Boundary Value Problem with Unbounded Random Flows

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Abstract. We modify Tartar's classical method to provide an alternative proof of a sharp homogenization theorem for diffusion in a turbulent flow. The flow has a stationary and square integrable stream matrix. This is accomplished by introducing the approximate correctors. We also obtain two-term asymptotics.

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1 Introduction

Let $\boldsymbol{b}(\boldsymbol{x})$, with $\nabla \cdot \boldsymbol{b}(\boldsymbol{x}) = 0$, be an incompressible velocity field in R^d, $d \geq 2$, and let $T(t, \boldsymbol{x})$ be the temperature distribution in the fluid with the velocity $\boldsymbol{b}(\boldsymbol{x})$ and some microscopic heat conductivity σ. The temperature satisfies the convection-diffusion equation

$$\frac{\partial T}{\partial t} + \boldsymbol{b}(\boldsymbol{x}) \cdot \nabla T = \sigma \Delta T, \tag{1.1}$$

with $T(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$.

We shall study the long time, large space scale behavior of $T(t, \boldsymbol{x})$ under the influence of both the velocity $\boldsymbol{b}(\boldsymbol{x})$ and the microscopic conductivity σ described by the equation (1.1). This is done by rescaling equation (1.1) by the diffusive scaling:

$$\boldsymbol{x} \to \boldsymbol{x}/\varepsilon, \quad t \to t/\varepsilon^2 \tag{1.2}$$

for small ε. Thus equation (1.1) becomes

$$\frac{\partial T_\varepsilon}{\partial t} + \frac{1}{\varepsilon} \boldsymbol{b}(\boldsymbol{x}/\varepsilon) \cdot \nabla T_\varepsilon = \sigma \Delta T_\varepsilon, \tag{1.3}$$

with slowly varying initial condition $T_\varepsilon(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$. This is particularly relevant when the velocity field has a repetitive structure as, for example, when it is a periodic or a stationary random function with mean zero. We expect then an overall diffusive behavior with an *effective* diffusion constant which is invariant under the scaling (1.2). When this happens, the equation (1.1) is said to *homogenize*.

The sharp condition under which the effective diffusion takes place is best formulated in terms of the stream matrix $\boldsymbol{H}(\boldsymbol{x}, \omega)$ which is skew-symmetric and satisfies

$$\nabla \cdot \boldsymbol{H}(\boldsymbol{x}, \omega) = \boldsymbol{b}(\boldsymbol{x}, \omega) \tag{1.4}$$

where ω denotes the randomness. Such matrix $\boldsymbol{H}$ always exists, because $\boldsymbol{b}(\boldsymbol{x}, \omega)$ is incompressible and has mean zero, but may not be stationary even $\boldsymbol{b}(\boldsymbol{x}, \omega)$ is stationary.

We assume throughout this paper that the velocity field comes from a *stationary* stream matrix $\boldsymbol{H}(\boldsymbol{x}, \omega)$ which is *square integrable* and (1.4) is meant in the weak sense. In terms of the stream matrix, the equation (1.3) can be written in divergence form

$$\frac{\partial T_\varepsilon(t, \boldsymbol{x}, \omega)}{\partial t} = \nabla \cdot [(\sigma \boldsymbol{I} + \boldsymbol{H}(\boldsymbol{x}/\varepsilon, \omega))\nabla T_\varepsilon(t, \boldsymbol{x}, \omega)], \tag{1.5}$$

where $\boldsymbol{I}$ is the identity matrix.

For stationary square integrable stream matrices,

$$\langle |\boldsymbol{H}_{i,j}|^2 \rangle < \infty, \quad \forall i, j \tag{1.6}$$

Fannjiang and Papanicolaou [4] proved homogenization of (1.5) in the sense that solution T_ε of equation (1.3) converges to $\bar{T}$ as $\varepsilon \to 0$:

$$\lim_{n\to\infty} \sup_{0\le t\le \tau} \int_{R^d} |T_\varepsilon(t, \boldsymbol{x}, \omega) - \bar{T}(t, \boldsymbol{x})|^2 d\boldsymbol{x} = 0 \tag{1.7}$$

for any $\tau < \infty$ and with probability one, where $\bar{T}$ is the solution of the effective diffusion equation

$$\frac{\partial \bar{T}(t, \boldsymbol{x})}{\partial t} = \sum_{i,j=1}^{d} \frac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}) \frac{\partial^2 \bar{T}(t, \boldsymbol{x})}{\partial x_i \partial x_j} \tag{1.8}$$

with $\bar{T}(0, \boldsymbol{x}) = T_0(\boldsymbol{x})$ where $\sigma^*_{i,j}$ is a constant matrix called the *effective diffusivity.* They were also able to characterize the asymptotic behavior of the *gradient* ∇T_ε in the strong norm. The sharpness of condition (1.6) is demonstrated in a class of random shear layer flows for which (1.6) is necessary and sufficient. The significance of this result is that it establishes the condition (1.6) as part of the boundary of homogenization regime.

Their proof relies on the novel variational principles for the *steady-state* problem derived from the original time evolution problem by the Laplace transform. The variational principles are nonlocal in nature.

In *bounded domains* where *boundary* conditions are present, the nonlocality of the variational principles requires subtle construction of cut-off functions to treat boundary behaviors. With this complication, the evaluation of nonlocal functionals is horrendous calculation but the benefits are the result of well-posedness in a suitable space which is not obvious at all for convection-diffusion equation with unbounded flows and, consequently, the characterization of the *gradient* of T_ε in the limit $\varepsilon \to 0$ in the *strong* L^2 norm. The latter one is also referred to as the two-term asymptotics.

In the fundamental paper of Papanicolaou and Varadhan [6] boundary value problems of divergence form equations with *bounded* random coefficients are considered. They demonstrated how Tartar's classical method [7] of oscillatory test functions can be effectively used for homogenization in random media. This method was reconsidered by Avellaneda and Majda [1] to address the convection-diffusion problem with unbounded random coefficients. They proved weak homogenization

$$\lim_{\varepsilon\to 0}\int_0^t \int_D \langle (T_\varepsilon(s,\boldsymbol{x},\omega) - \bar{T}(s,\boldsymbol{x}))\phi(s,\boldsymbol{x})\rangle = 0, \quad \forall \phi \in L^2((0,\infty), L^2(D)), \quad \forall t > 0 \tag{1.9}$$

for unbounded flows satisfying

$$\langle |\boldsymbol{b}_k(\boldsymbol{x},\cdot)|^{d+\delta}\rangle + \langle |\boldsymbol{H}_{i,j}(\boldsymbol{x},\cdot)|^p\rangle < \infty, \quad i,j,k = 1,2,...,d, \tag{1.10}$$

for some $\delta > 0$ where $p = 2+\delta$, if $d = 2$ and $p = d$ for $d \geq 3$. Here $\langle\cdot\rangle$ denotes the ensemble average w.r.t. ω.

Condition (1.10) is required to justify using Tartar's oscillatory test functions and passing to the limit. In view of the Hölder and Sobolev inequalities on which the proof in [1] is based, the mear square integrability of stream matrices is not sufficient since both the solution and Tartar's test functions are only known to be L^2 functions.

To make this line of argument work for square integrable stream matrix one clearly needs to work with approximate test functions rather than Tartar's exact test functions. The gap is then bridged by an approximation lemma 3.1 proved by using the variational method for abstract cell problems where there is no difficulty with boundaries. Completing this argument to show the following theorem without compromising the sharp condition (1.6) is the main motivation of this paper.

Theorem 1.1 *Suppose that the stream matrix is square integrable and that the sequence of solutions T_ε of (1.5) with homogeneous Dirichlet condition have uniformly bounded energy. Then*

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq\tau}\int_D |T_\varepsilon(t,\boldsymbol{x},\omega)-\bar{T}(t,\boldsymbol{x})|^2 d\boldsymbol{x}=0 \tag{1.11}$$

for almost all realizations, where $\bar{T}$ is the solutions of the initial-boundary value problem of (1.8).

The notion of *uniformly bounded energy* is defined in (2.22). The effective diffusivity σ^* is defined in (3.8).

With an additional assumption, we shall also establish the two-term asymptotics

Theorem 1.2 *Assume, in addition to those in Theorem 1.1, that the solutions T_ε satisfy the energy equality (stated in (2.25)). Then*

$$\lim_{\varepsilon\to\ 0}\sup_{t\in[0,\tau]}\int_D(\nabla T_\varepsilon(\boldsymbol{x})-\nabla\bar{T}(\boldsymbol{x})-\sum_i\frac{\partial\bar{T}}{\partial x_i}(\boldsymbol{x})\nabla\chi_i(\boldsymbol{x}/\varepsilon))^2=0 \tag{1.12}$$

for almost all ω. Here $\chi_i, i=1,2,3,...,d$ are the correctors defined in (3.12)-(3.13).

The solution produced by the variational methods in ([4]) satisfies the energy equality.

Other advantages of this approach include the flexibility of carrying over to the case of *space-time* random velocity $\boldsymbol{b}(\boldsymbol{x},t)$, where the initial-boundary value problem, not reducible to a boundary value problem, does not admit a variational formulation. We will address the problem with time dependent velocity in a forthcoming paper.

2 Formulation and Result

For simplicity we set σ to be one.

Since the stream matrix is time independent we work entirely with time independent problems through the Laplace transform of (1.5)

$$\hat{T}_\varepsilon(\lambda, \boldsymbol{x}, \omega) = \int_0^\infty e^{-\lambda t} T_\varepsilon(t, \boldsymbol{x}, \omega) dt \ , \quad \lambda > 0 \tag{2.1}$$

which satisfies

$$-\nabla \cdot [(\boldsymbol{I} + \boldsymbol{H}(\boldsymbol{x}/\varepsilon, \omega))\nabla \hat{T}_\varepsilon(\lambda, \boldsymbol{x}, \omega)] + \lambda \hat{T}_\varepsilon(\lambda, \boldsymbol{x}, \omega) = T_0(\boldsymbol{x}), \tag{2.2}$$

for $\boldsymbol{x} \in R^d$. Convergence of $\hat{T}_\varepsilon$ in L^2, with probability one, to the Laplace transform of $\bar{T}$ for each $\lambda > 0$ implies (1.7).

Dropping the hat and λ from $\hat{T}_\varepsilon(\lambda, \boldsymbol{x}, \omega)$, the Dirichlet problem has the weak form

$$\int_D (\boldsymbol{I} + \boldsymbol{H}(\boldsymbol{x}/\varepsilon, \omega))\nabla T_\varepsilon(\boldsymbol{x}, \omega) \cdot \nabla \phi(\boldsymbol{x}, \omega) d\boldsymbol{x} + \lambda \int_D T_\varepsilon(\boldsymbol{x}, \omega)\phi(\boldsymbol{x}, \omega) = \int_D T_0(\boldsymbol{x})\phi(\boldsymbol{x}, \omega) d\boldsymbol{x} \tag{2.3}$$

for every test function $\phi \in W_0^{1,\infty}(D)$. To make precise the meaning of (2.3) let us review the framework of stationary processes in the next section.

Theorem 1.1 then reduces to Theorem 2.1 stated below regarding the convergence of solution of the equation (2.3).

2.1 Random Stationary Processes and Associated Spaces

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\boldsymbol{H}(\boldsymbol{x}, \omega)$ be a *strictly stationary* random skew-symmetric matrix of $\boldsymbol{x} \in R^d$ such that each element $\boldsymbol{H}_{ij}$ is a L^2 function

$$\langle |\boldsymbol{H}_{ij}(\boldsymbol{x}, \cdot)|^2 \rangle < \infty, \quad \forall i, j, \tag{2.4}$$

where $\langle \cdot \rangle$ denotes the average or integral with respect to the measure P. By strict stationarity we mean that the joint distribution of $\boldsymbol{H}_{ij}(\boldsymbol{x}_1, \omega), \boldsymbol{H}_{ij}(\boldsymbol{x}_2, \omega), \ldots, \boldsymbol{H}_{ij}(\boldsymbol{x}_n, \omega)$

for any points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n \in R^d$ and that of $\boldsymbol{H}_{ij}(\boldsymbol{x}_1+\boldsymbol{l}, \omega), \boldsymbol{H}_{ij}(\boldsymbol{x}_2+\boldsymbol{l}, \omega), \ldots, \boldsymbol{H}_{ij}(\boldsymbol{x}_n+\boldsymbol{l}, \omega)$ for any $\boldsymbol{l} \in R^d$ is the same, so the averages in (2.4) are independent of $\boldsymbol{x}$. Without loss of generality (see Doob [3]), we may assume that there is a group of transformations $\tau_{\boldsymbol{x}}$, $\boldsymbol{x} \in R^d$ from Ω into Ω that is one to one and preserves the measure P. That is, $\tau_{\boldsymbol{x}}\tau_{\boldsymbol{y}} = \tau_{\boldsymbol{x}+\boldsymbol{y}}$ and $P(\tau_{\boldsymbol{x}}A) = P(A)$ for any $A \in \mathcal{F}$. We may also suppose that there is a square integrable (w.r.t. P) matrix function $\tilde{\boldsymbol{H}}(\omega)$ such that

$$\boldsymbol{H}(\boldsymbol{x}, \omega) = \tilde{\boldsymbol{H}}(\tau_{-\boldsymbol{x}}\,\omega)\ , \qquad \boldsymbol{x} \in R^d\ , \qquad \omega \in \Omega\ . \tag{2.5}$$

We assume that the group of transformations $\tau_{\boldsymbol{x}}$ is ergodic with respect to the probability measure P.

The random stationary divergence free velocity $\boldsymbol{b}$ which we consider in this paper is given by

$$-\boldsymbol{b}(\boldsymbol{x}, \omega) = \nabla \cdot \boldsymbol{H}(\boldsymbol{x}, \omega). \tag{2.6}$$

In two dimension the matrix $\boldsymbol{H}$ has the form

$$\boldsymbol{H} = \begin{pmatrix} 0 & -\boldsymbol{h} \\ \boldsymbol{h} & 0 \end{pmatrix} \tag{2.7}$$

where $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{x}, \omega)$ is the usual stream function. In three dimensions, $\boldsymbol{H}$ has the form

$$\boldsymbol{H} = \begin{pmatrix} 0 & -\boldsymbol{h}_3 & \boldsymbol{h}_2 \\ \boldsymbol{h}_3 & 0 & -\boldsymbol{h}_1 \\ -\boldsymbol{h}_2 & \boldsymbol{h}_1 & 0 \end{pmatrix}. \tag{2.8}$$

where $\boldsymbol{h}(\boldsymbol{x}, \omega) = (\boldsymbol{h}_1(\boldsymbol{x}, \omega), \boldsymbol{h}_2(\boldsymbol{x}, \omega), \boldsymbol{h}_3(\boldsymbol{x}, \omega))$ is the vector potential of the flow $\boldsymbol{b}$ so that $\nabla \cdot \boldsymbol{H} = \nabla \times \boldsymbol{h} = -\boldsymbol{b}$.

The group of transformations $\tau_{\boldsymbol{x}}$ acting on Ω induces a group of operators on the Hilbert space of real-valued functions $\mathcal{H} = L^2(\Omega, \mathcal{F}, P)$ with inner product

$$(\tilde{f}, \tilde{g}) \equiv \langle \tilde{f}\tilde{g} \rangle \equiv \int_\Omega P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega)\ , \qquad \tilde{f}, \tilde{g} \in \mathcal{H}. \tag{2.9}$$

The group of operators $T_{\boldsymbol{x}}$ on $\mathcal{H}$ is given by

$$(T_{\boldsymbol{x}}\tilde{f})(\omega) = \tilde{f}(\tau_{-\boldsymbol{x}}\omega) \ , \qquad \boldsymbol{x} \ \in R^d, \ \omega \in \Omega \tag{2.10}$$

Since $\tau_{\boldsymbol{x}}$ is measure preserving, the operators $T_{\boldsymbol{x}}$ form a unitary group. Therefore they have closed densely defined infinitesimal generators $\tilde{\nabla}_i$ in each direction $i = 1, 2, \ldots, d$ with domains $\mathcal{D}_i \subset \mathcal{H}$. Then,

$$\tilde{\nabla}_i = \frac{\partial}{\partial x_i} T_{\boldsymbol{x}}\Big|_{\boldsymbol{x}=0} \ , \qquad i = 1, \ldots, d \ , \tag{2.11}$$

with differentiation defined in the sense of convergence in $\mathcal{H}$ for elements of $\mathcal{D}_i$. The closed subset of $\mathcal{H}$

$$\mathcal{H}^1 = \bigcap_{i=1}^{d} \mathcal{D}_i \tag{2.12}$$

becomes a Hilbert space with the inner product

$$(\tilde{f}, \tilde{g})_1 \equiv \int_\Omega P(d\omega)\, \tilde{f}(\omega)\, \tilde{g}(\omega) + \sum_{i=1}^{a} \int_\Omega P(d\omega)\, \tilde{\nabla}_i \tilde{f}(\omega)\, \tilde{\nabla}_i \tilde{g}(\omega) \tag{2.13}$$

The hypothesis that the action of the translation group $\tau_{\boldsymbol{x}}$ is ergodic on Ω takes the following form in $\mathcal{H}$: the only functions in $\mathcal{H}$ that are invariant under $T_{\boldsymbol{x}}$ are the constant functions. Let $H_s(R^d; \mathcal{H})$ be the space of all stationary random processes $f(\boldsymbol{x}, \omega)$ on R^d, such that $\int_\Omega P(d\omega) f^2(\boldsymbol{x}, \omega) = \text{const.} < \infty$. Clearly $H_s(R^d; \mathcal{H})$ is in one-to-one correspondence with $\mathcal{H}$ since it is simply the space of all translates of $\mathcal{H}$, that is, $f(\boldsymbol{x}, \omega) \in H_s(R^d; \mathcal{H})$ iff $f(\boldsymbol{x}, \omega) = T_{\boldsymbol{x}}\tilde{f}(\omega)$, $\tilde{f}(\omega) \in \mathcal{H}$. Similarly, we may identify $\mathcal{H}^1$ with the set of mean square differentiable, stationary processes $H_s^1(R^d; \mathcal{H})$. In particular, if $f \in H_s^1$, then its derivatives are also a stationary processes and

$$\nabla_i f(\boldsymbol{x}, \omega) = \frac{\partial f(\boldsymbol{x}, \omega)}{\partial x_i} = \tilde{\nabla}_i f(\boldsymbol{x}, \omega) \tag{2.14}$$

with equality holding $d\boldsymbol{x} \times P$ almost everywhere. Thus, we have $H_s^1(R^d; \mathcal{H}) = H_s(R^d; \mathcal{H}^1)$.

We define also the spaces $\mathcal{H}_g$ and $\mathcal{H}_{g,\infty}$ of square integrable and bounded gradient fields, respectively,

$$\mathcal{H}_g = \left\{ \tilde{F}_i(\omega) \in \mathcal{H} \ , \ i = 1, \ldots, d \mid \tilde{\nabla}_i \tilde{F}_j = \tilde{\nabla}_j \tilde{F}_i, \forall i, j \quad \text{weakly} \right\} \tag{2.15}$$

$$\mathcal{H}_{g,\infty} = \mathcal{H}_g \cup (L^\infty)^d(\Omega, \mathcal{F}, P). \tag{2.16}$$

2.2 Weak Convergence Theorem

The existence of solutions to (2.3) can be shown by a truncation argument: First we introduce the level M truncation $\tilde{\boldsymbol{H}}^{(M)}$ of the stream matrix:

$$\tilde{\boldsymbol{H}}_{i,j}^{(M)} = \begin{cases} \tilde{\boldsymbol{H}}_{i.j}, & \text{for } |\tilde{\boldsymbol{H}}_{i,j}| < M \\ \mathrm{sig}(\tilde{\boldsymbol{H}}_{i,j})M, & \text{for } |\tilde{\boldsymbol{H}}_{i,j}| \geq M \end{cases} \tag{2.17}$$

for all i, j. Thus $|\tilde{\boldsymbol{H}}_{i,j}^{(M)}| \leq M, \forall i, j$.

By individual ergodic theorem [3], the space averages converge to the ensemble average

$$\frac{\varepsilon^d}{|D|} \int_{D/\varepsilon} \boldsymbol{H}_{i,j}^2(\boldsymbol{x}, \omega) = \frac{1}{|D|} \int_D \boldsymbol{H}_{i,j}^2(\frac{\boldsymbol{x}}{\varepsilon}, \omega) \to \langle \tilde{\boldsymbol{H}}_{i,j}^2 \rangle \tag{2.18}$$

as $\varepsilon \to 0$ for almost all realizations. Without loss of generality we may assume that $\boldsymbol{H}(\boldsymbol{x}, \omega)$ is locally square integrable for almost all realizations. Thus the truncated stream matrix converges to $\boldsymbol{H}(\boldsymbol{x}, \omega)$ locally in the L^2 sense.

We consider the similar boundary value problem (2.3) associated with the truncated stream matrix $\tilde{\boldsymbol{H}}^{(M)}$, namely,

$$\int_D (\boldsymbol{I} + \boldsymbol{H}^{(M)}(\boldsymbol{x}/\varepsilon, \omega)) \nabla T_\varepsilon^{(M)}(\boldsymbol{x}, \omega) \cdot \nabla \phi(\boldsymbol{x}, \omega) d\boldsymbol{x} + \lambda \int_D T_\varepsilon^{(M)}(\boldsymbol{x}, \omega) \phi(\boldsymbol{x}, \omega) = \int_D T_0(\boldsymbol{x}) \phi(\boldsymbol{x}, \omega) d\boldsymbol{x} \tag{2.19}$$

for every test function ϕ in $W^{1,\infty}(D, \mathcal{H})$. For $T_0(\boldsymbol{x}) \in L^2(D)$ this problem has a unique solution in $H_0^1(D)$ for each $M > 0$ by the Lax-Milgram lemma. Moreover, the space of test functions for (2.19) can be enlarged to include $H^1(D)$ functions. So subsituting $\phi = T_\varepsilon^{(M)}$ and using the Poincare inequality we obtain

$$\int_D \nabla T_\varepsilon^{(M)} \cdot \nabla T_\varepsilon^{(M)} + \lambda \int_D (T_\varepsilon^{(M)})^2 \leq C \int_D T_0^2 \tag{2.20}$$

for some constant $C > 0$ independent of M, ε and T_0. Here we have used the skew symmetry of $\boldsymbol{H}^{(M)}$. The estimate (2.20) is uniform in M thus we can pass to the limit $M \to \infty$ in $H_0^1(D)$ in the weak norm and extract convergent subsequence, still denoted by $T_\varepsilon^{(M)}$, converging to T_ε weakly in $H_0^1(D)$. Since $\boldsymbol{H}^{(M)}$ tends to $\boldsymbol{H}$ strongly in $L^2(D)$ for almost all ω we have

$$\boldsymbol{H}^{(M)}(\boldsymbol{x}/\varepsilon, \omega)\nabla T_\varepsilon^{(M)} \to \boldsymbol{H}(\boldsymbol{x}/\varepsilon, \omega)\nabla T_\varepsilon \tag{2.21}$$

weakly in $L^1(D)$. With this, passing to the limit $M \to \infty$ in (2.19), we obtain a solution for (2.3) for almost all ω. Furthermore, passing the limit in the inequality (2.20) we know the limit solution has *uniformly bounded energy*

$$\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon + \lambda \int_D T_\varepsilon^2 \leq C \int_D T_0^2 \tag{2.22}$$

for some constant $C > 0$ independent of ε. Namely, the energy bound is scale independent.

However, lacking stronger estimates for the solution T_ε the test functions for the limit equation (2.3) are restricted to $W_0^{1,\infty}(D, \mathcal{H})$. Moreover, the usual argument for uniqueness does not work due to lack of the Helmholtz decomposition for $L^1(D)$. In contrast, the solution obtained by variational methods in [4] satisfies a stronger estimate and is unique.

A sequence of solutions to (2.3) is said to have *uniformly bounded energy* if it has a scale independent bound on energy

$$\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon + \lambda \int_D T_\varepsilon^2 < C \tag{2.23}$$

where C is independent of ε. The preceding limit procedure produces a $H_0^1(D)$-solution with uniformly bounded energy to (2.3).

In this paper we concern with homogenization of any sequence of weak solutions with uniformly bounded energy. Now we state the theorem from which Theorem 1.1 follows.

Theorem 2.1 *For almost all ω any sequence of weak solutions T_ε of (2.3) with uniformly bounded energy converge weakly in $H_0^1(D)$ and strongly in $L^2(D)$ to the solution $\bar{T}$ of the deterministic problem*

$$\int_D \sum_{i,j} \frac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}) \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \lambda \int_D \bar{T}\phi = \int_D T_0 \phi \tag{2.24}$$

for all $\phi \in W_0^{1,\infty}(D)$. Here σ^* is a constant matrix with a positive definite symmetric part defined in Theorem 3.1.

A solution T_ε of (2.3) is said to satisfy the *energy equality* if

$$\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon + \lambda \int_D T_\varepsilon^2 = \int_D T_0 T_\varepsilon. \tag{2.25}$$

Clearly (2.25) implies (2.23). The solution produced by the variational methods in ([4]) always satisfies the energy equality (2.25). We do not know if the solution produced by the truncation argument satisfies (2.25). Assuming it does we shall prove the corrector result from which Theorem 1.2 follows.

Theorem 2.2 *Assume, in addition to those in Theorem 2.1, that the solutions T_ε satisfy the energy equality (2.25). Then*

$$\lim_{\varepsilon \to 0} \int_D (\nabla T_\varepsilon - \nabla \bar{T} - \sum_i \frac{\partial \bar{T}}{\partial x_i} \nabla \chi_i(\boldsymbol{x}/\varepsilon))^2 = 0 \tag{2.26}$$

for almost all ω. Here $\chi_i, i = 1, 2, 3, ..., d$ are the correctors defined in (3.12).

3 Cell Problem, Correctors and Effective Diffusivity

Following the similar line of arguments leading to the existence of solutions to (2.3) we see immediately that there exist functions $\tilde{\boldsymbol{E}}^k \in \mathcal{H}_g,\ k = 1, 2, 3, ..., d$ such that

$$\langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}^k + \tilde{\boldsymbol{E}}^k) \cdot \tilde{\boldsymbol{F}} \rangle = 0, \quad \forall \tilde{\boldsymbol{F}} \in \mathcal{H}_{g,\infty} \tag{3.1}$$

$$\langle \tilde{\boldsymbol{E}}^k \rangle = 0. \tag{3.2}$$

The point is the square integrability of $\tilde{\boldsymbol{H}}$ is just enough to define an effective matrix $\sigma^* = (\sigma^*_{i,j})$ by

$$\sigma^*_{i,j} = \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}^i + \tilde{\boldsymbol{E}}^i) \cdot \boldsymbol{e}^j \rangle, \quad i, j = 1, 2, ..., d. \tag{3.3}$$

Unfortunately like (2.3) we do not have uniqueness for (3.1)-(3.2). Consequently, we do not know if the effective matrix defined by (3.3) is unique. For question of homogenization, unlike above mentioned non-uniqueness for boundary value problem, this is a serious problem that one has to address.

A way out of this is to resort to the variational method alluded to in the Introduction. The variational principles for the cell problem defining σ^* can yield a stronger unique solution. Moreover, to complete Tartar's approach we will also need a crucial approximation lemma which we only know how to prove by the variational principles. This may be implicitly related to the nonuniqueness of σ^* defined otherwise.

Before stating Theorem 3.1 we define a new space $\mathcal{H}_g(\tilde{\boldsymbol{H}})$ for the solution by

$$\mathcal{H}_g(\tilde{\boldsymbol{H}}) = \overline{\{\tilde{\boldsymbol{G}} \in \mathcal{H}_g; \tilde{\boldsymbol{H}}\tilde{\boldsymbol{G}} \in (\mathcal{H})^d\}}. \tag{3.4}$$

completed with respect to the norm

$$\|\tilde{\boldsymbol{G}}\|^2_{\tilde{\boldsymbol{H}}} = \langle \tilde{\boldsymbol{G}} \cdot \tilde{\boldsymbol{G}} \rangle + \langle \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{G}} \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{G}} \rangle \tag{3.5}$$

where the operator $\tilde{\Gamma}$ is the projection of $\mathcal{H}^d$ unto the space $\mathcal{H}_g$.

Theorem 3.1 *There exist unique* $\tilde{\boldsymbol{E}}_k \in \mathcal{H}_g(\tilde{\boldsymbol{H}}),\ k = 1, 2, ..., d$ *such that*

$$\langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}_k + \tilde{\boldsymbol{E}}_k) \cdot \tilde{\boldsymbol{F}} \rangle = 0, \quad \forall \tilde{\boldsymbol{F}} \in \mathcal{H}_g \tag{3.6}$$

$$\langle \tilde{\boldsymbol{E}}_k \rangle = 0 \qquad\qquad . \tag{3.7}$$

The effective diffusivity σ^* *is uniquely defined by*

$$\sigma^*_{i,j} = \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot \boldsymbol{e}_j \rangle, \quad i, j = 1, 2, ..., d. \tag{3.8}$$

Similarily, there exist unique $\tilde{\boldsymbol{E}'}_k \in \mathcal{H}_g(\tilde{\boldsymbol{H}}),\ k = 1, 2, ..., d$ *such that*

$$\langle (\boldsymbol{I} - \tilde{\boldsymbol{H}})(\boldsymbol{e}_k + \tilde{\boldsymbol{E}'}_k) \cdot \tilde{\boldsymbol{F}} \rangle = 0, \quad \forall \tilde{\boldsymbol{F}} \in \mathcal{H}_g \tag{3.9}$$

$$\langle \tilde{\boldsymbol{E}'}_k \rangle = 0 \qquad\qquad . \tag{3.10}$$

The effective diffusivity σ^* *is uniquely defined by*

$$\sigma^*_{i,j} = \langle (\boldsymbol{I} - \tilde{\boldsymbol{H}})(\boldsymbol{e}_j + \tilde{\boldsymbol{E}'}_j) \cdot \boldsymbol{e}_i \rangle, \quad i, j = 1, 2, ..., d. \tag{3.11}$$

The primitive function χ_k of the solution $\tilde{\boldsymbol{E}}_k$ to the *abstract cell problem* (3.6)-(3.7)

$$\nabla \chi_k(\boldsymbol{x}, \omega) = \boldsymbol{E}_k(\boldsymbol{x}, \omega), \quad i = 1, 2, 3, ..., d \tag{3.12}$$

is called the *corrector*. The primitive function χ'_k of the solution $\tilde{\boldsymbol{E}'}_k$ to the *adjoint cell problem* (3.9)-(3.10) is called the *adjoint corrector*. They are nonstationary and are unique when normalized by

$$\chi_k(0, \omega) = 0, \quad i = 1, 2, 3, ..., d \tag{3.13}$$

$$\chi'_k(0, \omega) = 0, \quad i = 1, 2, 3, ..., d \tag{3.14}$$

for almost all ω. The connection between the cell problem and homogenization as stated in Theorem 2.1 comes about by the usual multiple scale arguments and is formally the same in the random as in the periodic case [2, 5]. On physical grounds, the cell problem can be understood as macroscopic concentration gradients $\boldsymbol{e}_k$ that induce through the flow microscopic concentration fluctuations χ_k which in turn lead to enhanced fluxes $-(\boldsymbol{I}+\boldsymbol{H})\nabla\chi_k$ by Fourier's law. The average of the enhanced flux is the macroscopic diffusivity (3.8).

The proof of Theorem 3.1 is given in the Appendix.

It is no surprise to learn later (Lemma A.1) that the unique solution to the abstract cell problem is the weak limit in $\mathcal{H}_g$ of the sequence of solutions to the truncated cell problem:

$$\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}}^{(M)})(\boldsymbol{e}_k+\tilde{\boldsymbol{E}}_k^{(M)})\cdot\tilde{\boldsymbol{F}}\rangle=0,\quad\forall\tilde{\boldsymbol{F}}\in\mathcal{H}_g \tag{3.15}$$

$$\langle\tilde{\boldsymbol{E}}_k^{(M)}\rangle=0 \tag{3.16}$$

where the stream matrix $\tilde{\boldsymbol{H}}^{(M)}$ is the level M truncation, defined by (2.17), of the original stream matrix $\tilde{\boldsymbol{H}}$.

Now we state the crucial estimate.

Lemma 3.1 *Given $\delta>0$ there exists bounded gradient field $\tilde{\boldsymbol{G}}_i$ and $\tilde{\boldsymbol{F}}_i$ in $\mathcal{H}_{g,\infty}$ such that*

$$\langle\tilde{\boldsymbol{G}}\rangle=0 \tag{3.17}$$

$$\langle\tilde{\boldsymbol{F}}\rangle=0 \tag{3.18}$$

$$\langle\left|\tilde{\boldsymbol{G}}_i+\tilde{\Gamma}\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{G}}_i+\boldsymbol{e}_i)\right|^2\rangle<\delta^2 \tag{3.19}$$

$$\langle\left|\tilde{\boldsymbol{F}}_i-\tilde{\Gamma}\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{F}}_i+\boldsymbol{e}_i)\right|^2\rangle<\delta^2 \tag{3.20}$$

$$|\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{G}}_i)\boldsymbol{e}_j\rangle-\sigma^*_{i,j}|<\delta\sqrt{\langle\tilde{\boldsymbol{E}}'_j\cdot\tilde{\boldsymbol{E}}'_j\rangle} \tag{3.21}$$

$$|\langle(\boldsymbol{I}-\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{F}}_i)\boldsymbol{e}_j\rangle-\sigma^*_{j,i}|<\delta\sqrt{\langle\tilde{\boldsymbol{E}}_j\cdot\tilde{\boldsymbol{E}}_j\rangle}. \tag{3.22}$$

Here $\tilde{\boldsymbol{E}}'_j$ and $\tilde{\boldsymbol{E}}_j$ are as defined in Theorem 3.1.

The proof of Lemma 3.1 is given in the Appendix.

Naturally we call the primitive functions $g_i(\boldsymbol{x},\omega)$ and $f_i(\boldsymbol{x},\omega)$ of bounded gradient fields $\tilde{\boldsymbol{G}}_i$ and $\tilde{\boldsymbol{F}}_i$ in Lemma 3.1 *approximate correctors* and *approximate adjoint correctors*, respectively.

For the approximate correctors and the adjoints we have the following L^∞ bound.

Lemma 3.2 *If $\tilde{\boldsymbol{F}}(\tau_{-\boldsymbol{x}}\omega) = \nabla f(\boldsymbol{x},\omega) \in \mathcal{H}_{g,\infty}$ has zero mean and $f(0,\omega)=0$, then*

$$\limsup_{\varepsilon\to 0}\sup_{\omega\in\Omega}\sup_{\boldsymbol{x}\in D}|\varepsilon f(\boldsymbol{x}/\varepsilon,\omega)| \le C_0 < \infty \tag{3.23}$$

$$\lim_{\varepsilon\to\infty}\sup_{\boldsymbol{x}\in D}|\varepsilon f(\boldsymbol{x}/\varepsilon,\omega)| = 0, \quad \text{for almost all } \omega \tag{3.24}$$

where the constant C_0 depends only on $\|\tilde{\boldsymbol{F}}\|_{L^\infty(\Omega,\mathcal{F},P)}$.

Proof. Set $f^\varepsilon(\boldsymbol{x},\omega) = \varepsilon f(\boldsymbol{x}/\varepsilon,\omega)$. Note that

$$\nabla f^\varepsilon(\boldsymbol{x},\omega) = \tilde{\boldsymbol{F}}(\tau_{-\boldsymbol{x}/\varepsilon}\omega) \in \mathcal{H}_{g,\infty}. \tag{3.25}$$

Thus for any ball B we have $f^\varepsilon \in W^{1,\infty}(B)$ uniformly in ω, i.e.,

$$\|f^\varepsilon\|_{W^{1,\infty}(B)} \le C < \infty \tag{3.26}$$

where the constant C is independent of ω. By the compact imbedding

$$W^{1,\infty}(B) \hookrightarrow \{g\in C^0(B) | g(0)=0\} \tag{3.27}$$

we know that (3.23) holds and there is a convergent subsequence, still denoted by f^ε, in $C^0(B)$.

By the averaging lemma 3.3 stated below we have that

$$\int_D \tilde{\boldsymbol{F}}(\tau_{-\boldsymbol{x}/\varepsilon}\omega)\boldsymbol{G}(\boldsymbol{x}) \to \langle\tilde{\boldsymbol{F}}\rangle \int_D \boldsymbol{G} = 0 \tag{3.28}$$

thus the function $\tilde{\boldsymbol{F}}(\tau_{-\boldsymbol{x}/\varepsilon}\omega)$ has a weak limit in $(L^\infty)^d(B)$ which is zero by the zero mean property of $\tilde{\boldsymbol{F}}$.

Therefore the convergent subsequence f^ε must also converge to a constant. But this constant must be zero in view of the normalization condition $f^\varepsilon(0,\omega)=0$. ∎

We need the following averaging lemma for the proof of Theorem 2.1.

Lemma 3.3 *Suppose $\tilde{f}(\omega) \in \mathcal{H}$ and $\phi(\boldsymbol{x}) \in L^2(D)$. Then*

$$\int_D \tilde{f}(\tau_{-\boldsymbol{x}/\varepsilon}\omega)\phi(\boldsymbol{x}) \to \langle \tilde{f} \rangle \int_D \phi \tag{3.29}$$

for almost all ω.

Proof. Without loss of generality we may assume that

$$\int_D \phi = 0 \tag{3.30}$$

otherwise it can be subtracted out by applying the individual ergodic theorem.

Let the space $\mathcal{H}$ be decomposed w.r.t. the Fourier spectrum of the translation group $\tau_{\boldsymbol{x}}$ and let $P(d\boldsymbol{k})$ be the orthogonal projection associated with $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. With this we obtain the resolution of identity

$$\boldsymbol{I} = \int P(d\boldsymbol{k}) \tag{3.31}$$

and the spectral representation for square integrable stationary function

$$f(\boldsymbol{x}/\varepsilon,\omega) = \tilde{f}(\tau_{-\boldsymbol{x}/\varepsilon}\omega) = \int e^{i\boldsymbol{k}/\varepsilon\cdot\boldsymbol{x}} P(d\boldsymbol{k})\tilde{f}. \tag{3.32}$$

With the spectral representation (3.32) the integral (3.29) becomes, after exchange of integrations,

$$\int_D f(\boldsymbol{x}/\varepsilon,\omega)\phi(\boldsymbol{x}) = \int P(d\boldsymbol{k})\tilde{f} \int_D e^{i\boldsymbol{k}/\varepsilon\cdot\boldsymbol{x}}\phi(\boldsymbol{x}). \tag{3.33}$$

By the Riemann-Lebesgue lemma and (3.30) we have

$$\int_D e^{i\boldsymbol{k}/\varepsilon\cdot\boldsymbol{x}}\phi(\boldsymbol{x}) \to 0, \quad \forall \boldsymbol{k} \tag{3.34}$$

as ε tends to zero. The integrand can be bounded by

$$|P(d\boldsymbol{k})\tilde{f}\int_D e^{i\boldsymbol{k}/\varepsilon\cdot\boldsymbol{x}}\phi(\boldsymbol{x})| \leq |P(d\boldsymbol{k})\tilde{f}|\int_D |\phi| \tag{3.35}$$

The right side of (3.35) is integrable by the square integrability of $\tilde{f}$. Thus by the Lebesgue dominated convergence theorem we have

$$\int_D f(\boldsymbol{x}/\varepsilon,\omega)\phi(\boldsymbol{x}) \to 0. \tag{3.36}$$

This concludes the proof. ∎

4 Proof of Theorem 2.1: Homogenization

We modify Tartar's argument ([7], Chapter 1, Section 3, [6]) with the use of approximate correctors.

By the condition (2.23) of uniformly bounded energy the solution T_ε is pre-compact weakly in $H_0^1(D,\mathcal{H})$ and pre-compact strongly in $L^2(D,\mathcal{H})$. Extracting a convergent subsequence in both senses, still denoted by T_ε, and let $\bar{T}$ the limit.

Define the flux

$$\boldsymbol{Q}_\varepsilon = -(\boldsymbol{I}+\boldsymbol{H}_\varepsilon)\nabla T_\varepsilon \tag{4.1}$$

with

$$\boldsymbol{H}_\varepsilon(\boldsymbol{x},\omega) = \boldsymbol{H}(\boldsymbol{x}/\varepsilon,\omega) \tag{4.2}$$

which converges in the weak L^1 norm since both $\boldsymbol{H}_\varepsilon$ and ∇T_ε have uniform bounds in $L^2(B)$. Extracting a convergent subsequence, still denoted by $\boldsymbol{Q}_\varepsilon$, and passing to the limit we obtain some limit flux $\bar{\boldsymbol{Q}}$.

The remaining question for homogenization is to identify the relation between the limit solution $\nabla\bar{T}$ and the limit flux $\bar{\boldsymbol{Q}}$. The goal is to show that they are linearly related with the proportionality given by the effective diffusivity matrix.

Passing to the limit in the equation (2.3) we obtain

$$-\int_D \bar{\boldsymbol{Q}} \cdot \nabla\phi + \lambda \int_D \bar{T}\phi = \int_D T_0\phi \tag{4.3}$$

for any $\phi \in W_0^{1,\infty}(D)$.

Let $f_i(\boldsymbol{x}, \omega)$ be an approximate adjoint corrector as stated in Lemma 3.1. Consider the test function

$$w_i(\boldsymbol{x}, \omega) = \boldsymbol{x} \cdot \boldsymbol{e}_i + f_i(\boldsymbol{x}, \omega) \tag{4.4}$$

and the scaled version of the test function

$$w_i^\varepsilon(\boldsymbol{x}, \omega) = \varepsilon w_i(\boldsymbol{x}/\varepsilon, \omega) = \boldsymbol{x} \cdot \boldsymbol{e}_i + \varepsilon f_i(\boldsymbol{x}/\varepsilon, \omega). \tag{4.5}$$

From the inequality (3.20) in Lemma 3.1 and the individual ergodic theorem [3] it follows that

$$\int_D [\nabla w_i^\varepsilon - \boldsymbol{e}_i - \boldsymbol{\Gamma}\boldsymbol{H}_\varepsilon \nabla w_i^\varepsilon]^2 \leq \delta^2 \tag{4.6}$$

for sufficiently small ε for almost all ω. Consequently we have that

$$|\int_D (\boldsymbol{I} - \boldsymbol{H}_\varepsilon)\nabla w_i^\varepsilon \cdot \nabla\phi| = |\int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon)\nabla\phi \cdot \nabla w_i^\varepsilon| < \delta\sqrt{\int_D |\nabla\phi|^2} \tag{4.7}$$

for any $\phi \in H_0^1(D)$. Here the skew symmetry of $\boldsymbol{H}_\varepsilon$ is used.

Let $\theta(\boldsymbol{x}, \omega)$ be any function in $C_0^\infty(D)$. It is clear that both θT_ε and θw_i^ε are admissible test functions for (4.7) and (2.3) respectively. Inserting them respectively we have

$$|\int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon)\nabla(\theta T_\varepsilon) \cdot \nabla w_i^\varepsilon| < \delta\sqrt{\int_D |\nabla(\theta T_\varepsilon)|^2} \tag{4.8}$$

$$-\int_D \boldsymbol{Q}_\varepsilon \cdot \nabla(\theta w_i^\varepsilon) + \lambda\int_D T_\varepsilon \theta w_i^\varepsilon = \int_D T_0 \theta w_i^\varepsilon. \tag{4.9}$$

Subtracting (4.8) from (4.9) we obtain

$$\begin{aligned} &|-\int_D T_\varepsilon \nabla\theta \cdot (\boldsymbol{I} - \boldsymbol{H}_\varepsilon)\nabla w_i^\varepsilon - \int_D w_i^\varepsilon \boldsymbol{Q}_\varepsilon \cdot \nabla\theta + \lambda\int_D T_\varepsilon\theta w_i^\varepsilon - \int_D T_0\theta w_i^\varepsilon| \\ &\leq \delta\sqrt{\int_D |\nabla(\theta T_\varepsilon)|^2} \\ &\leq C_1\delta \end{aligned} \tag{4.10}$$

where the constant C_1 depends on the energy bound C in (2.23) and the function θ. Lemma 3.2 (regarding the asymptotic behavior of w_i^ε) and the strong convergence of T_ε to $\bar{T}$ in $L^2(D)$ allow us to pass to the limit in the inequality (4.10):

$$\int_D T_\varepsilon \nabla\theta \cdot (\boldsymbol{I} - \boldsymbol{H}_\varepsilon)\nabla w_i^\varepsilon \to \int_D \bar{T}\nabla\theta \cdot \langle(\boldsymbol{I} - \tilde{\boldsymbol{H}})(\boldsymbol{e}_i + \tilde{\boldsymbol{F}}_i)\rangle \tag{4.11}$$

$$\int_D w_i^\varepsilon \boldsymbol{Q}_\varepsilon \cdot \nabla\theta \to \int_D \bar{\boldsymbol{Q}} \cdot \nabla\theta \boldsymbol{x} \cdot \boldsymbol{e}_i \tag{4.12}$$

$$\lambda \int_D T_\varepsilon \theta w_i^\varepsilon \to \lambda \int_D \bar{T}\theta \boldsymbol{x} \cdot \boldsymbol{e}_i \tag{4.13}$$

$$\int_D T_0 \theta w_i^\varepsilon \to \int_D T_0 \theta \boldsymbol{x} \cdot \boldsymbol{e}_i \tag{4.14}$$

and it follows that

$$\begin{aligned} &| - \int_D \bar{T}\langle(\boldsymbol{I} - \tilde{\boldsymbol{H}})(\boldsymbol{e}_i + \tilde{\boldsymbol{F}}_i)\rangle \cdot \nabla\theta - \int_D \bar{\boldsymbol{Q}} \cdot \nabla\theta \boldsymbol{x} \cdot \boldsymbol{e}_i + \lambda \int_D \bar{T}\theta \boldsymbol{x} \cdot \boldsymbol{e}_i - \int_D T_0 \theta \boldsymbol{x} \cdot \boldsymbol{e}_i| \\ &\leq C_1 \delta \end{aligned} \tag{4.15}$$

for any $\delta > 0$.

The convergence (4.11) is justified by the averaging lemma 3.3 and the strong convergence of T_ε in $L^2(D)$. The convergence of (4.12)-(4.14) is justified by the strong convergence of w_i^ε to $\boldsymbol{x} \cdot \boldsymbol{e}_i$ in $L^\infty(D)$ (Lemma A.1).

In view of (3.22) in Lemma 3.1, we let δ tend to zero and obtain the equality

$$-\int_D \bar{T} \sum_j \frac{\partial\theta}{\partial x_j} \sigma_{j,i}^* - \int_D \bar{\boldsymbol{Q}} \cdot \nabla\theta \boldsymbol{x} \cdot \boldsymbol{e}_i + \lambda \int_D \bar{T}\theta \boldsymbol{x} \cdot \boldsymbol{e}_i = \int_D T_0 \theta \boldsymbol{x} \cdot \boldsymbol{e}_i. \tag{4.16}$$

Inserting $\phi = \theta \boldsymbol{x} \cdot \boldsymbol{e}_i$ in the equation (4.3) we also have that

$$-\int_D \bar{\boldsymbol{Q}} \cdot \boldsymbol{e}_i \theta - \int_D \bar{\boldsymbol{Q}} \cdot \nabla\theta \boldsymbol{x} \cdot \boldsymbol{e}_i + \lambda \int_D \bar{T}\theta \boldsymbol{x} \cdot \boldsymbol{e}_i = \int_D T_0 \theta \boldsymbol{x} \cdot \boldsymbol{e}_i. \tag{4.17}$$

Subtracting (4.16) from (4.17) yields

$$\int_D \sum_j \sigma_{j,i}^* \frac{\partial\theta}{\partial x_j} \bar{T} = \int_D \bar{\boldsymbol{Q}} \cdot \boldsymbol{e}_i \theta \tag{4.18}$$

for any $\theta \in C_0^\infty(D)$. Thus the following relation holds, after integrating by parts,

$$-\sum_j \sigma^*_{j,i} \frac{\partial \bar{T}}{\partial x_j} = \bar{\boldsymbol{Q}} \cdot \boldsymbol{e}_i. \tag{4.19}$$

Inserting the identity (4.19) into the equation (4.3) gives

$$\int_D \sum_{i,j} \sigma^*_{i,j} \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \lambda \int_D \bar{T}\phi = \int_D T_0 \phi \tag{4.20}$$

for all $\phi \in W_0^{1,\infty}(D)$. Since only the symmetric part of $\sigma^*_{i,j}$ contributes to the integral we rewrite the equation (4.20) in the symmetric form

$$\int_D \sum_{i,j} \frac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}) \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \phi}{\partial x_j} + \lambda \int_D \bar{T}\phi = \int_D T_0 \phi. \tag{4.21}$$

The proof of Theorem 2.1 is now complete. ∎

5 Proof of Theorem 2.2: Two-Term Asymptotics

For any $\delta > 0$ let $g_i(\boldsymbol{x}, \omega)$ be the approximate correctors with gradients $\nabla g_i(\boldsymbol{x}, \omega) = \boldsymbol{G}_i(\boldsymbol{x}, \omega)$ as asserted in the approximation lemma 3.1. Then by the averaging lemma 3.3 and Lemma 3.1 we have that

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})(\nabla \chi_i(\boldsymbol{x}/\varepsilon) - \nabla g_i(\boldsymbol{x}/\varepsilon)) \right)^2 \\
&= \lim_{\varepsilon \to 0} \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})(\boldsymbol{E}_i(\boldsymbol{x}/\varepsilon) - \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) \right)^2 \\
&= \int_D \left(\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) \langle (\tilde{\boldsymbol{E}}_i - \tilde{\boldsymbol{G}}_i) \rangle \right)^2 \\
&\le d\delta^2 \int_D \sum_i (\frac{\partial \bar{T}}{\partial x_i})^2.
\end{aligned} \tag{5.1}$$

Thus, to prove Theorem 2.2 it is sufficient to show that

$$\limsup_{\varepsilon \to 0} \int_D \left(\nabla T_\varepsilon(\boldsymbol{x}) - \nabla \bar{T}(\boldsymbol{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) \nabla g_i(\boldsymbol{x}/\varepsilon) \right)^2 \le c\delta \tag{5.2}$$

for some $c > 0$ independent of δ.

Due to the skew symmetry of $\boldsymbol{H}_\varepsilon$ the integral on the left side of (5.2) is equal to

$$\begin{aligned}
&\int_D \left(\nabla T_\varepsilon(\boldsymbol{x}) - \nabla\bar{T}(\boldsymbol{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right)^2 \\
&= \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\left(\nabla T_\varepsilon(\boldsymbol{x}) - \nabla\bar{T}(\boldsymbol{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right) \\
&\quad \cdot \left(\nabla T_\varepsilon(\boldsymbol{x}) - \nabla\bar{T}(\boldsymbol{x}) - \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right) \\
&= \int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon - \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\nabla T_\varepsilon(\boldsymbol{x}) \cdot \left(\nabla\bar{T}(\boldsymbol{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right) \\
&- \int_D \nabla T_\varepsilon(\boldsymbol{x}) \cdot (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\left(\nabla\bar{T}(\boldsymbol{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right) \\
&\quad + \int_D \sum_{i,j} \frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}(\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon) + \boldsymbol{e}_i) \cdot (\boldsymbol{G}_j(\boldsymbol{x}/\varepsilon) + \boldsymbol{e}_j).
\end{aligned} \tag{5.3}$$

By the energy equality and the strong convergence of T_ε the first integral in (5.3) becomes, in the limit $\varepsilon \to 0$,

$$\lim_{\varepsilon\to 0}\int_D \nabla T_\varepsilon \cdot \nabla T_\varepsilon = \lim_{\varepsilon\to 0}\left[\int_D T_0 T_\varepsilon - \lambda\int_D T_\varepsilon^2\right] = \int_D T_0\bar{T} - \lambda\int_D \bar{T}^2. \tag{5.4}$$

Rewrite the second integral in (5.3) as

$$\begin{aligned}
&\int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\nabla T_\varepsilon(\boldsymbol{x}) \cdot \left(\nabla\bar{T}(\boldsymbol{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})\nabla g_i(\boldsymbol{x}/\varepsilon)\right) \\
&= \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\nabla T_\varepsilon(\boldsymbol{x}) \cdot \nabla\left(\bar{T}(\boldsymbol{x}) + \varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon)\right) \\
&\quad - \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\nabla T_\varepsilon(\boldsymbol{x}) \cdot \varepsilon\sum_i \frac{\partial \nabla\bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon).
\end{aligned} \tag{5.5}$$

In view of the equation (2.3), (5.5) now becomes

$$\begin{aligned}
&-\lambda\int_D T_\varepsilon(\boldsymbol{x})\left(\bar{T}(\boldsymbol{x}) + \varepsilon\sum_i \frac{\partial \nabla\bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon)\right) + \int_D T_0(\boldsymbol{x})\left(\bar{T} + \varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon)\right) \\
&- \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))\nabla T_\varepsilon(\boldsymbol{x}) \cdot \varepsilon\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon)
\end{aligned}$$

$$= -\lambda \int_D T_\varepsilon(\boldsymbol{x}) \left(\bar{T}(\boldsymbol{x}) + \varepsilon \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon) \right) + \int_D T_0(\boldsymbol{x}) \left(\bar{T}(\boldsymbol{x}) + \varepsilon \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon) \right)$$
$$+ \int_D \boldsymbol{Q}_\varepsilon(\boldsymbol{x}) \cdot \varepsilon \sum_i \frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x}) g_i(\boldsymbol{x}/\varepsilon). \tag{5.6}$$

Passing to the limit in (5.6) with the strong convergence of T_ε and Lemma 3.2 we have that

$$\lim_{\varepsilon \to 0} \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x})) \nabla T_\varepsilon(\boldsymbol{x}) \cdot \left(\nabla \bar{T}(\boldsymbol{x}) + \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) \nabla g_i(\boldsymbol{x}/\varepsilon) \right)$$
$$= -\lambda \int_D \bar{T}^2 + \int_D T_0 \bar{T}. \tag{5.7}$$

The third integral in (5.3) can be written as

$$\int_D \nabla T_\varepsilon(\boldsymbol{x}) \cdot \sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x}) (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i + \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon))$$
$$= \sum_i \int_D \nabla (T_\varepsilon \frac{\partial \bar{T}}{\partial x_i})(\boldsymbol{x}) \cdot (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i + \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon))$$
$$- \sum_i \int_D T_\varepsilon \frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x}) \cdot (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i + \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) \tag{5.8}$$

Note that, following Lemma 3.1, we have the inequality, similar to (4.7),

$$| \int_D (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i + \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) \cdot \nabla \phi | \le \delta \sqrt{\int_D |\nabla \phi|^2} \tag{5.9}$$

for all $\phi \in H_0^1(D)$ and sufficiently small ε. Consequently,

$$| \sum_i \int_D \nabla (T_\varepsilon \frac{\partial \bar{T}}{\partial x_i})(\boldsymbol{x}) \cdot (\boldsymbol{I} + \boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i + \boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) |$$
$$\le \delta \sum_i \sqrt{\int_D |\nabla (T_\varepsilon \frac{\partial \bar{T}}{\partial x_i})|^2}$$
$$\le c_1 \delta \tag{5.10}$$

due to the energy bound (2.23).

Passing the limit in the second term in (5.8) with the strong convergence of T_ε and the averaging lemma 3.3 we have that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\sum_i \int_D T_\varepsilon \frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x})\cdot(\boldsymbol{I}+\boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i+\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) \\
&= \sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x})\cdot\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{G}})\rangle. \qquad (5.11)
\end{aligned}$$

Using the approximation lemma 3.1 and the equation (2.24) yields

$$\begin{aligned}
&|\lim_{\varepsilon\to 0}\sum_i \int_D T_\varepsilon \frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x})\cdot(\boldsymbol{I}+\boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i+\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) - \sum_{i,j}\int_D \bar{T}\frac{\partial^2 \bar{T}}{\partial x_i \partial x_j}\sigma^*_{i,j}| \\
&= |\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x})\cdot\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{G}}_i)\rangle + \int_D \sum_{i,j}\sigma^*_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}| \\
&= |\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}(\boldsymbol{x})\cdot\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{G}}_i)\rangle - \lambda\int_D \bar{T}^2 + \int_D T_0\bar{T}| \\
&\le c_2\delta. \qquad (5.12)
\end{aligned}$$

Thus,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}|\int_D \nabla T_\varepsilon(\boldsymbol{x})\cdot\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})(\boldsymbol{I}+\boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i+\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) \\
&+\sum_i \int_D \bar{T}\frac{\partial \nabla \bar{T}}{\partial x_i}\cdot\langle(\boldsymbol{I}+\tilde{\boldsymbol{H}})(\boldsymbol{e}_i+\tilde{\boldsymbol{G}}_i)\rangle| \\
&\le \limsup_{\varepsilon\to 0}|\int_D \nabla T_\varepsilon(\boldsymbol{x})\cdot\sum_i \frac{\partial \bar{T}}{\partial x_i}(\boldsymbol{x})(\boldsymbol{I}+\boldsymbol{H}_\varepsilon(\boldsymbol{x}))(\boldsymbol{e}_i+\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)) + \lambda\int_D \bar{T}^2 - \int_D T_0\bar{T}| \\
&\le c_3\delta. \qquad (5.13)
\end{aligned}$$

For the fourth integral in (5.3) we apply the averaging lemma 3.3 in passing to the limit and Lemma 3.1

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}(\boldsymbol{G}_i(\boldsymbol{x}/\varepsilon)+\boldsymbol{e}_i)\cdot(\boldsymbol{G}_j(\boldsymbol{x}/\varepsilon)+\boldsymbol{e}_j) \\
&= \int_D \sum_{i,j}\frac{\partial \bar{T}}{\partial x_i}\frac{\partial \bar{T}}{\partial x_j}\langle(\boldsymbol{e}_i+\tilde{\boldsymbol{G}}_i)\cdot(\boldsymbol{e}_j+\tilde{\boldsymbol{G}}_j)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \int_D \sum_{i,j} \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \bar{T}}{\partial x_j} \langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle \\
&+ \int_D \sum_{i,j} \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \bar{T}}{\partial x_j} \Big(\langle (\boldsymbol{e}_i + \tilde{\boldsymbol{G}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{G}}_j) \rangle - \langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle \Big) \\
&= \int_D \sum_{i,j} \frac{1}{2} (\sigma^*_{i,j} + \sigma^*_{j,i}) \frac{\partial \bar{T}}{\partial x_i} \frac{\partial \bar{T}}{\partial x_j} + c_4 \delta \\
&= -\lambda \int_D \bar{T}^2 + \int_D T_0 \bar{T} + c_4 \delta. \qquad (5.14)
\end{aligned}
$$

Here we have used also the identity (A.19).

(5.2) clearly follows from (5.4), (5.7), (5.13) and (5.14). And this concludes the proof. ∎.

Appendix

A Cell Problem and Variational Principles

For convenience and completeness we present a brief account of the variational formulation for the cell problem and its consequences, including the proof of the approximation lemma 3.1. Readers should consult [4] for the details of the variational methods for the convection-diffusion problem.

A.1 Proof of Theorem 3.1

Consider the min-max principle

$$S(\boldsymbol{e}_i,\boldsymbol{e}_j) = \inf_{\substack{\tilde{\boldsymbol{F}}\in\mathcal{H}_g(\tilde{\boldsymbol{H}})\\ \langle\tilde{\boldsymbol{F}}\rangle=(\boldsymbol{e}_i+\boldsymbol{e}_j)/2}} \sup_{\substack{\tilde{\boldsymbol{F}}'\in\mathcal{H}_g(\tilde{\boldsymbol{H}})\\ \langle\tilde{\boldsymbol{F}}'\rangle=(\boldsymbol{e}_i-\boldsymbol{e}_j)/2}} \left\langle \begin{pmatrix} -\boldsymbol{I} & -\tilde{\boldsymbol{H}} \\ \tilde{\boldsymbol{H}} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{F}}' \\ \tilde{\boldsymbol{F}} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\boldsymbol{F}}' \\ \tilde{\boldsymbol{F}} \end{pmatrix} \right\rangle . \tag{A.1}$$

Clearly, $S(\boldsymbol{e}_i,\boldsymbol{e}_j)$ exists because of the square integrability of $\boldsymbol{H}$.

The supremum in (A.1) is given by the equation

$$\tilde{\nabla}\cdot\tilde{\boldsymbol{F}}' + \tilde{\nabla}\cdot\tilde{\boldsymbol{H}}\tilde{\boldsymbol{F}} = 0 \tag{A.2}$$

which can be solved for $\tilde{\boldsymbol{F}}' \in \mathcal{H}_g$ in the form

$$\tilde{\boldsymbol{F}}' = -\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{F}} + (\boldsymbol{e}_i - \boldsymbol{e}_j)/2 \tag{A.3}$$

in terms of the projection operator

$$\tilde{\boldsymbol{\Gamma}} = \tilde{\nabla}\tilde{\Delta}^{-1}\tilde{\nabla} \tag{A.4}$$

from square integrable vector fields unto the space $\mathcal{H}_g$. Substituting the expression (A.3) in (A.1) we get

$S(\boldsymbol{e}_i,\boldsymbol{e}_j)$

$$= \inf_{\substack{\boldsymbol{F} \in \mathcal{H}_g(\tilde{\boldsymbol{H}}) \\ \langle \tilde{\boldsymbol{F}} \rangle = (\boldsymbol{e}_i + \boldsymbol{e}_j)/2}} \left\{ \langle \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}} \rangle - 2 \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}' \rangle - \langle \tilde{\boldsymbol{F}}' \cdot \tilde{\boldsymbol{F}}' \rangle \right\}$$

$$= \inf_{\substack{\boldsymbol{F} \in \mathcal{H}_g(\tilde{\boldsymbol{H}}) \\ \langle \tilde{\boldsymbol{F}} \rangle = (\boldsymbol{e}_i + \boldsymbol{e}_j)/2}} \left\{ \langle \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}} \rangle + \langle \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \rangle - \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \cdot (\boldsymbol{e}_i - \boldsymbol{e}_j) \rangle - \left| \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2} \right|^2 \right\} \tag{A.5}$$

Similarly, we can eliminate the infimum in (A.1) and derive a nonlocal maximum principle

$$S(\boldsymbol{e}_i, \boldsymbol{e}_j)$$

$$= \sup_{\substack{\boldsymbol{F}' \in \mathcal{H}_g(\tilde{\boldsymbol{H}}) \\ \langle \tilde{\boldsymbol{F}}' \rangle = (\boldsymbol{e}_i - \boldsymbol{e}_j)/2}} \left\{ -\langle \tilde{\boldsymbol{F}}' \cdot \tilde{\boldsymbol{F}}' \rangle - \langle \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}}' \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}}' \rangle + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}}' \cdot (\boldsymbol{e}_i + \boldsymbol{e}_j) \rangle + \left| \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2} \right|^2 \right\} \tag{A.6}$$

Due to the quadratic nature of the functionals, both the minimizer and maximizer of (A.5) and (A.6) respectively exist in the space $\mathcal{H}_g(\tilde{\boldsymbol{H}})$ by Friedrichs' extension of symmetric quadratic forms.

Observe that the minimaximizing pair $\tilde{\boldsymbol{E}}_{i,j}, \tilde{\boldsymbol{E}}'_{i,j}$ of (A.1) satisfies the identities

$$\tilde{\boldsymbol{E}}'_{i,j} = -\tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_{i,j} + (\boldsymbol{e}_i - \boldsymbol{e}_j)/2 \tag{A.7}$$

$$\tilde{\boldsymbol{E}}_{i,j} = -\tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}'_{i,j} + (\boldsymbol{e}_i + \boldsymbol{e}_j)/2. \tag{A.8}$$

Adding (A.7) and (A.8) we have

$$\tilde{\boldsymbol{E}}^+ = -\tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}^+ + \boldsymbol{e}_i \tag{A.9}$$

where $\tilde{\boldsymbol{E}}^+$ is the sum

$$\tilde{\boldsymbol{E}}^+ = \tilde{\boldsymbol{E}}_{i,j} + \tilde{\boldsymbol{E}}'_{i,j}. \tag{A.10}$$

The equation (A.9) is equivalent to the following one

$$\tilde{\nabla} \cdot (\boldsymbol{I} + \tilde{\boldsymbol{H}}) \tilde{\boldsymbol{E}}^+ = 0 \tag{A.11}$$

which is the cell problem and is understood in the weak sense. Thus we have that

$$\tilde{\boldsymbol{E}}^{+} = \tilde{\boldsymbol{E}}_i + \boldsymbol{e}_i \in \mathcal{H}_g(\tilde{\boldsymbol{H}}). \tag{A.12}$$

The uniqueness follows from reversing the above argument. The punch line is that the solution $\tilde{\boldsymbol{E}}_i+\boldsymbol{e}_i$ of the cell problem, if in $\mathcal{H}_g(\tilde{\boldsymbol{H}})$, must be the sum of the minimaximizing pair $\tilde{\boldsymbol{E}}_{i,j}, \tilde{\boldsymbol{E}}'_{i,j}$. The uniqueness of $\tilde{\boldsymbol{E}}_i + \boldsymbol{e}_i$ follows immediately from the uniqueness of the minimaximizing pair.

This concludes the proof of Theorem 3.1. ∎

A.2 Variational Principles for Effective Diffusivity

We shall show that the value $S(\boldsymbol{e}_i, \boldsymbol{e}_j)$ defined by (A.1) is equal to $\sigma^*_{i,j}$.

In terms of the sum and difference of $\tilde{\boldsymbol{E}}_{i,j}$ and $\tilde{\boldsymbol{E}}'_{i,j}$

$$\tilde{\boldsymbol{E}}^{+} = \tilde{\boldsymbol{E}}_{i,j} + \tilde{\boldsymbol{E}}'_{i,j}, \quad \tilde{\boldsymbol{E}}^{-} = \tilde{\boldsymbol{E}}_{i,j} - \tilde{\boldsymbol{E}}'_{i,j}. \tag{A.13}$$

the value $S(\boldsymbol{e}_i, \boldsymbol{e}_j)$ can be expressed as

$$S(\boldsymbol{e}_i, \boldsymbol{e}_j) = \left\langle \begin{pmatrix} -\boldsymbol{I} & -\tilde{\boldsymbol{H}} \\ \tilde{\boldsymbol{H}} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{E}}'_{i,j} \\ \tilde{\boldsymbol{E}}_{i,j} \end{pmatrix} \cdot \begin{pmatrix} \tilde{\boldsymbol{E}}'_{i,j} \\ \tilde{\boldsymbol{E}}_{i,j} \end{pmatrix} \right\rangle \tag{A.14}$$

$$= -\langle \tilde{\boldsymbol{E}}'_{i,j} \cdot \tilde{\boldsymbol{E}}'_{i,j} \rangle - 2\langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_{i,j} \cdot \tilde{\boldsymbol{E}}'_{i,j} \rangle + \langle \tilde{\boldsymbol{E}}_{i,j} \cdot \tilde{\boldsymbol{E}}_{i,j} \rangle \tag{A.15}$$

$$= \langle \tilde{\boldsymbol{E}}^{+} \cdot \tilde{\boldsymbol{E}}^{-} \rangle + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}^{+} \cdot \tilde{\boldsymbol{E}}^{-} \rangle \tag{A.16}$$

$$= \langle \left(\boldsymbol{I} + \tilde{\boldsymbol{H}}\right) \tilde{\boldsymbol{E}}^{+} \cdot \tilde{\boldsymbol{E}}^{-} \rangle \tag{A.17}$$

$$= \langle \left(\boldsymbol{I} + \tilde{\boldsymbol{H}}\right) \tilde{\boldsymbol{E}}^{+} \cdot \boldsymbol{e}_j \rangle \tag{A.18}$$

using the weak form of the equation (A.11) in the last identity. It is now clear from the identity (A.12) and the definition (3.8) that $S(\boldsymbol{e}_i, \boldsymbol{e}_j) = \sigma^*_{i,j}$.

In general the effective diffusivity matrix σ^* is not symmetric but only the symmetric part appears in the homogenized equation.

Before entering the proof of Lemma 3.1 let us show an identity for the effective diffusivity

$$\frac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}) = \langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle. \tag{A.19}$$

We break (A.19) into two cases. When $i = j$, it follows from the equation (A.11) and the skew symmetry of $\tilde{\boldsymbol{H}}$ that

$$\begin{aligned}\sigma^*_{i,i} &= \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot \boldsymbol{e}_i \rangle \\ &= \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \rangle \\ &= \langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \rangle. \end{aligned} \tag{A.20}$$

When $i \neq j$, let us first note that $\langle \boldsymbol{e}_i \cdot \boldsymbol{e}_j \rangle = 0$. Using the mean zero property of $\tilde{\boldsymbol{E}}_i, \tilde{\boldsymbol{E}}_j$ and $\tilde{\boldsymbol{H}}$, and the equation (A.9) we get

$$\begin{aligned} &\langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle \\ &= \langle \tilde{\boldsymbol{E}}_i \cdot \tilde{\boldsymbol{E}}_j \rangle \\ &= -\langle \tilde{\boldsymbol{E}}_i \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle \\ &= \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle \\ &= \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot \boldsymbol{e}_j \rangle + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot \tilde{\boldsymbol{E}}_j \rangle \\ &= \langle \tilde{\boldsymbol{H}} (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot \boldsymbol{e}_j \rangle + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot \tilde{\boldsymbol{E}}_j \rangle \\ &= \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}}) (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot \boldsymbol{e}_j \rangle + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot \tilde{\boldsymbol{E}}_j \rangle \\ &= \sigma^*_{i,j} + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_i \cdot \tilde{\boldsymbol{E}}_j \rangle. \end{aligned} \tag{A.21}$$

Similarly, since $\langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle$ is a symmetric expression, one can show that

$$\langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle = \sigma^*_{j,i} + \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{E}}_j \cdot \tilde{\boldsymbol{E}}_i \rangle. \tag{A.22}$$

Adding (A.21) and (A.22) and cancelling the second term by the skew symmetry of $\tilde{\boldsymbol{H}}$ give the identity

$$\langle (\boldsymbol{e}_i + \tilde{\boldsymbol{E}}_i) \cdot (\boldsymbol{e}_j + \tilde{\boldsymbol{E}}_j) \rangle = \frac{1}{2}(\sigma^*_{i,j} + \sigma^*_{j,i}). \tag{A.23}$$

A.3 Intermediate Approximation

Since the space $\mathcal{H}_g(\tilde{\boldsymbol{H}})$ is the closure of the space

$$\mathcal{D} = \{\boldsymbol{G} \in \mathcal{H}_g; \tilde{\boldsymbol{H}}\boldsymbol{G} \in (\mathcal{H})^d\} \tag{A.24}$$

the variations in (A.1), (A.5) and (A.6) can be restricted to D which is the actual domain of the operator $\tilde{\boldsymbol{\Gamma}}\tilde{\boldsymbol{H}}$ defining (A.5) and (A.6).

Note that for any $\tilde{\boldsymbol{F}} \in \mathcal{D}$ we have

$$\langle[(\tilde{\boldsymbol{H}} - \tilde{\boldsymbol{H}}^{(M)})\tilde{\boldsymbol{F}}]^2\rangle \to 0. \tag{A.25}$$

as $M \to \infty$. Consequently,

$$\langle\tilde{\Gamma}(\tilde{\boldsymbol{H}} - \tilde{\boldsymbol{H}}^{(M)})\tilde{\boldsymbol{F}} \cdot \tilde{\Gamma}(\tilde{\boldsymbol{H}} - \tilde{\boldsymbol{H}}^{(M)})\tilde{\boldsymbol{F}}\rangle \to 0 \tag{A.26}$$

as $M \to \infty$. Here the stream matrix $\tilde{\boldsymbol{H}}^{(M)}$ is the level M truncation of $\tilde{\boldsymbol{H}}$.

Homogenization with the bounded stream matrix $\tilde{\boldsymbol{H}}^{(M)}$ (see [6]) leads to the effective diffusivity which can be put in the same variational forms as (A.5) and (A.6), namely,

$$\sigma_{i,j}^{(M)} = \inf_{\substack{\boldsymbol{F} \in D \\ \langle\tilde{\boldsymbol{F}}\rangle = (\boldsymbol{e}^i + \boldsymbol{e}^j)/2}} \left\{\langle\tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}}\rangle + \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}} \cdot \tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}\rangle - \langle\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}} \cdot (\boldsymbol{e}^i - \boldsymbol{e}^j)\rangle - |\frac{\boldsymbol{e}^i - \boldsymbol{e}^j}{2}|^2\right\} \tag{A.27}$$

$$= \sup_{\substack{\tilde{\nabla} \times \tilde{\boldsymbol{F}}' = 0 \\ \langle\tilde{\boldsymbol{F}}'\rangle = (\boldsymbol{e}^i - \boldsymbol{e}^j)/2}} \left\{-\langle\tilde{\boldsymbol{F}}' \cdot \tilde{\boldsymbol{F}}'\rangle - \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}' \cdot \tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\rangle + \langle\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}' \cdot (\boldsymbol{e}^i + \boldsymbol{e}^j)\rangle + |\frac{\boldsymbol{e}^i + \boldsymbol{e}^j}{2}|^2\right\} \tag{A.28}$$

A crucial step toward establishing the approximation lemma 3.1 is the convergence of $\sigma^{(M)}$ as $M \to \infty$.

Lemma A.1

$$\lim_{M \to \infty} \sigma^{(M)} = \sigma^*. \tag{A.29}$$

Lemma A.1 resolves the difficulty with nonuniqueness of the cell problem: The cell problem may not have a unique solution in $\mathcal{H}_g$ but the limit procedure of M tending to infinity does provide a unique limit. By the previous analysis we know this limit is actually in the space $\mathcal{H}_g(\tilde{\boldsymbol{H}})$.

Proof. We first show the upper bound: $\limsup_{M\to\infty} \sigma_{i,j}^{(M)} \leq \sigma_{i,j}^*, \forall i, j$, using the minimum principles (A.5) and (A.27).

By the previous remark and (A.5), for given $\delta > 0$ there exists a $\tilde{\boldsymbol{F}} \in \mathcal{D}$ such that

$$\langle \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}} \rangle + \langle \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \rangle - \langle \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}} \cdot (\boldsymbol{e}^i - \boldsymbol{e}^j) \rangle - |\frac{\boldsymbol{e}^i - \boldsymbol{e}^j}{2}|^2 \leq \sigma^* + \delta. \tag{A.30}$$

By (A.25) and (A.26), we also know that the left-side of (A.30) is bigger than

$$\langle \tilde{\boldsymbol{F}} \cdot \tilde{\boldsymbol{F}} \rangle + \langle \tilde{\Gamma} \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}} \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}} \rangle - \langle \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}} \cdot (\boldsymbol{e}^i - \boldsymbol{e}^j) \rangle - |\frac{\boldsymbol{e}^i - \boldsymbol{e}^j}{2}|^2 - \delta \tag{A.31}$$

which in turn is bigger than

$$\sigma^{(M)} - \delta \tag{A.32}$$

in view of (A.27). Thus we have that

$$\sigma_{i,j}^* \geq \sigma_{i,j}^{(M)} + 2\delta \tag{A.33}$$

for sufficiently large M. This proves the upper bound.

We turn to the lower bound: $\liminf_{M\to\infty} \sigma_{i,j}^{(M)} \geq \sigma_{i,j}^*, \forall i, j$.

By the maximum principle (A.6), there exists $\tilde{\boldsymbol{F}}' \in \mathcal{D}$ for given $\delta > 0$, such that

$$\sigma_{i,j}^* - \delta \leq \langle -\tilde{\boldsymbol{F}}' \cdot \tilde{\boldsymbol{F}}' \rangle - \langle \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}}' \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{F}}' \rangle + \langle \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}}' \cdot (\boldsymbol{e}^i + \boldsymbol{e}^j) \rangle + |\frac{\boldsymbol{e}^i + \boldsymbol{e}^j}{2}|^2. \tag{A.34}$$

In view of (A.25) and (A.26), the right side of (A.34) in turn is bounded by

$$-\langle \tilde{\boldsymbol{F}}' \cdot \tilde{\boldsymbol{F}}' \rangle - \langle \tilde{\Gamma} \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}}' \cdot \tilde{\Gamma} \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}}' \rangle + \langle \tilde{\boldsymbol{H}}^{(M)} \tilde{\boldsymbol{F}}' \cdot (\boldsymbol{e}^i + \boldsymbol{e}^j) \rangle + |\frac{\boldsymbol{e}^i + \boldsymbol{e}^j}{2}|^2 + \delta \tag{A.35}$$

for sufficiently large M. Combining (A.34), (A.35) and (A.28), we have

$$\sigma_{i,j}^* - 2\delta \leq \liminf_{M\to\infty} \sigma_{i,j}^{(M)} \tag{A.36}$$

for any $\delta > 0$. This concludes the proof. ∎

Using Lemma A.1 we now show that the trial functions for the variational principles (A.5) and (A.5) can be restricted to bounded gradient fields.

Lemma A.2 *The minimum and maximum in (A.5) and (A.6) respectively are achieved in the space of bounded gradient fields $\mathcal{H}_{g,\infty}$.*

Proof. For the minimum principle, suffice it to show that given $\delta > 0$, there exists bounded gradient field $\tilde{\boldsymbol{F}}$ with $\langle\tilde{\boldsymbol{F}}\rangle = \frac{\boldsymbol{e}_i+\boldsymbol{e}_j}{2}$ such that

$$\sigma^*_{i,j} + \delta \geq \langle\tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}\rangle + \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{F}}\cdot\tilde{\Gamma}\tilde{\boldsymbol{H}}\tilde{\boldsymbol{F}}\rangle - \langle\tilde{\boldsymbol{H}}\tilde{\boldsymbol{F}}\cdot(\boldsymbol{e}^i-\boldsymbol{e}^j)\rangle - |\frac{\boldsymbol{e}^i-\boldsymbol{e}^j}{2}|^2. \qquad (A.37)$$

By Lemma A.1, we have that

$$\sigma^*_{i,j} + \frac{\delta}{2} \geq \sigma^{(M)}_{i,j} \qquad (A.38)$$

for sufficiently large M. By the remark in the beginning of the section, there exists $\tilde{\boldsymbol{F}} \in \mathcal{H}_{g,\infty}$ such that

$$\sigma^{(M)}_{i,j} + \frac{\delta}{2} \geq \langle\tilde{\boldsymbol{F}}\cdot\tilde{\boldsymbol{F}}\rangle + \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}\cdot\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}\rangle - \langle\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}\cdot(\boldsymbol{e}^i-\boldsymbol{e}^j)\rangle - |\frac{\boldsymbol{e}_i-\boldsymbol{e}_j}{2}|^2 \quad (A.39)$$

Moreover, (A.39) is valid uniformly in M by (A.25) and (A.26). Equations (A.38), (A.39) together with (A.25), (A.26) imply (A.37).

We turn to the maximum principle. Suffice it to show th at given $\delta > 0$, there exists $\tilde{\boldsymbol{F}}' \in \mathcal{H}_{g,\infty}$ wi th $\langle\tilde{\boldsymbol{F}}'\rangle = \frac{\boldsymbol{e}_i-\boldsymbol{e}_j}{2}$ such that

$$\sigma^*_{i,j} - \delta \leq -\langle\tilde{\boldsymbol{F}}'\cdot\tilde{\boldsymbol{F}}'\rangle - \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\cdot\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\rangle + \langle\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\cdot(\boldsymbol{e}_i+\boldsymbol{e}_j)\rangle + |\frac{\boldsymbol{e}_i+\boldsymbol{e}_j}{2}|^2. \quad (A.40)$$

By Lemma A.1, we have that

$$\sigma^*_{i,j} - \frac{\delta}{2} \leq \sigma^{(M)}_{i,j} \qquad (A.41)$$

for sufficiently large M. Thus it follows from (A.28) and the density of $\mathcal{H}_{g,\infty}$ for bounded $\tilde{\boldsymbol{H}}^{(M)}$ that there exists $\boldsymbol{F}' \in \mathcal{H}_{g,\infty}$ such that

$$\sigma^*_{i,j} - \delta \leq -\langle\tilde{\boldsymbol{F}}'\cdot\tilde{\boldsymbol{F}}'\rangle - \langle\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\cdot\tilde{\Gamma}\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\rangle + \langle\tilde{\boldsymbol{H}}^{(M)}\tilde{\boldsymbol{F}}'\cdot(\boldsymbol{e}^i+\boldsymbol{e}^j)\rangle + |\frac{\boldsymbol{e}^i+\boldsymbol{e}^j}{2}|^2. \quad (A.42)$$

Note that (A.42) is valid for all sufficiently large M due to (A.25) and (A.26). Thus, (A.40) follows by passing to the limit $M \to \infty$. $\blacksquare$

Let us turn to the proof of Lemma 3.1.

A.4 Proof of the Approximation Lemma 3.1

Lemma A.2 is equivalent to the following statement: Given $\delta > 0$ there exist $\tilde{\boldsymbol{G}}_{i,i}, \tilde{\boldsymbol{G}'}_{i,i}$ in the space $\mathcal{H}_{g,\infty}$ such that

$$\langle \left| \tilde{\boldsymbol{G}}_{i,i} + \boldsymbol{e}_i - \tilde{\boldsymbol{E}}_{i,i} \right|^2 \rangle < \delta^2/2 \tag{A.43}$$

$$\langle \left| \tilde{\Gamma} \tilde{\boldsymbol{H}} (\tilde{\boldsymbol{G}}_{i,i} + \boldsymbol{e}_i - \tilde{\boldsymbol{E}}_{i,i}) \right|^2 \rangle < \delta^2/2 \tag{A.44}$$

$$\langle \left| \tilde{\boldsymbol{G}}'_{i,i} + \boldsymbol{e}_i - \tilde{\boldsymbol{E}}'_{i,i} \right|^2 \rangle < \delta^2/2 \tag{A.45}$$

$$\langle \left| \tilde{\Gamma} \tilde{\boldsymbol{H}} (\tilde{\boldsymbol{G}}'_{i,i} + \boldsymbol{e}_i - \tilde{\boldsymbol{E}}'_{i,i}) \right|^2 \rangle < \delta^2/2. \tag{A.46}$$

Thus it follows from (A.7) and (A.8) that

$$\langle \left| \tilde{\boldsymbol{G}}_{i,i} + \tilde{\Gamma} \tilde{\boldsymbol{H}} \tilde{\boldsymbol{G}}'_{i,i} \right|^2 \rangle < \delta^2 \tag{A.47}$$

$$\langle \left| \tilde{\boldsymbol{G}}'_{i,i} + \tilde{\Gamma} \tilde{\boldsymbol{H}} (\tilde{\boldsymbol{G}}_{i,i} + \boldsymbol{e}_i) \right|^2 \rangle < \delta^2. \tag{A.48}$$

And therefore we have for the sum $\tilde{\boldsymbol{G}}_i = \tilde{\boldsymbol{G}}_{i,i} + \tilde{\boldsymbol{G}}'_{i,i}$ that

$$\langle \left| \tilde{\boldsymbol{G}}_i + \tilde{\Gamma} \tilde{\boldsymbol{H}} (\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \right|^2 \rangle < \delta^2 \tag{A.49}$$

which is the inequality (3.19).

The proof of (3.20) follows from the similar argument and the version of Lemma A.2 for the adjoint cell problem (i.e. with $\tilde{\boldsymbol{H}}$ replaced by $-\tilde{\boldsymbol{H}}$).

To show the inequality (3.21) let us consider the following identities:

$$\begin{aligned} \sigma^*_{i,j} &= \langle (\boldsymbol{I} - \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{E}'}_j + \boldsymbol{e}_j) \cdot \boldsymbol{e}_i \rangle \\ &= \langle (\boldsymbol{I} - \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{E}'}_j + \boldsymbol{e}_j) \cdot (\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle (\tilde{\boldsymbol{E}}'_j + \boldsymbol{e}_j) \cdot (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \rangle \\
&= \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle + \langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\boldsymbol{E}}'_j \rangle. \qquad \text{(A.50)}
\end{aligned}
$$

This leads to the identity

$$
\begin{aligned}
\langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle - \sigma^*_{i,j} &= -\langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\boldsymbol{E}}'_j \rangle \\
&= -\langle \tilde{\Gamma}(\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \tilde{\boldsymbol{E}}'_j \rangle \\
&= -\langle (\tilde{\boldsymbol{G}}_i + \tilde{\Gamma}\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i)) \cdot \tilde{\boldsymbol{E}}'_j \rangle. \qquad \text{(A.51)}
\end{aligned}
$$

Thus we have by the Cauchy-Schwarz inequality that

$$
\begin{aligned}
|\langle (\boldsymbol{I} + \tilde{\boldsymbol{H}})(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i) \cdot \boldsymbol{e}_j \rangle - \sigma^*_{i,j}| &\le \sqrt{\langle \tilde{\boldsymbol{E}}'_j \cdot \tilde{\boldsymbol{E}}'_j \rangle} \sqrt{\langle |\tilde{\boldsymbol{G}}_i + \tilde{\Gamma}\tilde{\boldsymbol{H}}(\tilde{\boldsymbol{G}}_i + \boldsymbol{e}_i)|^2 \rangle} \\
&\le \delta \sqrt{\langle \tilde{\boldsymbol{E}}'_j \cdot \tilde{\boldsymbol{E}}'_j \rangle} \qquad \text{(A.52)}
\end{aligned}
$$

which is the desired inequality (3.21). The inequality (3.22) can be similarily proved. ∎

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