions $\mathbf{x}'$ as the transversal coordinates and z' as the longitudinal or propagation direction. We let L_x and L_z be characteristic length scales and $1/k_0$ represent a typical wavelength scale. The associated Fresnel number is defined by

$$\gamma_0 \quad = \quad \frac{L_z}{k_0 L_x^2} \, ,$$

and we introduce non-dimensionalized coordinates by

$$t = k_0 \bar{c} t', \quad \mathbf{x} = \frac{\mathbf{x}'}{\sqrt{\gamma_0} L_x}, \quad z = \frac{z'}{L_z}.$$

In non-dimensionalized coordinates we then get

$$u_{zz} + c_0^{-1} \triangle u - c_0^{-2}(1+\eta) u_{tt} = 0 \, ,$$

for

$$c_0 := \frac{1}{L_z k_0} \, ,$$

and with Δ the Laplacian in the $\mathbf{x}$ coordinates. We shall consider narrow beam propagation in the z direction and accordingly write

$$u(t, \mathbf{x}, z) \quad = \quad \int e^{\frac{ik(z/c_0 - t)}{\gamma}} \Psi(k, \mathbf{x}, z) \, dk \, , \tag{1}$$

for $k_0 k$ corresponding to a wave number in original coordinates and γ a small non-dimensional parameter which here will determine the high frequency scaling. We thus also have

$$\Psi(k, \mathbf{x}, z) \quad = \quad \frac{1}{2\pi\gamma} \int e^{\frac{-ik(z/c_0 - t)}{\gamma}} u(t, \mathbf{x}, z) \, dt \, , \tag{2}$$

and the decomposition (1) gives for the complex wave amplitude

$$\Psi_{zz} + \frac{2ik}{\gamma c_0} \Psi_z + \frac{1}{c_0} \triangle \Psi + \left(\frac{k}{\gamma c_0} \right)^2 \eta \Psi \quad = \quad 0.$$

The paraxial approximation corresponds to dropping the Ψ_{zz} term and with a slight abuse of notation we shall henceforth let Ψ solve

$$i\gamma \Psi_z + \frac{\gamma^2}{2k} \triangle \Psi + \frac{k}{2c_0} \eta \Psi \quad = \quad 0. \tag{3}$$

We write

$$\frac{1}{\varepsilon} V \left(\frac{z}{\varepsilon^2}, \mathbf{x} \right) \quad := \quad \frac{1}{2c_0} \eta \left(\frac{z L_z}{\ell_z}, \frac{\mathbf{x} \sqrt{\gamma_0} L_x}{\ell_x} \right) \, , \tag{4}$$

where

$$\varepsilon := \sqrt{\frac{\ell_z}{L_z}} \, ,$$

is a small parameter that gives a “white noise scaling” and in Section 4 we will introduce the important white noise model that enables us to derive field moment equations in the limit $\varepsilon \to 0$. The normalized refractive index field fluctuations V has a spectral density denoted by Φ.

With this notation (3) becomes

$$i\gamma\Psi_z + \frac{\gamma^2}{2k}\triangle\Psi + \frac{k}{\varepsilon}V\left(\frac{z}{\varepsilon^2},\mathbf{x}\right)\Psi \quad = \quad 0. \tag{5}$$

2.1. **Spatial Diversity Scaling.** The scaling (4) corresponds to

$$\begin{aligned}\frac{\sqrt{\gamma_0}L_x}{\ell_x} &= \sqrt{\frac{L_z}{k_0}}\frac{1}{\ell_x} \;=\; \mathcal{O}(1)\,,\\ \varepsilon\frac{\sqrt{\mathbb{E}[\eta^2]}}{c_0} &= k_0\sqrt{\ell_z L_z \mathbb{E}[\eta^2]} \;=\; \mathcal{O}(1).\end{aligned}$$

The scaling with *lateral diversity*, however, corresponds to

$$\begin{aligned}\frac{\sqrt{\gamma_0}L_x}{\ell_x} &= \sqrt{\frac{L_z}{k_0}}\frac{1}{\ell_x} \;=\; \frac{1}{\delta} \gg 1\,,\\ \varepsilon\frac{\sqrt{\mathbb{E}[\eta^2]}}{c_0} &= k_0\sqrt{\ell_z L_z \mathbb{E}[\eta^2]} \;=\; \delta \ll 1.\end{aligned} \tag{6}$$

The lateral diversity $\delta \ll 1$ gives rise to various self-averaging scaling limits which is important for statistical stability of time-reversal experiments. For a systematic treatment of the self-averaging scaling limits see [4, 18].

2.2. **The Initial Field.** In original coordinates we write the initial condition on the plane $z' = z'_s$ for the paraxial field propagating in the positive z direction as

$$u(t',\mathbf{x}',z'_s) \quad = \quad \Phi_0\left(\frac{\mathbf{x}'}{\sqrt{\gamma_0}L_{\mathbf{x}}}\right) f(k_0\bar{c}(t'-t'_s))\cos\left(\frac{k_0\bar{k}\bar{c}(t'-t'_s)}{\gamma}\right).$$

Here $k_0\bar{k}\bar{c}/\gamma$ is the *high* carrier frequency. The parameter $\bar{k}$ is assumed to be an order one parameter that scales the carrier and it plays no essential role in the analysis. The bandwidth of the source is of the order of $k_0\bar{c}$. In non-dimensionalized coordinates the source becomes

$$u(t,\mathbf{x},z_s) \quad = \quad \Phi_0(\mathbf{x})\, f(t-t_s)\cos\left(\frac{\bar{k}(t-t_s)}{\gamma}\right). \tag{7}$$

The associated initial condition on the envelope Ψ is

$$\Psi(z_s,\mathbf{x};k) \quad = \quad \frac{\Phi_0(\mathbf{x})}{2\gamma}\left(\hat{f}\left(\frac{k+\bar{k}}{\gamma}\right)+\hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)\right)e^{-\frac{ik(z_s/c_0-t_s)}{\gamma}}. \tag{8}$$

In Sections 6 and 7 we will use initial fields of this particular type when analyzing the properties of the transmitted and time reversed wave fields.

2.3. **Two Frequency Formulation.** In the paraxial approximation for the wave the complex amplitude field at two different wavenumbers k_j are denoted $\Psi_j, j = 1,2$, and are given as the solutions of the parabolic wave equation (5) at the respective wavenumber k_j. We explicitly indicate the dependence on the wavenumber and write as

$$i\gamma\frac{\partial\Psi_j^\varepsilon}{\partial z} + \frac{\gamma^2}{2k_j}\Delta\Psi_j^\varepsilon + \frac{k_j}{\varepsilon}V\left(\frac{z}{\varepsilon^2},\mathbf{x}\right)\Psi_j^\varepsilon = 0\,, \tag{9}$$

with the white noise limit $\varepsilon \to 0$ of this equation being discussed in [8]. Rather than taking the white noise limit at this level, we will follow an alternative route and take the white noise limit at the level of the Wigner distribution as discussed below. Note that although we do not assume isotropic spectral densities, the spectral density Φ of the fluctuations V always satisfies the basic symmetry:

$$\Phi(\xi, \mathbf{k}) = \Phi(-\xi, \mathbf{k}) = \Phi(\xi, -\mathbf{k}), \quad \forall (\xi, \mathbf{k}) \in \mathbb{R}^{d+1}, \tag{10}$$

because the refractive-index field is real-valued. Observe that we will denote the number of transversal spatial dimensions by d. We also assume that $V_z(\mathbf{x}) \equiv V(z, \mathbf{x})$ is a centered, square-integrable, z-stationary and $\mathbf{x}$-homogeneous process with the (partial) spectral representation

$$V_z(\mathbf{x}) = \int \exp(i\mathbf{p} \cdot \mathbf{x}) \widehat{V}_z(d\mathbf{p}), \tag{11}$$

where the process $\widehat{V}_z(d\mathbf{p})$ is the $z-$stationary orthogonal spectral measure satisfying

$$\mathbb{E}\left[\hat{V}_z(d\mathbf{p})\hat{V}_z(d\mathbf{q})\right] = \delta(\mathbf{p}+\mathbf{q}) \left[\int \Phi(w, \mathbf{p}) dw\right] d\mathbf{p} d\mathbf{q}. \tag{12}$$

3. **Wigner Distribution and Two-Frequency Wigner-Moyal Equation.** We introduce the two-frequency Wigner distributions

$$W^\varepsilon(z, \mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p}\cdot\mathbf{y}} \Psi_1^\varepsilon\left(z, \frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\gamma \mathbf{y}}{2\sqrt{k_1}}\right) \Psi_2^{\varepsilon *}\left(z, \frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\gamma \mathbf{y}}{2\sqrt{k_2}}\right) d\mathbf{y}, \tag{13}$$

and its complex conjugate $W^{\varepsilon *}$ which are ideally suited for analyzing the two-frequency problem.

The Wigner distribution has the following obvious properties.

$$\int W^\varepsilon(z, \mathbf{x}, \mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{y}} d\mathbf{p} = \Psi_1\left(z, \frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\gamma \mathbf{y}}{2\sqrt{k_1}}\right) \Psi_2^{\varepsilon *}\left(z, \frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\gamma \mathbf{y}}{2\sqrt{k_2}}\right) \tag{14}$$

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} W^\varepsilon(z, \mathbf{x}, \mathbf{p}) e^{-i\mathbf{x}\cdot\mathbf{q}} d\mathbf{x}$$
$$= \left(\frac{\sqrt{k_1 k_2}}{\gamma}\right)^d \widehat{\Psi}_1\left(z, \frac{\mathbf{p}\sqrt{k_2}}{\gamma} + \frac{\sqrt{k_1}\mathbf{q}}{2}\right) \widehat{\Psi}_2^{\varepsilon *}\left(z, \frac{\mathbf{p}\sqrt{k_2}}{\gamma} - \frac{\sqrt{k_1}\mathbf{q}}{2}\right).$$

Hence from W^ε and $W^{\varepsilon *}$ one can recover all but an overall phase factor about Ψ_1^ε and Ψ_2^ε.

Furthermore, the Wigner distribution $W_z^\varepsilon(\cdot) = W^\varepsilon(z, \cdot)$ satisfies the Wigner-Moyal equation

$$\frac{\partial W_z^\varepsilon}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} W_z^\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_z^\varepsilon W_z^\varepsilon = 0, \tag{15}$$

with the initial data

$$W_0(\mathbf{x}, \mathbf{p}) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p}\cdot\mathbf{y}} \Psi_{1,0}\left(z, \frac{\mathbf{x}}{\sqrt{k_1}} + \frac{\gamma \mathbf{y}}{2\sqrt{k_1}}\right) \Psi_{2,0}^*\left(z, \frac{\mathbf{x}}{\sqrt{k_2}} - \frac{\gamma \mathbf{y}}{2\sqrt{k_2}}\right) d\mathbf{y}, \tag{16}$$

where the operator $\mathcal{L}_z^\varepsilon$ is formally given as

$$\mathcal{L}_z^\varepsilon W_z^\varepsilon \tag{17}$$
$$= i \int \gamma^{-1} \left[e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{k_1}} k_1 W_z^\varepsilon\left(\mathbf{x}, \mathbf{p} + \frac{\gamma \mathbf{q}}{2\sqrt{k_1}}\right) - e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{k_2}} k_2 W_z^\varepsilon\left(\mathbf{x}, \mathbf{p} - \frac{\gamma \mathbf{q}}{2\sqrt{k_2}}\right)\right] \widehat{V}\left(\frac{z}{\varepsilon^2}, d\mathbf{q}\right).$$

The complex conjugate $W^{\varepsilon *}(z, \mathbf{x}, \mathbf{p})$ satisfies a similar equation

$$\frac{\partial W^{\varepsilon *}_z}{\partial z} + \mathbf{p} \cdot \nabla_{\mathbf{x}} W^{\varepsilon *}_z + \frac{1}{\varepsilon} \mathcal{L}_z^{\varepsilon *} W^{\varepsilon *}_z = 0, \tag{18}$$

where

$$\mathcal{L}_z^{\varepsilon *} W_z^{\varepsilon *} \tag{19}$$
$$= i\int \gamma^{-1}\left[e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{k_2}} k_2 W_z^{\varepsilon *}\left(\mathbf{x},\mathbf{p}+\frac{\gamma\mathbf{q}}{2\sqrt{k_2}}\right) - e^{i\mathbf{q}\cdot\mathbf{x}/\sqrt{k_1}} k_1 W_z^{\varepsilon *}\left(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2\sqrt{k_1}}\right)\right]\widehat{V}\left(\frac{z}{\varepsilon^2}, d\mathbf{q}\right).$$

Note that we use the following definition of the Fourier transform and inversion:

$$\hat{g}(\mathbf{p}) = \mathcal{F}g(\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{x}\cdot\mathbf{p}} g(\mathbf{x})d\mathbf{x}\,,$$
$$g(\mathbf{x}) = \mathcal{F}^{-1}\hat{g}(\mathbf{x}) = \int e^{i\mathbf{p}\cdot\mathbf{x}}\hat{g}(\mathbf{p})d\mathbf{p}\,,$$

for the lateral space variable and its dual and

$$\hat{f}(k) = \mathcal{F}f(k) = \frac{1}{2\pi}\int e^{ikt} f(t)dt\,,$$
$$f(t) = \mathcal{F}^{-1}\hat{f}(k) = \int e^{-ikt}\hat{f}(k)dk\,,$$

for the time variable and its dual in this wave propagation context.

When making a *partial* (inverse) Fourier transform on a phase-space function we will write $\mathcal{F}_1$ (respectively $\mathcal{F}_1^{-1}$) and $\mathcal{F}_2$ (respectively $\mathcal{F}_2^{-1}$) to denote the (resp. inverse) transform w.r.t. $\mathbf{x}$ and $\mathbf{p}$.

For every realization of V_z the operator $\mathcal{L}_z^{\varepsilon *}$ is defined as

$$\mathcal{L}_z^{\varepsilon} W_z^{\varepsilon}(\mathbf{x},\mathbf{p}) \equiv i\gamma^{-1}\mathcal{F}_2\left[\delta_\gamma V_z\left(\frac{z}{\varepsilon^2},\mathbf{x},\mathbf{y}\right)\mathcal{F}_2^{-1} W_z^{\varepsilon}(\mathbf{x},\mathbf{y})\right], \tag{20}$$

with the difference operator δ_γ given by

$$\delta_\gamma V\left(\frac{z}{\varepsilon^2},\mathbf{x},\mathbf{y}\right) \equiv k_1 V\left(\frac{z}{\varepsilon^2},\frac{\mathbf{x}}{\sqrt{k_2}}+\frac{\gamma\mathbf{y}}{2\sqrt{k_2}}\right) - k_2 V\left(\frac{z}{\varepsilon^2},\frac{\mathbf{x}}{\sqrt{k_1}}-\frac{\gamma\mathbf{y}}{2\sqrt{k_1}}\right). \tag{21}$$

The operator $\mathcal{L}_z^{\varepsilon *}$ is defined similarly. In order to characterize the transmitted and time reversed field below we will need the second order moment equations for the time harmonic wave field. These equations will derive from the equations for the Wigner distribution. The problem (15) is not explicitly solvable, however, we consider next the narrow band white noise regime corresponding to $\gamma \to \infty$ and $\varepsilon \to 0$ where we will obtain explicit expressions for the Wigner distribution which will lead to expressions for the mutual coherence function that describe the second order wave statistics.

Consider the simultaneous limit

$$\gamma \to 0, \quad k_1, k_2 \to k \neq 0\,, \tag{22}$$

with

$$\begin{aligned} k_1 &= k - \gamma\beta/2\,, \\ k_2 &= k + \gamma\beta/2. \end{aligned} \tag{23}$$

Here the parameter $\beta < \infty$ has the physical meaning of a normalized bandwidth.

In the narrow-band, geometrical optics limit $\mathcal{L}_z^{\varepsilon}$ becomes

$$\lim_{\gamma\to 0}\mathcal{L}_z^{\varepsilon} W_z^{\varepsilon}(\mathbf{x},\mathbf{p}) \tag{24}$$
$$= -\mathcal{F}_2\left[\nabla_{\mathbf{x}} V_z^{\varepsilon}(\mathbf{x})\cdot\left[i\mathbf{y}\mathcal{F}_2^{-1} W_z^{\varepsilon}(\mathbf{x},\mathbf{y})\right]\right]$$
$$= -\sqrt{k}\nabla V_z^{\varepsilon}\left(\frac{\mathbf{x}}{\sqrt{k}}\right)\cdot\nabla_{\mathbf{p}} W_z^{\varepsilon}(\mathbf{x},\mathbf{p}) + i\beta W_z^{\varepsilon}(\mathbf{x},\mathbf{p})\left[V_z^{\varepsilon}\left(\frac{\mathbf{x}}{\sqrt{k}}\right) - \frac{\mathbf{x}}{2\sqrt{k}}\cdot\nabla V_z^{\varepsilon}\left(\frac{\mathbf{x}}{\sqrt{k}}\right)\right].$$

4. **The White-Noise, Markovian model.** The convergence of the weak solution of the Wigner-Moyal equation, described in the preceding section, to the weak solution of the two-frequency white-noise, Markovian model has been proved in [5], [6]. Below we give only the essential feature of the two frequency Markovian model which will be further analyzed later.

Formerly the model is governed by the Wigner-Itô equation

$$dW_z + (\mathbf{p}\cdot\nabla_{\mathbf{x}}W_z - Q_0 W_z)dz = dB_z W_z\,, \tag{25}$$

where the operator $\mathcal{Q}_0$ is given by

$$\begin{aligned}&\mathcal{Q}_0 W_z(\mathbf{x},\mathbf{p}) \qquad (26)\\ &= \int \Phi(\mathbf{q})\gamma^{-2}\Big[k_1k_2 e^{-i(k_1^{-1/2}-k_2^{-1/2})\mathbf{q}\cdot\mathbf{x}} W_z\Big(\mathbf{x},\mathbf{p}-\Big(k_1^{-1/2}+k_2^{-1/2}\Big)\gamma\mathbf{q}/2\Big)\\ &+k_1k_2 e^{i(k_1^{-1/2}-k_2^{-1/2})\mathbf{q}\cdot\mathbf{x}} W_z\Big(\mathbf{x},\mathbf{p}+\Big(k_1^{-1/2}+k_2^{-1/2}\Big)\gamma\mathbf{q}/2\Big)-\big(k_1^2+k_2^2\big)W_z(\mathbf{x},\mathbf{p})\Big]d\mathbf{q}\,,\end{aligned}$$

with Φ being the spectrum of V and B_z is an operator-valued Brownian motion (see [6] for details) which vanishes after ensemble-averaging:

$$\frac{\partial \bar{W}_z}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}_z \;=\; \mathcal{Q}_0\bar{W}_z. \tag{27}$$

For the narrow-band geometrical optics limit (22)-(23) we obtain

$$\mathcal{Q}_0 W \;\approx\; k\nabla_{\mathbf{p}}\cdot\mathbf{D}\cdot\nabla_{\mathbf{p}}W + i\beta\mathbf{x}\cdot\mathbf{D}\cdot\nabla_{\mathbf{p}}W - \frac{\beta^2}{4k}\mathbf{x}\cdot\mathbf{D}\cdot\mathbf{x}W - \beta^2 D_0 W\,,$$

which can be written as

$$\begin{aligned}\mathcal{Q}_0 W_z(\mathbf{x},\mathbf{p}) \;&\approx \qquad (28)\\ &-k\Big(-i\nabla_{\mathbf{p}}+\frac{\beta}{2k}\mathbf{x}\Big)\cdot\mathbf{D}\cdot\Big(-i\nabla_{\mathbf{p}}+\frac{\beta}{2k}\mathbf{x}\Big)W_z(\mathbf{x},\mathbf{p}) - \beta^2 D_0 W_z(\mathbf{x},\mathbf{p})\,,\end{aligned}$$

where

$$\mathbf{D} \;=\; \int \Phi(0,\mathbf{q})\mathbf{q}\otimes\mathbf{q}d\mathbf{q}\,, \tag{29}$$

$$D_0 \;=\; \int \Phi(0,\mathbf{q})d\mathbf{q}. \tag{30}$$

In the case of a power-law spectrum with the Hurst exponent H, inner scale ℓ_0 and the outer scale L_0 the diffusion matrix has the following asymptotics for $\ell_0 \ll 1, L_0 \gg 1$

$$\mathbf{D} = O(\ell_0^{2H-1} + L_0^{2H-1})\,,$$

and hence has a finite limit as $L_0 \to \infty$ for $H < 1/2$ and as $\ell_0 \to 0$ for $H > 1/2$.

The evolution equation for the two-frequency mutual coherence function

$$\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2) = \mathbb{E}[\Psi_1(z,\mathbf{x}_1)\Psi_2^*(z,\mathbf{x}_2)]\,, \tag{31}$$

as defined in [15] can using (14) be obtained by setting

$$\mathbf{x} \;=\; \frac{1}{2}(\sqrt{k_1}\mathbf{x}_1 + \sqrt{k_2}\mathbf{x}_2)\,, \tag{32}$$

$$\mathbf{y} \;=\; \frac{1}{\gamma}(\sqrt{k_1}\mathbf{x}_1 - \sqrt{k_2}\mathbf{x}_2)\,, \tag{33}$$

and applying $\mathcal{F}_2^{-1}$ to the mean field equation (27), that is

$$\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2) \;=\; \int \bar{W}(z,\mathbf{x},\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{y}}d\mathbf{p}\,,$$

for $\mathbf{x}$ and $\mathbf{y}$ defined by (32) and (33) respectively. The result then corresponds to the one obtained for the two-frequency mutual coherence function in the literature (see [15]). The advantage of working with the two-frequency Wigner distribution lies in the ease of taking the geometrical optics limit (28).

5. **Exact and Asymptotic Solutions for Narrow-Band Geometrical Optics.** Neither (27) nor the resulting equation for Γ_{12} is exactly solvable and various approximations have been proposed (see [3], [10], [13],[16], [17], [19], [21]). However, the geometrical optics limit

$$\frac{\partial \bar{W}_z}{\partial z} + \mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}_z = -k\left(-i\nabla_{\mathbf{p}} + \frac{\beta}{2k}\mathbf{x}\right)\cdot\mathbf{D}\cdot\left(-i\nabla_{\mathbf{p}} + \frac{\beta}{2k}\mathbf{x}\right)W_z(\mathbf{x},\mathbf{p}) - \beta^2 D_0 W_z(\mathbf{x},\mathbf{p}),$$

is exactly solvable. Let us construct the Green function.

For simplicity of notation, let us assume isotropy of the medium, namely $\Phi(0,\mathbf{p}) = \Phi(0,|\mathbf{p}|)$ and hence $\mathbf{D} = D$, a scalar. In the case that the fluctuations are stationary with covariance $C(\Delta\mathbf{x},\Delta z)$ we find that the correlation parameters have the interpretation

$$D_0 = \int C(z,\mathbf{0})\,dz,$$

$$D = -\int \triangle_{\mathbf{x}} C(z,\mathbf{x})|_{\mathbf{x}=0}\,dz.$$

Taking the inverse Fourier transform $\mathcal{F}_2^{-1}$ in $\mathbf{p}$ we obtain

$$\frac{\partial}{\partial z}\hat{W} = i\nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{y}}\hat{W} - \frac{D}{k}\left|-k\mathbf{y} + \frac{\beta}{2}\mathbf{x}\right|^2\hat{W} - \beta^2 D_0\hat{W}. \quad (34)$$

Introducing the new variables

$$\mathbf{y}_1 = k\mathbf{y} + \frac{\beta}{2}\mathbf{x}, \quad (35)$$

$$\mathbf{y}_2 = k\mathbf{y} - \frac{\beta}{2}\mathbf{x}, \quad (36)$$

we rewrite the above equation in the new coordinates as

$$\frac{\partial}{\partial z}\check{W} = \frac{ik\beta}{2}\left(\nabla_1^2 - \nabla_2^2\right)\check{W} - \frac{D}{k}|\mathbf{y}_2|^2\check{W} - \beta^2 D_0\check{W}, \quad (37)$$

where ∇_1, ∇_2 are the gradients with respect to $\mathbf{y}_1, \mathbf{y}_2$, respectively.

Consider the function

$$\widetilde{W}(z,\mathbf{p}_1,\mathbf{y}_2) = e^{\beta^2 D_0 z} e^{ik\beta|\mathbf{p}_1|^2 z/2}\frac{1}{(2\pi)^d}\int \check{W}\left(z, \frac{\mathbf{y}_1-\mathbf{y}_2}{\beta}, \frac{\mathbf{y}_1+\mathbf{y}_2}{2k}\right)e^{-i\mathbf{y}_1\cdot\mathbf{p}_1}d\mathbf{y}_1, \quad (38)$$

which satisfies the equation

$$\frac{\partial}{\partial z}\widetilde{W} = -\frac{ik\beta}{2}\nabla_2^2\widetilde{W} - \frac{D}{k}|\mathbf{y}_2|^2\widetilde{W}. \quad (39)$$

Equation (39) is just the Schrödinger equation with an imaginary, quadratic potential.

First we look for the Green function in the coordinates $\mathbf{y}_1, \mathbf{y}_2$. To this end, we set

$$A(z) = -idk\beta\int_1^z B(s)ds + \frac{D}{k}\int_\infty^z |\mathbf{C}|^2(s)ds, \quad z > 0,$$

with

$$B(z) = \frac{-1}{k(1+i)}\sqrt{\frac{D}{\beta}}\cot\left[\sqrt{D\beta}(1+i)z\right],$$

and $\mathbf{C}(z)$ given by the formula

$$\mathbf{C}(z) = \mathbf{C}(0)\exp\left[-\frac{D}{k}\int_0^z B^{-1}(s)ds\right], \quad \mathbf{C}(0) = \mathbf{y}_2'.$$

We find that functions of the form

$$e^{-A(z)-B(z)|\mathbf{y}_2-\mathbf{C}(z)|^2},$$

in fact solves (39) and it follows that the Green function of (37) is given by

$$\begin{aligned}
& G_{\breve{W}}(z,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_1',\mathbf{y}_2';k,\beta) \qquad (40)\\
= \; & c_0 e^{-\beta^2 D_0 z} e^{-A(z)-B(z)|\mathbf{y}_2-\mathbf{C}(z)|^2}\int e^{-ik\beta|\mathbf{p}_1|^2 z/2} e^{i\mathbf{p}_1\cdot(\mathbf{y}_1-\mathbf{y}_1')}d\mathbf{p}_1\\
= \; & \left(\frac{1}{\pi k\beta z}\right)^{d/2}\left(\frac{\sqrt{D}(1+i)}{\pi k\sqrt{\beta}\sin(\sqrt{D\beta}(1+i)z)}\right)^{d/2} e^{-\beta^2 D_0 z}\exp\left[\frac{i|\mathbf{y}_1-\mathbf{y}_1'|^2}{2zk\beta}\right]\\
& \times\exp\left[-\frac{|\mathbf{y}_2'|^2}{(1+i)k}\sqrt{\frac{D}{\beta}}\tan(\sqrt{D\beta}(1+i)z)\right]\\
& \times\exp\left[\frac{1}{k(1+i)}\sqrt{\frac{D}{\beta}}\cot\left(\sqrt{D\beta}(1+i)z\right)\left|\mathbf{y}_2-\frac{\mathbf{y}_2'}{\cos\left(\sqrt{D\beta}(1+i)z\right)}\right|^2\right].
\end{aligned}$$

The general solution for equation (34) can then be expressed as

$$\hat{W}(z,\mathbf{x},\mathbf{y}) = (k\beta)^d\int \hat{W}_0(\mathbf{x}',\mathbf{y}')G_{\breve{W}}\left(z,k\mathbf{y}+\frac{\beta}{2}\mathbf{x},k\mathbf{y}-\frac{\beta}{2}\mathbf{x},k\mathbf{y}'+\frac{\beta}{2}\mathbf{x}',k\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right)d\mathbf{x}'d\mathbf{y}'.$$

We thus find the expression

$$\begin{aligned}
\bar{W}_z(\mathbf{x},\mathbf{p}) \; &= \; \mathcal{F}_2\{\hat{W}(z,\mathbf{x},\mathbf{y})\} \qquad (41)\\
&= \; \left(\frac{k\beta}{2\pi}\right)^d\int e^{-i\mathbf{p}\cdot\mathbf{y}}e^{i\mathbf{p}'\cdot\mathbf{y}'}\bar{W}_0(\mathbf{x}',\mathbf{p}') \qquad (42)\\
& \quad \times G_{\breve{W}}\left(z,k\mathbf{y}+\frac{\beta}{2}\mathbf{x},k\mathbf{y}-\frac{\beta}{2}\mathbf{x},k\mathbf{y}'+\frac{\beta}{2}\mathbf{x}',k\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right)d\mathbf{x}'d\mathbf{y}'d\mathbf{y}d\mathbf{p}',
\end{aligned}$$

for the mean Wigner transform.

5.1. **Coherence Function and Schrödinger Spectrum.** From (13) and (41) we obtain for the mutual coherence function the expression

$$\begin{aligned}
\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2) &= \mathcal{F}_2^{-1}\{\bar{W}(z,\mathbf{x},\mathbf{p})\} \;=\; \hat{W}(z,\mathbf{x},\mathbf{y}) \\
&= (k\beta)^d \int \mathcal{F}_2^{-1}\{\bar{W}\}(0,\mathbf{x}',\mathbf{y}') \\
&\quad \times G_{\check{W}}\left(z, k\mathbf{y}+\frac{\beta}{2}\mathbf{x}, k\mathbf{y}-\frac{\beta}{2}\mathbf{x}, k\mathbf{y}'+\frac{\beta}{2}\mathbf{x}', k\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right) d\mathbf{x}'d\mathbf{y}' \\
&= (k\beta)^d \int \Psi_{1,0}\left(\frac{\mathbf{x}'}{\sqrt{k_1}}+\frac{\gamma\mathbf{y}'}{2\sqrt{k_1}}\right)\Psi_{2,0}^*\left(\frac{\mathbf{x}'}{\sqrt{k_2}}-\frac{\gamma\mathbf{y}'}{2\sqrt{k_2}}\right) \\
&\quad \times G_{\check{W}}\left(z, k\mathbf{y}+\frac{\beta}{2}\mathbf{x}, k\mathbf{y}-\frac{\beta}{2}\mathbf{x}, k\mathbf{y}'+\frac{\beta}{2}\mathbf{x}', k\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right) d\mathbf{x}'d\mathbf{y}' \\
&= \left(\frac{\beta k\sqrt{k_1k_2}}{\gamma}\right)^d \int \Psi_{1,0}\left(\tilde{\mathbf{x}}\right)\Psi_{2,0}^*\left(\tilde{\mathbf{y}}\right) \\
&\quad \times G_{\check{W}}\left(z, k\mathbf{y}+\frac{\beta}{2}\mathbf{x}, k\mathbf{y}-\frac{\beta}{2}\mathbf{x}, k\mathbf{y}'+\frac{\beta}{2}\mathbf{x}', k\mathbf{y}'-\frac{\beta}{2}\mathbf{x}'\right) d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}
\end{aligned}$$

for

$$\mathbf{x}=\frac{1}{2}(\sqrt{k_1}\mathbf{x}_1+\sqrt{k_2}\mathbf{x}_2)\,,\quad \mathbf{y}=\frac{1}{\gamma}(\sqrt{k_1}\mathbf{x}_1-\sqrt{k_2}\mathbf{x}_2)\,,\quad \tilde{\mathbf{x}}=\frac{\mathbf{x}'}{\sqrt{k_1}}+\frac{\gamma\mathbf{y}'}{2\sqrt{k_1}}\,,\quad \tilde{\mathbf{y}}=\frac{\mathbf{x}'}{\sqrt{k_2}}-\frac{\gamma\mathbf{y}'}{2\sqrt{k_2}}\,,$$

so that

$$\mathbf{x}'=\frac{1}{2}(\sqrt{k_1}\tilde{\mathbf{x}}+\sqrt{k_2}\tilde{\mathbf{y}})\,,\quad \mathbf{y}'=\frac{1}{\gamma}(\sqrt{k_1}\tilde{\mathbf{x}}-\sqrt{k_2}\tilde{\mathbf{y}}).$$

Here we have assumed that the source is given at $z_s=0$.

Let G_ψ be the Green function associated with the Schrödinger equation (9). Define the Schrödinger spectral function as

$$\tilde{\Lambda}(z,\mathbf{x}_1,\mathbf{x}_2,\tilde{\mathbf{x}},\tilde{\mathbf{y}},k_1,k_2) \;=\; \mathbb{E}\left[G_\Psi(z,\mathbf{x}_1,\tilde{\mathbf{x}};k_1)G_\Psi^*(z,\mathbf{x}_2,\tilde{\mathbf{y}};k_2)\right] \tag{43}$$

namely the two-point coherence function of Schrödinger Green functions of two different frequencies.

In the high frequency limit $\gamma \ll 1$ we have the following asymptotic

$$\begin{aligned}
&\tilde{\Lambda}(z,\mathbf{x}_1,\mathbf{x}_2,\tilde{\mathbf{x}},\tilde{\mathbf{y}};k_1,k_2) \qquad (44)\\
&= \left(\frac{\beta k^2}{\gamma}\right)^d G_{\check{W}}\left(z,\frac{k^{3/2}}{\gamma}(\mathbf{x}_1-\mathbf{x}_2),\frac{\sqrt{k}}{\gamma}(k_1\mathbf{x}_1-k_2\mathbf{x}_2),\frac{k^{3/2}}{\gamma}(\tilde{\mathbf{x}}-\tilde{\mathbf{y}}),\frac{\sqrt{k}}{\gamma}(k_1\tilde{\mathbf{x}}-k_2\tilde{\mathbf{y}})\right)\\
&\equiv \Lambda(z,\mathbf{x}_1,\mathbf{x}_2,\tilde{\mathbf{x}},\tilde{\mathbf{y}},k_1,\beta).
\end{aligned}$$

Recall that

$$k_1=k-\gamma\beta/2\,,\quad k_2=k+\gamma\beta/2\,,$$

so that we have

$$\frac{\sqrt{k}}{\gamma}(k_1\mathbf{x}_1-k_2\mathbf{x}_2)\sim\frac{k^{3/2}}{\gamma}\left((\mathbf{x}_1-\mathbf{x}_2)-(\mathbf{x}_1+\mathbf{x}_2)\left(\frac{\gamma\beta}{2k}\right)\right). \tag{45}$$

We can now express the mutual coherence function in terms of the Schrödinger spectral function as follows.

$$\begin{aligned}\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2) &= \int \mathbb{E}\left[G_\Psi(z,\mathbf{x}_1,\tilde{\mathbf{x}};k_1)G^*_\Psi(z,\mathbf{x}_2,\tilde{\mathbf{y}};k_2)\right]\Psi_{1,0}\left(\tilde{\mathbf{x}}\right)\Psi^*_{2,0}\left(\tilde{\mathbf{y}}\right)d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}\\ &\sim \int \tilde{\Lambda}(z,\mathbf{x}_1,\mathbf{x}_2,\tilde{\mathbf{x}},\tilde{\mathbf{y}},k_1,k_2)\Psi_{1,0}(\tilde{\mathbf{x}})\Psi^*_{2,0}(\tilde{\mathbf{y}})d\tilde{\mathbf{x}}d\tilde{\mathbf{y}}.\end{aligned}$$

5.2. **Long Distance Asymptotics.** We have assumed a scaling with the distance of propagation z being an order one quantity and with the source being located at the origin so that $z_s = 0$. Since we are interested in narrow beam long distance propagation we will next evaluate the expression for the mutual coherence function in the limit that z is large. To this effect we note the following asymptotics in the limit $z \to \infty$ for fixed $D > 0$:

$$\begin{aligned}\sin\left(\sqrt{D\beta}(1+i)z\right) &\sim -\frac{e^{\sqrt{D\beta}(1-i)z}}{2i},\\ \left(\cos\left(\sqrt{D\beta}(1+i)z\right)\right)^{-1} &\sim 0,\\ \tan\left(\sqrt{D\beta}(1+i)z\right) &\sim i,\\ \cot\left(\sqrt{D\beta}(1+i)z\right) &\sim -i.\end{aligned}$$

Therefore, the Green function has the long distance asymptotics

$$\begin{aligned}G_{\breve{W}}(z,\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}'_1,\mathbf{y}'_2) \sim& \left(\frac{1}{\pi k\beta z}\right)^{d/2}\left(\frac{2\sqrt{D}(1-i)}{\pi k\sqrt{\beta}}\right)^{d/2} e^{-\beta^2 D_0 z}\exp\left[-\frac{1-i}{2}d\sqrt{\beta D}z\right]\\ &\times\exp\left[\frac{i|\mathbf{y}_1-\mathbf{y}'_1|^2}{2zk\beta}\right]\exp\left[-\frac{1+i}{2k}\sqrt{\frac{D}{\beta}}\left(|\mathbf{y}_2|^2+|\mathbf{y}'_2|^2\right)\right], \quad (46)\end{aligned}$$

from which it follows that the long distance asymptotics for the mutual coherence function Γ_{12} is:

$$\begin{aligned}&\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2)\\ &\sim \left(\frac{\beta k^2}{\gamma}\right)^{d}\left(\frac{1}{\pi k\beta z}\right)^{d/2}\left(\frac{2\sqrt{D}(1-i)}{\pi k\sqrt{\beta}}\right)^{d/2} e^{-\beta^2 D_0 z}\exp\left[-\frac{1-i}{2}d\sqrt{\beta D}z\right]\\ &\quad\times\int\exp\left[\frac{ik^2}{2\gamma^2\beta z}|\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}'_1+\mathbf{x}'_2|^2\right]\exp\left[-\frac{1+i}{2\gamma^2}\sqrt{\frac{D}{\beta}}\left(\left|k_1\mathbf{x}_1-k_2\mathbf{x}_2\right|^2+\left|k_1\mathbf{x}'_1-k_2\mathbf{x}'_2\right|^2\right)\right]\\ &\quad\times\Psi_{1,0}\left(\mathbf{x}'_1\right)\Psi^*_{2,0}\left(\mathbf{x}'_2\right)d\mathbf{x}'_1 d\mathbf{x}'_2.\end{aligned}$$

We now evaluate this expression for $\Psi_{j,0}$ having spatial *point support*:

$$\Psi_{j,0}(\mathbf{x}) = \delta(\mathbf{x})f(k_j)\,, \quad (47)$$

and find that

$$\begin{aligned}
&\Gamma_{12}(z,\mathbf{x}_1,\mathbf{x}_2) \sim \Lambda(z,\mathbf{x}_1,\mathbf{x}_2,0,0,k_1,\beta) f(k_1) f^*(k_2)\\
&= \left(\frac{\beta k^2}{\gamma}\right)^d \left(\frac{1}{\pi k \beta z}\right)^{d/2} \left(\frac{2\sqrt{D}(1-i)}{\pi k\sqrt{\beta}}\right)^{d/2} e^{-\beta^2 D_0 z} \exp\left[-\frac{1-i}{2} d\sqrt{\beta D} z\right]\\
&\times \exp\left[\frac{ik^2}{2\gamma^2\beta z}|\mathbf{x}_1-\mathbf{x}_2|^2\right] \exp\left[-\frac{1+i}{2\gamma^2}\sqrt{\frac{D}{\beta}}\,|k_1\mathbf{x}_1 - k_2\mathbf{x}_2|^2\right] f(k_1) f^*(k_2).
\end{aligned}$$

One sees from the above expression that the coherent bandwidth β_c is given by

$$\beta_c \quad \sim \quad \frac{1}{Dz^2}\,, \tag{48}$$

so that the wave field at nearby frequencies decorrelate rapidly in the regime of large propagation distance and strong medium fluctuations. The two-frequency coherence length ℓ_c is given by

$$\ell_c \quad \sim \quad \frac{\gamma}{k}\left(\frac{\beta}{D}\right)^{1/4}, \tag{49}$$

which determine the lateral spatial scale at which the wave field decorrelate. Note that this coherence length depends on the frequency separation β. At the limit of the coherence bandwidth, for $\beta = \beta_c$, we find

$$\ell_c \quad \sim \quad \frac{\gamma}{k}\frac{1}{\sqrt{zD}}\,, \tag{50}$$

with k/γ being the *carrier wavenumber*.

6. **Time Reversal Operation and Refocusing.** The time reversal operation is illustrated in Figure 6 and can be described as follows. In the plane $z = L$ a pulse traveling in the *negative* z direction is being emitted. The transmitted field is recorded, stored and time reversed at the time reversal mirror located in the plane $z = 0$, and then sent back toward the source point.

The initial condition is as in (7)

$$u^b(t,\mathbf{x},L) = u_0(t,\mathbf{x}) = \Phi_0(\mathbf{x})\, f(t-t_s)\cos\left(\frac{\bar{k}(t-t_s)}{\gamma}\right).$$

This gives the following initial condition for the *backpropagating* modulation Ψ:

$$\begin{aligned}
\Psi^b(k,\mathbf{x},L) &= \frac{1}{2\pi\gamma}\int e^{\frac{ik(L/c_0+t)}{\gamma}} u_0(t,\mathbf{x})\,dt \qquad (51)\\
&= \frac{\Phi_0(\mathbf{x})}{2\gamma}\left(\hat{f}\left(\frac{k+\bar{k}}{\gamma}\right) + \hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)\right) e^{\frac{ik(L/c_0+t_s)}{\gamma}}\,,
\end{aligned}$$

with the associated wave field being

$$u^b(t,\mathbf{x},z) \quad = \quad \int e^{\frac{-ik(z/c_0+t)}{\gamma}} \Psi^b(k,x,z)\,dk.$$

We then find that the field transmitted to the plane $z = 0$ is

$$u^b(t,\mathbf{x}_m,0) \quad = \quad \frac{u_I(t,\mathbf{x}_m) + u_I^*(t,\mathbf{x}_m)}{2}\,, \tag{52}$$

with

$$u_I(t,\mathbf{x}_m) \quad = \quad \int G_L(\mathbf{x}_s;\mathbf{x}_m,k)\Phi_0(\mathbf{x}_s)\,\hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)\frac{1}{\gamma} e^{\frac{ik(L/c_0+(t_s-t))}{\gamma}}\, d\mathbf{x}_s dk.$$

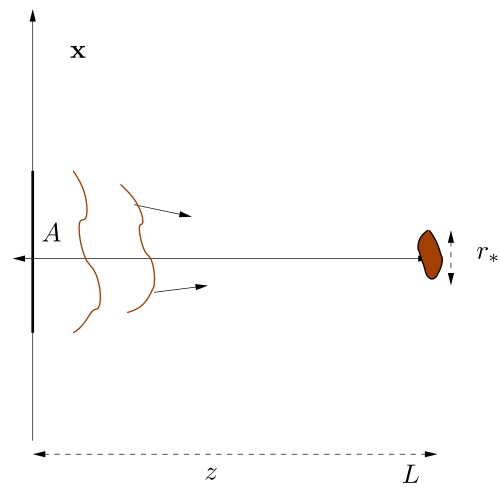

FIGURE 1. The time reversal procedure. A source emits a pulse in the plane $z = L$. The transmitted field is recorded and time reversed at the mirror of size A at $z = 0$, and then sent back toward the source point. There it refocuses on a spot with the lateral size r_*. Medium heterogeneity typically enhances the refocusing resolution.

In the derivation we have used the reciprocity of the Green function G_H of the Helmholtz equation:

$$G_H(\mathbf{x}_2, L; \mathbf{x}_1, 0, k) \quad = \quad G_H(\mathbf{x}_1, 0; \mathbf{x}_2, L, k)\,,$$

with parabolic approximations

$$G_H(\mathbf{x}_2, L; \mathbf{x}_1, 0, k) \sim G_L(\mathbf{x}_2; \mathbf{x}_1, k)e^{ikL/\gamma}\,, \quad G_H(\mathbf{x}_1, 0; \mathbf{x}_2, L, k) \sim G_L(\mathbf{x}_1; \mathbf{x}_2, k)e^{ikL/\gamma}.$$

The time reversal operation now corresponds to a re-emission at the mirror with the "new source" at the mirror being

$$u_{tr,I}(t, \mathbf{x}_m) \quad = \quad u_I(-t, \mathbf{x}_m)\mathbb{I}_A(\mathbf{x}_m)g(-t)\,,$$

with $\mathbb{I}_A$ being the spatial aperture function with characteristic dimension A and g being a temporal window aperture function that we assume is compactly supported. We then find

$$\begin{aligned} u_{tr,I}(t, \mathbf{x}_m) &= \mathbb{I}_A(\mathbf{x}_m)\int G_L(\mathbf{x}_s; \mathbf{x}_m, \bar{k} + \gamma\beta)\hat{f}(\beta)\hat{g}^*(\tilde{\beta})\Phi_0\left(\mathbf{x}_s\right) \\ &\quad \times e^{i\beta(L/c_0+t_s)}e^{i(\beta-\tilde{\beta})t}d\mathbf{x}_s d\beta d\tilde{\beta} e^{\frac{i\bar{k}(L/c_0+t_s+t))}{\gamma}} \\ &= \mathbb{I}_A(\mathbf{x}_m)\int G_L(\mathbf{x}_s; \mathbf{x}_m, \bar{k} + \gamma\beta)\hat{f}(\beta)\hat{g}(\tilde{\beta} - \beta)\Phi_0\left(\mathbf{x}_s\right) \\ &\quad \times e^{i\beta(L/c_0+t_s)}e^{\frac{i\bar{k}(L/c_0+t_s)}{\gamma}}d\mathbf{x}_s d\beta e^{-\frac{i(-(\bar{k}+\gamma\tilde{\beta})t)}{\gamma}}d\tilde{\beta}. \end{aligned}$$

From (1) it now follows that the associated modulation field in the mirror plane is

$$\begin{aligned} &\Psi_{tr,I}(0, \mathbf{x}_m, -(\bar{k} + \gamma\tilde{\beta})) \\ &= \mathbb{I}_A(\mathbf{x}_m)\int G_L(\mathbf{x}_s; \mathbf{x}_m, \bar{k} + \gamma\beta)\hat{f}(\beta)\hat{g}(\tilde{\beta} - \beta)\Phi_0\left(\mathbf{x}_s\right)e^{i\beta(L/c_0+t_s)}e^{\frac{i\bar{k}(L/c_0+t_s)}{\gamma}}\,d\mathbf{x}_s d\beta/\gamma. \end{aligned}$$

This gives the expression for the wave field in the original source plane $z = L$:

$$u_{tr,I}(t,\mathbf{x},L) = \int G_L(\mathbf{x}_s;\mathbf{x}_m,\bar{k}+\gamma\beta)G_L^*(\mathbf{x};\mathbf{x}_m,\bar{k}+\gamma\tilde{\beta})\hat{f}(\beta)\hat{g}(\tilde{\beta}-\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0(\mathbf{x}_s)\, d\mathbf{x}_s d\mathbf{x}_m \times e^{i\beta(L/c_0+t_s)}e^{i\tilde{\beta}(-L/c_0+t)}\, d\beta d\tilde{\beta} e^{\frac{i\bar{k}(t+t_s)}{\gamma}},$$

where we used the fact that $G_z(\cdot;\cdot,k) = G_z^*(\cdot;\cdot,-k)$. The corresponding refocused mean signal is

$$\mathbb{E}[u_{tr,I}(t,\mathbf{x},L)] = \int \Lambda(L,\mathbf{x}_s,\mathbf{x},\mathbf{x}_m,\mathbf{x}_m,\bar{k}+\gamma\beta,\tilde{\beta}-\beta)\hat{f}(\beta)\hat{g}(\tilde{\beta}-\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0(\mathbf{x}_s)\, d\mathbf{x}_s d\mathbf{x}_m \times e^{i\beta(L/c_0+t_s)}e^{i\tilde{\beta}(-L/c_0+t)}\, d\beta d\tilde{\beta} e^{\frac{i\bar{k}(t+t_s)}{\gamma}}.$$

We now use the expression for the Schrödinger spectral function in (44) to obtain a characterization of the mean back-propagated field $\mathbb{E}[u_{tr}]$ in the narrow-band geometrical optics limit.

We let the original source emission time t_s be chosen so that the pulse reaches the origin approximately at time zero:

$$t_s = -L/c_0\,,$$

and observe the back-propagated pulse in a window centered at the deterministic (background) travel time to depth L from the origin:

$$t = L/c_0 + \tau.$$

We then find

$$\begin{aligned}
&\mathbb{E}[u_{tr,I}(L/c_0+\tau,\mathbf{x},L)]e^{\frac{-i\bar{k}\tau}{\gamma}} \\
&\sim \int \Lambda(L,\mathbf{x}_s,\mathbf{x},\mathbf{x}_m,\mathbf{x}_m,\bar{k}+\gamma\beta,\tilde{\beta}-\beta)\hat{f}(\beta)\hat{g}(\tilde{\beta}-\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0(\mathbf{x}_s)\, d\mathbf{x}_s d\mathbf{x}_m e^{i\tilde{\beta}\tau}\, d\beta d\tilde{\beta} \\
&= \int \Lambda(L,\mathbf{x}_s,\mathbf{x},\mathbf{x}_m,\mathbf{x}_m,\bar{k}+\gamma(\tilde{\beta}-\beta),\beta)\hat{f}(\tilde{\beta}-\beta)\hat{g}(\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0(\mathbf{x}_s)\, d\mathbf{x}_s d\mathbf{x}_m e^{i\tilde{\beta}\tau}\, d\beta d\tilde{\beta}.
\end{aligned}$$

We now scale the source support and observation point by γ as:

$$\mathbf{x}_s = \gamma\mathbf{x}_s'\,, \quad \mathbf{x} = \gamma\mathbf{x}'.$$

and then

$$\begin{aligned}
&\mathbb{E}[u_{tr,I}(L/c_0+\tau,\gamma\mathbf{x}',L)]e^{\frac{-i\bar{k}\tau}{\gamma}} \\
&\sim \gamma^d \int \Lambda(L,\gamma\mathbf{x}_s',\gamma\mathbf{x}',\mathbf{x}_m,\mathbf{x}_m,\bar{k}-\gamma\beta/2,\beta)\hat{f}(\tilde{\beta}-\beta)\hat{g}(\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0(\gamma\mathbf{x}_s')\, d\mathbf{x}_s' d\mathbf{x}_m e^{i\tilde{\beta}\tau}\, d\beta d\tilde{\beta}.
\end{aligned} \tag{53}$$

The associated Green function coordinates according to the expression (44) for the Schrödinger spectral function are then to leading order:

$$\mathbf{y}_1 = (\mathbf{x}_s' - \mathbf{x}')\bar{k}^{3/2}\,, \quad \mathbf{y}_2 = (\mathbf{x}_s' - \mathbf{x}')\bar{k}^{3/2}\,, \quad \mathbf{y}_1' = 0 \quad \mathbf{y}_2' = -\beta\sqrt{\bar{k}}\mathbf{x}_m\,,$$

and it follows that the refocused pulse then can be characterized by

$$\begin{aligned}
&\mathbb{E}[u_{tr,I}(L/c_0+\tau,\gamma\mathbf{x}',L)]e^{\frac{-i\bar{k}\tau}{\gamma}}\\
&\sim \int\left(\frac{\bar{k}^2}{\pi^2 L}\right)^{d/2}\left(\frac{(1+i)\sqrt{\beta D}}{\sin(\sqrt{D\beta}(1+i)L)}\right)^{d/2} e^{-\beta^2 D_0 L}\, e^{\frac{i\bar{k}^2|\mathbf{x}'_s-\mathbf{x}'|^2}{2\beta L}}\\
&\quad\times\exp\left[\frac{(1-i)}{2\sqrt{\beta}}\sqrt{D}\bar{k}^2\cot\left(\sqrt{D\beta}(1+i)L\right)\left|\mathbf{x}'_s-\mathbf{x}'+\frac{\beta\mathbf{x}_m}{\bar{k}\cos\left(\sqrt{D\beta}(1+i)L\right)}\right|^2\right]\\
&\quad\times\exp\left[\frac{(i-1)}{2}\sqrt{D}\beta^{3/2}|\mathbf{x}_m|^2\tan\left(\sqrt{D\beta}(1+i)L\right)\right]\hat{f}(\tilde{\beta}-\beta)\hat{g}(\beta)\\
&\quad\times\mathbb{I}_A(\mathbf{x}_m)\Phi_0\left(\gamma\mathbf{x}'_s\right)\,d\mathbf{x}'_s d\mathbf{x}_m e^{i\tilde{\beta}\tau}\,d\beta d\tilde{\beta}.
\end{aligned}$$

Remark 1. *In the above expression the spectrum for $\beta<0$ is found by: evaluating it at $|\beta|$ and then taking its complex conjugate.*

We now use the long distance asymptotical approximation for this expression to characterize the mean refocused signal in that regime.

6.1. **Long Propagation Limit.** For large propagation distances $L\gg 1$ we have the following asymptotic

$$\begin{aligned}
&\mathbb{E}[u_{tr,I}(L/c_0+\tau,\gamma\mathbf{x}',L)]e^{\frac{-i\bar{k}\tau}{\gamma}}\\
&\sim \int\left(\frac{2(1-i)\bar{k}^2\sqrt{D\beta}}{\pi^2 L}\right)^{d/2}\hat{f}(\tilde{\beta}-\beta)\hat{g}(\beta)\mathbb{I}_A(\mathbf{x}_m)\Phi_0\left(\gamma\mathbf{x}'_s\right)e^{-(1-i)d\sqrt{D\beta}L/2}e^{-\beta^2 D_0 L}\\
&\quad\times e^{\frac{i\bar{k}^2|\mathbf{x}'_s-\mathbf{x}'|^2}{2\beta L}}\exp\left[\frac{-(i+1)}{2\sqrt{\beta}}\sqrt{D}\bar{k}^2\left(|\mathbf{x}'_s-\mathbf{x}'|^2+|\beta\mathbf{x}_m/\bar{k}|^2\right)\right]d\mathbf{x}'_s d\mathbf{x}_m e^{i\tilde{\beta}\tau}\,d\beta d\tilde{\beta}.
\end{aligned}$$

We simplify this expression further by assuming a point mirror and source:

$$\Phi_0(\mathbf{x}')=\gamma^d\delta(\mathbf{x}'),\quad \mathbb{I}_A(\mathbf{x}')=\delta(\mathbf{x}'). \tag{54}$$

The scaling of the source is chosen so that the refocused signal is of order one and from linearity of the problem this choice plays no essential role. We then obtain

$$\begin{aligned}
&\mathbb{E}[u_{tr,I}(L/c_0+\tau,\gamma\mathbf{x}',L)]e^{\frac{-i\bar{k}\tau}{\gamma}} \qquad (55)\\
&\sim\int\left(\frac{2(1-i)\bar{k}^2\sqrt{D\beta}}{\pi^2 L}\right)^{d/2}e^{-(1-i)d\sqrt{D\beta}L/2}e^{-\beta^2 D_0 L}e^{\frac{i\bar{k}^2|\mathbf{x}'|^2}{2\beta L}}\,e^{\frac{-(i+1)\sqrt{D}\bar{k}^2|\mathbf{x}'|^2}{2\sqrt{\beta}}}\\
&\quad\times\hat{f}(\tilde{\beta}-\beta)\hat{g}(\beta)\,e^{i\tilde{\beta}\tau}\,d\beta d\tilde{\beta}\\
&= f(-\tau)\int\left(\frac{2(1-i)\bar{k}^2\sqrt{D\beta}}{\pi^2 L}\right)^{d/2}e^{-(1-i)d\sqrt{D\beta}L/2}e^{-\beta^2 D_0 L}e^{\frac{i\bar{k}^2|\mathbf{x}'|^2}{2\beta L}}\,e^{\frac{-(i+1)\sqrt{D}\bar{k}^2|\mathbf{x}'|^2}{2\sqrt{\beta}}}\\
&\quad\times\hat{g}(\beta)\,e^{i\beta\tau}\,d\beta.
\end{aligned}$$

This gives the following result

Theorem 1. *In the high frequency ($\gamma\to 0$) limit the ensemble-averaged backpropagated pulse of the white-noise model has the following large distance ($L\gg 1$) asymptotic for $D>0$:*

$$\mathbb{E}[u_{tr,I}(L/c_0+\tau,\gamma\mathbf{x}',L)]e^{\frac{-i\bar{k}\tau}{\gamma}}\sim f(-\tau)\left\{f_g(\cdot)*f_{D_0}(\cdot)*f_D(\cdot;\mathbf{x}')\right\}(-\tau) \tag{56}$$

with

$$f_g(\tau) = \frac{1}{2\pi}\left(\frac{2\sqrt{D}(1-i)\bar{k}^2}{\pi^2 L}\right)^{d/2}\int \beta^{d/4}\hat{g}(\beta)e^{-i\beta\tau}\,d\beta\,,$$

$$f_{D_0}(\tau) = \frac{1}{\sqrt{4\pi D_0 L}}e^{-\tau^2/(4D_0 L)}$$

and

$$f_D(\tau;\mathbf{x}') = \int e^{-(1-i)\sqrt{D\beta}dL/2}e^{\frac{i\bar{k}^2|\mathbf{x}'|^2}{2\beta L}}e^{\frac{-(i+1)\sqrt{D}\bar{k}^2|\mathbf{x}'|^2}{2\sqrt{\beta}}}e^{-i\beta\tau}\,d\beta. \tag{57}$$

Thus, the original source pulse shape is modified via multiplication with a pulse modulation function that depends on the lateral offset $\mathbf{x}'$ of the observation point. In fact, this function is the convolution of three functions. The function f_g contains the signature of the temporal aperture function g. We have assumed that this function is compactly supported. In the case that the whole time trace is recorded at the mirror, so that $g \equiv 1$, we cannot factorize as in (56). In fact, in this case the above analysis simplifies since we do not have to consider nearby frequency correlations.

The second function f_{D_0} is the Gaussian pulse of variance $2D_0L$ with D_0 characterizing the magnitude of the medium fluctuations. A large magnitude D_0 does *not* lead to a temporal spread-out of the refocused signal in the high frequency limit $\gamma \to 0$ of the parabolic approximation. This is contrary to the situation with strong longitudinal scattering in a layered medium which leads to temporal spreading of the pulse [12].

The third function $f_D(\cdot,\mathbf{x}')$ has a support that depends on the correlation parameter D and also on the lateral observation point $\mathbf{x}'$. We shall see that large values for D and large offset leads to a diminishing magnitude for this function and hence to spatial refocusing of the back-propagated signal. Consider first the case $\mathbf{x}' = 0$. We have

$$f_D(\tau;0) = \int e^{-(1-i)\sqrt{D\beta}dL/2}e^{-i\beta\tau}\,d\beta.$$

The support of the function $f_D(\tau;0)$ is of the order DL^2, and increases rapidly with the propagation distance L and the magnitude of this function is of the order $1/(DL^2)$. On the other hand the support of f_g is independent of D and L, and its magnitude is of the order $D^{d/4}L^{-d/2}$. Also the support of f_{D_0} is of the order $\sqrt{D_0 L}$ and has the magnitude $(D_0L)^{-1/2}$. Thus, we conclude

Remark 2. *The amplitude of the refocused signal has the long distance asymptotic*

$$\frac{1}{L^{2+d/2}}.$$

The lateral refocusing of the back-propagated signal depends on the lateral support scale for the function $f(\cdot,\mathbf{x}')$ and we next identify this scale. We rewrite (57) as

$$f_D(\tau;\mathbf{x}') = D^{-1}L^{-2}\int e^{-(1-i)\sqrt{\beta}d/2}e^{\frac{iDL\bar{k}^2|\mathbf{x}'|^2}{2\beta}}e^{\frac{-(i+1)DL\bar{k}^2|\mathbf{x}'|^2}{2\sqrt{\beta}}}e^{-i\beta\tau}\,d\beta\,,$$

which leads to the following observations

Remark 3. *The refocused pulse has the following lateral scale*

$$\triangle \mathbf{x} \quad = \gamma \triangle \mathbf{x}' \quad \sim \frac{\gamma}{\bar{k}\sqrt{DL}}.$$

Note that this is the coherence length at the coherence bandwidth as introduced in (50).

Remark 4. *In the case that the Brownian field B in (25) decorrelate rapidly in the lateral dimension as in the scaling (6) the time reversed field will actually be self-averaging, meaning that the random fluctuations in the refocused wave field are small, see* [9].

7. **Application to Transmitted Field.** In this section we analyze the spreading and decorrelation of the transmitted field in the case that the initial field is specified at $z_s = 0, t_s = 0$ so that (8) becomes

$$\Psi(k, \mathbf{x}, 0) \quad = \quad \Phi_0\left(\mathbf{x}\right) \frac{\hat{f}\left(\frac{k+\bar{k}}{\gamma}\right) + \hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)}{2\gamma}.$$

The transmitted envelope field can then be expressed as

$$\Psi(k, \mathbf{x}, z) \quad = \quad \int G_z(\mathbf{x}; \mathbf{x}_s, k)\Phi_0\left(\mathbf{x}_s\right) \frac{\hat{f}\left(\frac{k+\bar{k}}{\gamma}\right) + \hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)}{2\gamma} \, d\mathbf{x}_s \,,$$

and the wavefield by

$$\begin{aligned} u(t, \mathbf{x}, z) \quad &= \quad \int G_z(\mathbf{x}; \mathbf{x}_s, k)\Phi_0\left(\mathbf{x}_s\right) \frac{\hat{f}\left(\frac{k+\bar{k}}{\gamma}\right) + \hat{f}\left(\frac{k-\bar{k}}{\gamma}\right)}{2\gamma} e^{\frac{ik(z/c_0-t)}{\gamma}} \, d\mathbf{x}_s dk \\ &= \quad \frac{u_I(t, \mathbf{x}, z) + u_I^*(t, \mathbf{x}, z)}{2}\,, \end{aligned}$$

with

$$\begin{aligned} u_I(t, \mathbf{x}, z) \quad &= \quad \int G_z(\mathbf{x}; \mathbf{x}_s, k)\Phi_0\left(\mathbf{x}_s\right) \hat{f}\left(\frac{k-\bar{k}}{\gamma}\right) \frac{1}{\gamma} e^{\frac{ik(z/c_0-t)}{\gamma}} \, d\mathbf{x}_s dk \\ &= \quad \int G_z(\mathbf{x}; \mathbf{x}_s, \bar{k}+\gamma\beta)\Phi_0\left(\mathbf{x}_s\right) \hat{f}\left(\beta\right) e^{i\beta(z/c_0-t)} \, d\mathbf{x}_s d\beta e^{\frac{i\bar{k}(z/c_0-t)}{\gamma}}. \end{aligned}$$

7.1. **The Transmitted Cross-moment.** The quantity of interest we will examine here is the cross-moment defined by

$$C(z, \mathbf{x}, t; \Delta\mathbf{x}, \Delta t) \quad = \quad \mathbb{E}\left[u(t, \mathbf{x}, z)u(t+\Delta t, \mathbf{x}+\Delta\mathbf{x}, z)\right]. \tag{58}$$

In order to analyze this moment we consider

$$
\begin{aligned}
&C_I(z,\mathbf{x},t;\Delta\mathbf{x},\Delta t) = \mathbb{E}\left[u_I(t,\mathbf{x},z)u_I^*(t+\Delta t,\mathbf{x}+\Delta\mathbf{x},z)\right] \\
&= \int \mathbb{E}\left[G_z(\mathbf{x};\mathbf{x}_s,\bar{k}+\gamma\beta)G_z^*(\mathbf{x}+\Delta\mathbf{x};\tilde{\mathbf{x}}_s,\bar{k}+\gamma\tilde{\beta})\right] \\
&\times \Phi_0\left(\mathbf{x}_s\right)\Phi_0^*\left(\tilde{\mathbf{x}}_s\right)\hat{f}\left(\beta\right)\hat{f}^*(\tilde{\beta})e^{i\beta(z/c_0-t)}e^{-i\tilde{\beta}(z/c_0-t-\Delta t)}\,d\mathbf{x}_s d\beta d\tilde{\mathbf{x}}_s d\tilde{\beta} e^{\frac{i\bar{k}\Delta t}{\gamma}} \\
&= \int \mathbb{E}\left[G_z(\mathbf{x};\mathbf{x}_s,\bar{k}+\gamma(\beta+\tilde{\beta}))G_z^*(\mathbf{x}+\Delta\mathbf{x};\tilde{\mathbf{x}}_s,\bar{k}+\gamma\tilde{\beta})\right] \\
&\times \Phi_0\left(\mathbf{x}_s\right)\Phi_0^*\left(\tilde{\mathbf{x}}_s\right)\hat{f}\left(\beta+\tilde{\beta}\right)\hat{f}^*(\tilde{\beta})e^{i\beta(z/c_0-t)}e^{i\tilde{\beta}\Delta t}\,d\mathbf{x}_s d\beta d\tilde{\mathbf{x}}_s d\tilde{\beta} e^{\frac{i\bar{k}\Delta t}{\gamma}} \\
&\sim \int \Lambda\left(z,\mathbf{x},\mathbf{x}+\Delta\mathbf{x},\mathbf{x}_s,\tilde{\mathbf{x}}_s,\bar{k}+\gamma(\beta+\tilde{\beta}),-\beta\right) \\
&\times \Phi_0\left(\mathbf{x}_s\right)\Phi_0^*\left(\tilde{\mathbf{x}}_s\right)\hat{f}\left(\beta+\tilde{\beta}\right)\hat{f}^*(\tilde{\beta})e^{i\beta(z/c_0-t)}e^{i\tilde{\beta}\Delta t}\,d\mathbf{x}_s d\beta d\tilde{\mathbf{x}}_s d\tilde{\beta} e^{\frac{i\bar{k}\Delta t}{\gamma}}\,,
\end{aligned}
$$

with Λ defined in (44). We are interested in this cross-moment at the arrival time for the coherent front when $t = z/c_0$ and when

$$\Delta\mathbf{x} \;=\; \gamma\Delta\mathbf{x}'\,,$$

we then find that

$$
\begin{aligned}
&C_I(z,\mathbf{x},z/c_0;\gamma\Delta\mathbf{x}',\Delta t)e^{-\frac{i\bar{k}\Delta t}{\gamma}} \\
&\sim \quad \int \Lambda\left(z,\mathbf{x},\mathbf{x}+\gamma\Delta\mathbf{x}',\mathbf{x}_s,\tilde{\mathbf{x}}_s,\bar{k}+\gamma(\beta+\tilde{\beta}),-\beta\right)\Phi_0\left(\mathbf{x}_s\right)\Phi_0^*\left(\tilde{\mathbf{x}}_s\right) \\
&\qquad \times\hat{f}\left(\beta+\tilde{\beta}\right)\hat{f}^*(\tilde{\beta})e^{i\tilde{\beta}\Delta t}\,d\mathbf{x}_s d\beta d\tilde{\mathbf{x}}_s d\tilde{\beta} \\
&= \quad \int \Lambda\left(z,\mathbf{x},\mathbf{x}+\gamma\Delta\mathbf{x}';\mathbf{x}_s,\tilde{\mathbf{x}}_s,\bar{k}-\gamma\beta/2,-\beta\right)\Phi_0\left(\mathbf{x}_s\right)\Phi_0^*\left(\tilde{\mathbf{x}}_s\right)\,d\mathbf{x}_s d\tilde{\mathbf{x}}_s \\
&\qquad \times \int f(\tau)f(\tau+\Delta t)e^{i\tau\beta}\,d\tau d\beta/(2\pi)\,,
\end{aligned}
$$

as $\gamma \to 0$. Note that the decorrelation in the time lag parameter is determined by source pulse shape f rather than the random effects captured by the Λ. This is consistent with the fact that we consider the parabolic regime where lateral rather than longitudinal scattering is important. We simplify the expression further by assuming a unit point source located at the origin as in (54) and that $\Delta t = 0$ and for $\mathbf{x} = 0$, then

$$C_I(z,0,z/c_0;\gamma\Delta\mathbf{x}',0) \sim \gamma^d \int \Lambda\left(z,0,\gamma\Delta\mathbf{x}',0,0,\bar{k}-\gamma\beta/2,-\beta\right)\ \hat{f}\star\hat{f}(\beta)\,d\beta\,,$$

with $\star$ indicating the convolution. We consider a long propagation distance regime and use (43)- (46) for the arguments

$$\mathbf{y}_1 = -\Delta\mathbf{x}'\bar{k}^{3/2}\,,\quad \mathbf{y}_2 = -\Delta\mathbf{x}'\bar{k}^{3/2}\,,\quad \mathbf{y}_1' = 0 \quad \mathbf{y}_2' = 0\,.$$

We then obtain the following.

Theorem 2. *In the high frequency and long propagation limit for the white noise model, the forward propagating pulse has the following asymptotic for $D > 0$:*

$$
\begin{aligned}
C_I(z,0,z/c_0;\gamma\Delta\mathbf{x}',0) \quad \sim \quad & \int \left(\frac{2(1-i)\sqrt{\beta D}\bar{k}^2}{\pi^2 z}\right)^{d/2} e^{-(1-i)d\sqrt{D\beta}z/2} \qquad (59)\\
& \times e^{-\beta^2 D_0 z} e^{\frac{i\bar{k}^2|\Delta\mathbf{x}'|^2}{2\beta z}} e^{\frac{-(i+1)\sqrt{D}\bar{k}^2|\Delta\mathbf{x}'|^2}{2\sqrt{\beta}}}\ \hat{f}\star\hat{f}(\beta)\,d\beta.
\end{aligned}
$$

Remark 5. *The expression (59) is exactly the same type of expression as (55). Hence the lateral decoherence length in the forward propagation is on the scale of the lateral size of the refocal spot in the time reversal experiment. In particular the lateral decoherence length has the asymptotic*

$$\triangle \mathbf{x} \quad = \gamma \triangle \mathbf{x}' \quad \sim \frac{\gamma}{\bar{k}\sqrt{Dz}},$$

with the carrier frequency being $\bar{k}/\gamma$.

This extends a similar duality relation discovered in [7] *for monochromatic waves to multi-frequency waves.*

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E-mail address: fannjian@math.ucdavis.edu; ksolna@math.uci.edu