Quenching of Reaction by Cellular Flows

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September 25, 2004

Abstract

We consider a reaction-diffusion equation in a cellular flow. We prove that in the strong flow regime there are two possible scenario for the initial data that is compactly supported and the size of the support is large enough. If the flow cells are large compared to the reaction length scale, propagating fronts will always form. For the small cell size, any finitely supported initial data will be quenched by a sufficiently strong flow. We estimate that the flow amplitude required to quench the initial data of support L_0 is $A > CL_0^4 \ln(L_0)$. The essence of the problem is the question about the decay of the L^∞ norm of a solution to the advection-diffusion equation, and the relation between this rate of decay and the properties of the Hamiltonian system generated by the two-dimensional incompressible fluid flow.

1 Introduction

It has been well understood since the classical work of G.I. Taylor that the presence of a fluid flow may greatly increase the mixing properties of diffusion. This phenomenon is known as "eddy diffusivity" or "enhanced diffusion". The mathematical approach to the problem is usually via the homogenization techniques that concentrate on the long time-large scale behavior: see [25] for a recent extensive review. This approach is appropriate when there are no other time scales in the problem so that one may wait as long as needed for the mixing effects to become prominent.

Recently there has been a lot of interest in the effect of flows on the qualitative and quantitative behavior of solutions of reaction-diffusion equations. Intuitively, there may be two opposite effects of the additional mixing by the flow: on one hand, it may increase the spreading rate of the chemical reaction (the "wind spreading the fire" effect), or it may extinguish the reaction (the "try to light the campfire in a wind" effect). The first effect is related to the behavior of front-like solutions, and has been extensively studied recently: traveling fronts have been shown to exist in various flows [5, 6, 7, 33, 34, 35, 36], and flows have been shown to speed-up the front propagation due to the improved mixing [3, 4, 9, 20, 37], see [4, 36] for recent reviews of the mathematical results in the area. This problem has also attracted a significant attention in the

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physical literature, we mention [1, 2, 3, 17, 18, 19, 22, 23] among the recent papers and refer to [30] as a general reference. The present paper addresses the second phenomenon mentioned above: the possibility of flame extinction by a flow. The basic idea is that if the reaction process may occur only at the temperatures T above a critical threshold θ_0, then mixing by a strong flow coupled to diffusion may drop the temperature everywhere below θ_0 and hence extinguish the flame. However, unlike the usual linear advection-diffusion homogenization problems, one may not wait for this to happen beyond the time t_c it takes for the chemical reaction to occur – the mixing has to happen before this time. The question we address is: "Given a threshold θ_0, a time t_c, and the support L_0 of the initial data, can we find a flow amplitude $A_0(L_0)$ so that if the flow amplitude $A > A_0(L_0)$ is sufficiently large then $\sup_{\mathbf{x}} T(t = t_c, \mathbf{x}) \leq \theta_0$?" This problem has been first considered in [10] for unidirectional, or shear, flows that have open streamlines. Even in this simple situation the answer is non-trivial: in order for quenching to be possible, the profile $u(y)$ should not be constant on intervals larger than a prescribed size. The answer has been shown to be sharp in [21].

In this paper, we study quenching by a different class of flows with a more complex structure: incompressible cellular flows. These are flows such that the whole plane $\mathbb{R}^2$ is separated into invariant regions bounded by the separatrices of the flow that connect the flow saddle points. Many types of instabilities in fluids lead to cellular flows, making them ubiquitous in nature. We only mention Rayleigh-Bénard instability in heat convection, Taylor vortices in Couette flow between rotating cylinders or heat expansion driven Landau-Darrieus instability. The fact that the cellular flows have closed streamlines make the effect of advection more subtle. An important role in the possibility of quenching is played by a thin boundary layer which forms along the separatrices of the flow. Our main results show that the cellular flow is quenching if and only if the size of the minimal invariant regions (the flow cells) is smaller than a certain critical size of the order of laminar flame length scale.

The simplest mathematical model that describes a chemical reaction in a fluid is a single equation for temperature T of the form

$$\begin{aligned} & T_t + Au(\mathbf{x}) \cdot \nabla T = \Delta T + Mf(T) \\ & T(0, \mathbf{x}) = T_0(\mathbf{x}) \end{aligned} \tag{1}$$

where the flow $u(\mathbf{x})$ is prescribed. We are interested in the effect of a strong advection, and accordingly have written the velocity as a product of an amplitude A and a fixed flow $u(\mathbf{x})$. In this paper we consider nonlinearity of the ignition type, that is, we assume that

$$\begin{aligned} & \text{(i) } f(T) \text{ is Lipschitz continuous on } 0 \leq T \leq 1, \\ & \text{(ii) } f(1) = 0, \;\; \exists \theta_0 \text{ such that } f(T) = 0 \text{ for } T \in [0, \theta_0],\; f(T) > 0 \text{ for } T \in (\theta_0, 1) \\ & \text{(iii) } \;\; f(T) \leq T. \end{aligned} \tag{2}$$

The threshold θ_0 is called the ignition temperature. The last condition in (2) is just a normalization. We consider the reaction-diffusion equation (1) in a two-dimensional strip $D = \{x \in \mathbb{R},\; y \in [0, 2\pi l]\}$ with the periodic boundary conditions at the vertical boundaries:

$$T(x, y + 2\pi l) = T(x, y).$$

The initial data $T_0(\mathbf{x}) = T(0, \mathbf{x})$ is assumed to satisfy $0 \le T_0(\mathbf{x}) \le 1$. The maximum principle implies that then $0 \le T(t, \mathbf{x}) \le 1$ for all $t \ge 0$. We will say that regions with temperature close to one are "hot", and those with temperature close to zero are "cold".

The problem of extinction and flame propagation in (1) with the ignition type nonlinearity (2) was first studied by Kanel [16] in one dimension and with no advection. Assume for simplicity that the initial data are given by a characteristic function: $T_0(\mathbf{x}) = \chi_{[0,L]}(x)$. Kanel showed that, in the absence of fluid motion, there exist two length scales $L_0 < L_1$ such that the flame becomes extinct for $L < L_0$, and propagates for $L > L_1$. More precisely, he has shown that there exist L_0 and L_1 such that

$$\begin{aligned} &T(t, \mathbf{x}) \to 0 \text{ as } t \to \infty \text{ uniformly in } D \text{ if } L < L_0 \\ &T(t, \mathbf{x}) \to 1 \text{ as } t \to \infty \text{ for all } (x, y) \in D \text{ if } L > L_1. \end{aligned} \tag{3}$$

In the absence of advection, the flame extinction is achieved by diffusion alone, given that the support of initial data is small compared to the scale of the laminar front width $l_c = M^{-1/2}$. However, in many applications quenching is a result of strong wind, intense fluid motion and operates on larger scales. Kanel's result was extended to non-zero advection by shear flows by Roquejoffre [31] who has shown that (3) holds also for $u \ne 0$ with L_0 and L_1 depending, in particular, on A and $u(y)$ in an uncontrolled way.

As we have mentioned, the question of the dependence of the strength of advection A which is necessary for quenching the initial data of a given size L_0 has been recently studied in [10] and [21] in the case of a unidirectional (shear) flow $(Au(y), 0)$. Following [10], we call the flow $u(x, y)$ quenching if for every L_0 there exists $A_0(L_0)$ such that the solution of (1) with the initial data of size L_0 and advection strength $A > A_0(L_0)$ quenches. It turns out [10, 21] that the shear flow $u(y)$ is quenching if and only if $u(y)$ does not have a plateau of size larger than a certain critical threshold (comparable with the length scale $M^{-1/2}$ which characterizes the width of a laminar flame). The intuition behind this result is that shear flows are very effective in stretching the front and exposing the hot initial data to cool-off effects of diffusion unless there is a long, flat part in their profile, where this phenomenon is obviously not present.

Here we consider (1) for the domain $D = \mathbb{R} \times [-\pi l, \pi l]$ with the $2\pi l$-periodic boundary conditions in y and decay conditions in x:

$$T(t, x, y) = T(t, x, y + 2\pi l), \quad T(t, x, y) \to 0 \;\text{ as } x \to \pm\infty. \tag{4}$$

We restrict ourselves to a particular example of a cellular flow $u(x, y)$ that has the form $u_l(x, y) = \nabla^\perp h_l(x, y)$, where $\nabla^\perp h = (h_y, -h_x)$. Here l defines the size of a flow cell, and we take the stream function h_l to be

$$h_l(x, y) = l \sin\frac{x}{l} \sin\frac{y}{l} \tag{5}$$

whose streamline structure are shown in Fig. 1.

We will usually omit the index l in notation for u_l, h_l for the fluid flow; it will be clear from the context what the scaling is. The initial data $T_0(x, y)$ is non-negative, and bounded above by one: $0 \le T_0(x, y) \le 1$.

The first theorem shows that cellular flows with large cells do not have the quenching property.

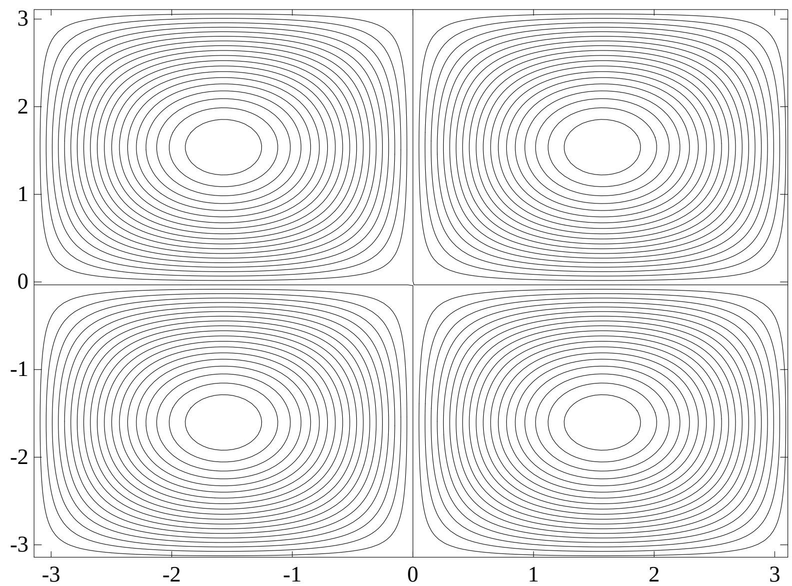

Figure 1: Cellular flow

Theorem 1.1 *Assume that $T_0(x,y)=1$ for $(x,y)\in[0,\pi l]\times[0,\pi l]$. There exists a critical cell size $l_0\sim M^{-1/2}$ so that if $l\geq l_0$, then under the above assumptions on the advection u, we have $T(t,x,y)\to 1$ as $t\to+\infty$, uniformly on compact sets, for all $A\in\mathbb{R}$.*

The notation $l_0\sim M^{-1/2}$ means that $C_1M^{-1/2}\leq l_0\leq C_2M^{-1/2}$ with $C_{1,2}$ some positive universal constants. The proof of Theorem 1.1 is simple and is based on a construction of an explicit time-independent sub-solution.

Next, we show that if the cell size l is sufficiently small then a sufficiently strong flow will quench a flame. More precisely, we have the following result.

Theorem 1.2 *Assume that $T_0(x,y)=0$ outside an interval $-L_0\leq x\leq L_0$ and $0\leq T_0(x,y)\leq 1$ for all $(x,y)\in D$. There exists a critical cell size $l_0\sim M^{-1/2}$ so that if $l\leq l_0$, then there exists $A_0(L_0)$ such that we have $T(t,x,y)\to 0$ as $t\to+\infty$, uniformly in D, for all $A\geq A_0(L_0)$. For large L_0, we have $A_0(L_0)\leq C(l)L_0^4\ln(L_0)$.*

A formal argument based on the homogenization theory predicts that $A_0\sim L_0^4$ without the factor of $\ln L_0$ – this follows from the effective diffusivity scaling $\kappa_*\sim\sqrt{A}$ that was first shown formally in [8] and later proved in [11, 24, 28]. The same scaling may be obtained from the formal predictions $V_A\sim A^{1/4}$ for the front speed V_A in a cellular flow [1, 2, 3, 32] – this implies that the front width is of the order $A^{1/4}$. Hence one might expect that initial data with the support less than the front width $L_0<A^{1/4}$ to be quenched. Therefore, the rigorously proved bound of Theorem 1.2 is likely to be sharp up to a logarithmic factor.

The particular choice of the stream function (5) is not important for the proof but it does simplify some of the estimates – it is straightforward to generalize our result to other cellular flows. However, the proof of Theorem 1.2 does use the periodicity of the flow in an essential way. We believe that the quenching property should hold for sufficiently regular non-periodic flows with small cells as well. We have preliminary results in this direction using different techniques; however, these results give much weaker upper bound for $A_0(L_0)$, and will appear elsewhere.

At the heart of the proof of Theorem 1.2 is the question about the rate of decay of L^∞ norm (and its dependence on A) of the solution to a passive advection-diffusion equation. Thus, our main object of study is a natural question about the effects of a combination of two fundamental and separately well-understood processes: advection by a fixed incompressible flow and diffusion. Yet their interaction is well known to produce subtle phenomena. The issues we study are directly related to the work of Freidlin and Wentzell [13, 14, 15] on the random perturbations of the Hamiltonian systems. They show that in the limit of large A the process converges to a diffusion on the Reeb graph of the background Hamiltonian. The relation to that work is very natural as any incompressible flow in two dimensions is a Hamiltonian system (the stream function is the Hamiltonian). However, the cellular flow that we consider does not satisfy the assumptions of the Freidlin-Wentzell theory, which requires growth of the Hamiltonian at infinity and does not allow existence of hetero-clinic orbits. Nevertheless, one may restate the small cell assumption in Theorem 1.2 as a requirement that the Reeb graph of the Hamiltonian has a sufficiently small diameter in its natural metric. The proof of Theorem 1.2 is based on two observations: first, temperature becomes approximately constant on the whole skeleton of separatrices, as the skeleton is just one point on the Reeb graph. Because of that, solution inside each cell may be split into two parts. One solves the initial value problem with zero data on the boundary. Another solves a nearly identical boundary value problem in each cell. The first part decays because the cell is small – this is where we use the size restriction. The second one is nearly identical on all cells, hence it has to be small in order not to violate the preservation of the L^1-norm. The technical part of the proof is in making this informal scenario rigorous.

The paper is organized as follows: in Section 2 we prove Theorem 1.1 by constructing an appropriate sub-solution and using certain PDE estimates to prove convergence of solution to unity. In Section 3 we give the proof of Theorem 1.2, which is more involved. It uses uniform in A estimates on the evolution of advection-diffusion equation, a boundary layer argument, and probabilistic estimates for an auxiliary cell heating problem.

Acknowledgment. The research of AF is supported in part by The Centennial Fellowship from American Mathematical Society and U.S. National Science Foundation (NSF) grant DMS-0306659. AK has been supported in part by NSF grants DMS-0321952 and DMS-0314129. LR has been supported in part by NSF grant DMS-0203537 and ONR grant N00014-02-1-0089. Both AK and LR acknowledge support by Alfred P. Sloan fellowships.

2 Absence of quenching by large cells

We prove in this section Theorem 1.1, that is, we show that cellular flows with sufficiently large cells do not have the quenching property. The proof consists of two steps. First, we construct a time-independent sub-solution $\Phi(\mathbf{x})$ to (1) in a cell $\mathcal{C}_1 = [0, \pi l] \times [0, \pi l]$: the function Φ satisfies

$$Au_l(\mathbf{x}) \cdot \nabla\Phi \le \Delta\Phi + Mf(\Phi) \tag{6}$$

and is $2\pi l$-periodic in y. It is also positive on an open set inside $\mathcal{C}_1$ and negative on $\partial\mathcal{C}_1$. We normalize Φ so that $\Phi \le 1$. As $T_0 = 1$ on $\mathcal{C}_1$ by assumption, we have $T_0(\mathbf{x}) \ge \Phi(\mathbf{x})$. Then the maximum principle implies that $T(t, \mathbf{x}) \ge \Phi(\mathbf{x})$ for all $t \ge 0$ and $\mathbf{x} \in \mathcal{C}_1$. It follows that $T(t, \mathbf{x})$ does not vanish as $t \to +\infty$. In the second step we show that actually $T(t, \mathbf{x}) \to 1$. We begin with the construction of the sub-solution $\Phi(\mathbf{x})$. First, we rescale equation (1) by $\mathbf{x} \to l\mathbf{x}$ so that a sub-solution in the rescaled coordinates should satisfy

$$\frac{A}{l} u(\mathbf{x}) \cdot \nabla\Phi \le \frac{1}{l^2}\Delta\Phi + Mf(\Phi). \tag{7}$$

Lemma 2.1 *If l is sufficiently large then there exists a C^1 function $\Phi(\mathbf{x})$ that is constant on the streamlines of the flow $u(\mathbf{x})$ and satisfies (7) for $\mathbf{x} \in \mathcal{C}_1 = [-\pi, \pi]^2$. Moreover, $\Phi(\mathbf{x}) \le 1$ for all $\mathbf{x} \in \mathcal{C}_1$, and $\Phi(\mathbf{x}) < 0$ for $\mathbf{x} \in \partial\mathcal{C}_1$.*

Proof of Lemma 2.1. We may choose two numbers θ_1 and θ_2 so that $\theta_0 < \theta_1 < \theta_2 < 1$ and such that the straight line that connects the point $(\theta_1, 0)$ to the point $(\theta_2, f(\theta_2))$ lies below the graph of $f(T)$. More precisely, that means that the function

$$g(T) = \begin{cases} 0, & T \le \theta_1 \\ \alpha(T - \theta_1), & \theta_1 \le T \le \theta_2 \\ f(T), & T \ge \theta_2 \end{cases} \tag{8}$$

with $\alpha = f(\theta_2)/(\theta_2 - \theta_1)$, satisfies $g(T) \le f(T)$. Such modification of $f(T)$ for the construction of sub-solutions was first used in [16], and then in [10]. A function $\Phi(h(\mathbf{x}))$ satisfies (7) if

$$\frac{1}{l^2}\left(|\nabla h|^2 \frac{d^2\Phi}{dh^2} + \Delta h \frac{d\Phi}{dh}\right) + Mg(\Phi) \ge 0,$$

where $h(x, y) = \sin x \sin y$ is the stream function. Note that the advection term vanishes identically for such functions. We use the fact that $\Delta h = -2h$ to obtain

$$|\nabla h|^2 \frac{d^2\Phi}{dh^2} - 2h\frac{d\Phi}{dh} + \left(\frac{l}{l_c}\right)^2 g(\Phi) \ge 0. \tag{9}$$

Here $l_c = M^{-1/2}$ is the laminar front width, the length scale associated to the chemical reaction strength. Relation (9) indicates that the ratio l/l_c has to be sufficiently large for a sub-solution to exist. Note that

$$2h(1 - h) \le |\nabla h|^2 \le 2(1 - h^2).$$

Indeed, we have

$$\begin{aligned} &|\nabla h(x, y)|^2 = \sin^2 x \cos^2 y + \cos^2 x \sin^2 y = \sin^2 x + \sin^2 y - 2\sin^2 x \sin^2 y \\ &\ge 2\sin x \sin y(1 - \sin x \sin y) = 2h(x, y)(1 - h(x, y)) \end{aligned}$$

and

$$|\nabla h(x,y)|^2 \leq 2 - 2\sin^2 x \sin^2 y = 2(1-h^2(x,y)).$$

Therefore it suffices to construct an increasing function $\Phi(h)$ satisfying

$$\frac{d^2\Phi}{dh^2} - \frac{2h}{2h(1-h)}\frac{d\Phi}{dh} + \frac{1}{2(1-h^2)}\left(\frac{l}{l_c}\right)^2 g(\Phi) \geq 0,$$

which would in turn follow from

$$\frac{d^2\Phi}{dh^2} - \frac{1}{1-h}\frac{d\Phi}{dh} + \frac{1}{2}\left(\frac{l}{l_c}\right)^2 g(\Phi) = 0.$$

Make a change of variables $R = \frac{l}{l_c\sqrt{2}}(1-h)$ so that the above becomes

$$\frac{d^2\Phi}{dR^2} + \frac{1}{R}\frac{d\Phi}{dR} + g(\Phi) = 0. \tag{10}$$

The center of the cell corresponds now to $R=0$, while the boundary $h=0$ becomes $R = \frac{l}{l_c\sqrt{2}}$. We impose the following "initial data" for (10):

$$\Phi(0) = \theta_2, \quad \frac{d\Phi(0)}{dR} = 0.$$

The explicit form (8) of the function $g(\Phi)$ implies that the solution $\Phi(R)$ is given explicitly by

$$\Phi(R) = \theta_1 + (\theta_2 - \theta_1)J_0\left(R\sqrt{\alpha}\right), \quad \text{for } R \leq R_1 = \frac{\xi_1}{\sqrt{\alpha}} \tag{11}$$

with α as in (8). Here $J_0(\xi)$ is the Bessel function of order zero, and ξ_1 is its first zero.

Furthermore, we have

$$\Phi(R) = B\ln\frac{R_2}{R}, \quad \text{for } R_1 \leq R. \tag{12}$$

The constants B and R_2 are determined by matching the functions (11) and (12), and their derivatives at $R = R_1$. Then we get

$$B = (\theta_2-\theta_1)\xi_1|J_0'(\xi_1)|, \quad R_2 = \xi_1\sqrt{\frac{\theta_2-\theta_1}{f(\theta_2)}}\exp\left[\frac{\theta_1}{(\theta_2-\theta_1)\xi_1|J_0'(\xi_1)|}\right].$$

Observe that the function $\Phi(R)$ constructed above is negative on the boundary of the cell only provided that $R_2 < \frac{l}{l_c\sqrt{2}}$, which means that the cell size

$$l \geq l_c\sqrt{2}\xi_1\sqrt{\frac{\theta_2-\theta_1}{f(\theta_2)}}\exp\left[\frac{\theta_1}{(\theta_2-\theta_1)\xi_1|J_0'(\xi_1)|}\right]$$

has to be sufficiently large for this construction to be applicable. This proves Lemma 2.1. □

In order to finish the proof of Theorem 1.1 we have to show that $T(t,\mathbf{x}) \to 1$ as $t \to +\infty$ provided that $T_0(\mathbf{x}) = 1$ on a cell. For such initial data we have

$$T_0(\mathbf{x}) \geq \Phi_0(\mathbf{x}) = \max\{\Phi(h(\mathbf{x})), 0\},$$

where the function $\Phi(\mathbf{x})$ is the sub-solution constructed in Lemma 2.1. It follows from the parabolic maximum principle that then $T(t,\mathbf{x}) \geq \Phi_0(\mathbf{x})$ for all $t \geq 0$. Furthermore, we have $T(t,\mathbf{x}) \geq \Psi(t,\mathbf{x})$, where the function $\Psi(t,\mathbf{x})$ satisfies (1) with the initial data $\Phi_0(\mathbf{x})$. Note that $\Psi(t,\mathbf{x}) \geq \Phi_0(\mathbf{x})$ for all $t \geq 0$. The maximum principle applied to the finite differences $\Psi_h(t,x) = \Psi(t+h,\mathbf{x}) - \Psi(t,\mathbf{x})$ implies that $\Psi(t,\mathbf{x})$ is a point-wise increasing function of time that is bounded above by one. Therefore the point-wise limit $\overline{\Psi}(\mathbf{x}) = \lim_{t\to+\infty}\Psi(t,\mathbf{x})$ exists, moreover, $\overline{\Psi}(\mathbf{x}) \geq \Phi_0(\mathbf{x})$, and $\overline{\Psi}(\mathbf{x})$ satisfies the stationary problem

$$Au(\mathbf{x}) \cdot \nabla\overline{\Psi} = \Delta\overline{\Psi} + Mf(\overline{\Psi}). \tag{13}$$

with the $2\pi l$-periodic boundary conditions in y

$$\overline{\Psi}(t,x,0) = \overline{\Psi}(t,x,2\pi l)$$

(here $2N$ is the number of cells in y direction). We have the following lemma.

Lemma 2.2 *The function $\overline{\Psi}(x,y)$ satisfies the following bound:*

$$\int_D |\nabla\overline{\Psi}(\mathbf{x})|^2\,d\mathbf{x} + \int_D f(\overline{\Psi}(\mathbf{x}))\,d\mathbf{x} < +\infty, \tag{14}$$

where $D = \mathbb{R}_x \times [-\pi l, \pi l]_y$.

Proof. The function $\Psi(t,\mathbf{x})$ satisfies an a priori bound

$$\frac{1}{\tau}\int_0^\tau \left(\int_D f(\Psi(t,\mathbf{x}))\,d\mathbf{x}\right)dt \leq C_0, \quad \tau \geq CM^{-1},$$

that may be easily proved as in [9]. Here the constant C_0 may depend on the flow amplitude A. Therefore, there exists a sequence of times $t_n \to +\infty$ so that

$$\int_D f(\Psi(t_n,\mathbf{x}))\,d\mathbf{x} \leq C_0.$$

This implies that

$$\int_D f(\overline{\Psi}(\mathbf{x}))\,d\mathbf{x} \leq C_0,$$

and it remains to obtain the bound on $\|\nabla\overline{\Psi}\|_{L^2}$ in (14). We multiply (13) by $\overline{\Psi}$ and integrate in x between $-X+\zeta$ and $X+\zeta$ with X large and $\zeta \in [0,l_c]$, and in $y \in \mathbb{R}$. We get

$$\begin{aligned}
&\frac{A}{2}\int_{-\pi l}^{\pi l}\left[u_1(X+\zeta,y)|\overline{\Psi}(X+\zeta,y)|^2 - u_1(-X+\zeta,y)|\overline{\Psi}(-X+\zeta,y)|^2\right]dy \\
&= \int_{-\pi l}^{\pi l}\left[\overline{\Psi}(X+\zeta,y)\overline{\Psi}_x(X+\zeta,y) - \overline{\Psi}(-X+\zeta,y)\overline{\Psi}_x(-X+\zeta,y)\right]dy \\
&+M\int_{-\pi l}^{\pi l}dy\int_{-X+\zeta}^{X+\zeta}\overline{\Psi}(x,y)f(\overline{\Psi}(x,y))dx - \int_{-\pi l}^{\pi l}dy\int_{-X+\zeta}^{X+\zeta}|\nabla\overline{\Psi}(x,y)|^2dx
\end{aligned}$$

and average this equation in $\zeta \in [0, l_c]$. This provides the bound

$$\frac{1}{l_c}\int_0^{l_c} d\zeta \int_{-\pi l}^{\pi l} dy \int_{-X+\zeta}^{X+\zeta} |\nabla\overline{\Psi}(x,y)|^2 dx \le \pi A l \|u\|_\infty + \frac{2\pi l}{l_c} + M\int_D f(\overline{\Psi}(x,y))dxdy$$

Taking X to infinity, we obtain

$$\int_D |\nabla\overline{\Psi}(x,y)|^2 dx \le \pi A l\|u\|_\infty + \frac{2\pi l}{l_c} + M\int_D f(\overline{\Psi}(x,y))dxdy$$

and the bound on $\nabla\overline{\Psi}$ in (14) follows. □

Now we are ready to complete the proof of Theorem 1.1.

Proof. Lemma 2.2 implies that there exist two sequences of points $x_n \to -\infty$ and $z_n \to +\infty$ so that

$$\int_{-\pi l}^{\pi l} \left(|\nabla\overline{\Psi}(x_n,y)|^2 + |\nabla\overline{\Psi}(z_n,y)|^2\right) dy \to 0 \ \text{ as } n \to +\infty. \tag{15}$$

We integrate (13) in y and in x between x_n and z_n to obtain

$$\begin{aligned} &A\int_{-\pi l}^{\pi l} \left[u_1(z_n,y)\overline{\Psi}(z_n,y) - u_1(x_n,y)\overline{\Psi}(x_n,y)\right] dy \\ &= \int_{-\pi l}^{\pi l} \left[\overline{\Psi}_x(z_n,y) - \overline{\Psi}_x(x_n,y)\right] dy + M\int_{-\pi l}^{\pi l} dy \int_{x_n}^{z_n} f(\overline{\Psi}(x,y))dx. \end{aligned} \tag{16}$$

We pass to the limit $n \to \infty$ in (16). Observe that

$$\left|\int_{-\pi l}^{\pi l} \overline{\Psi}_x(z_n,y)dy\right| \le \sqrt{\pi l}\left(\int_{-\pi l}^{\pi l} |\overline{\Psi}_x(z_n,y)|^2 dy\right)^{1/2} \to 0 \ \text{ as } n \to +\infty$$

as follows from (15), and similarly

$$\left|\int_{-\pi l}^{\pi l} \overline{\Psi}_x(x_n,y)dy\right| \to 0 \ \text{ as } n \to +\infty.$$

Furthermore, since

$$\int_{-\pi l}^{\pi l} u_1(x,y)dy = 0$$

for all $x \in \mathbb{R}$, (15) and the Cauchy-Schwartz inequality imply that

$$\int_{-\pi l}^{\pi l} u_1(z_n,y)\overline{\Psi}(z_n,y)dy = \int_{-\pi l}^{\pi l} u_1(z_n,y)\left(\int_0^y \overline{\Psi}_\xi(z_n,\xi)d\xi\right) dy \to 0, \ \text{ as } n \to +\infty.$$

Therefore in the limit $n \to +\infty$ equation (16) becomes

$$\int_D f(\overline{\Psi}(x,y))dxdy = 0$$

and hence $f(\overline{\Psi}(x,y)) = 0$ for all $(x,y) \in D$. However, since $\overline{\Psi}(x,y) \ge \Phi_0(x,y)$, while, on the other hand, $\max \Phi_0(x,y) = \theta_2 > \theta_0$, and the function $\overline{\Psi}(x,y)$ is continuous, we conclude that $\overline{\Psi}(x,y) \equiv 1$. This finishes the proof of Theorem 1.1. □

3 Quenching by small cells

In this section we show that the cellular flow with small cells is quenching. The proof proceeds in several steps. First, we reduce the problem to a linear advection-diffusion equation. Indeed, as $f(T) \leq T$ we have the following upper bound for T :

$$T(t, \mathbf{x}) \leq e^{Mt} \phi(t, \mathbf{x}). \tag{17}$$

The function $\phi(t, \mathbf{x})$ satisfies the advection-diffusion equation

$$\frac{\partial \phi}{\partial t} + Au \cdot \nabla \phi = \Delta \phi \tag{18}$$

with the same initial data $\phi(0, \mathbf{x}) = T_0(\mathbf{x})$ and the $2\pi l$-periodic boundary conditions in y: $\phi(t, x, y) = \phi(t, x, y + 2\pi l)$. Note that if at some time $t_0 > 0$ we have $\phi(t_0, \mathbf{x}) \leq \theta_0$ everywhere, then the maximum principle implies that $\phi(t, \mathbf{x}) \leq \theta_0$ and T satisfies the linear equation (18) for all $t \geq t_0$. Then the conclusion of Theorem 1.2 follows. Hence, the upper bound (17) implies that it suffices to show $\phi(t = M^{-1}, x, y) \leq \theta_0' = \theta_0 e^{-1}$ and this is what we will do.

Heuristically, the proof relies on the observation that solution of (18) should generally become constant along the streamlines of the flow if its amplitude is large. Moreover, the value of the solution on the streamlines $h = h_0$ very near the boundary in two neighboring cells have to be close (as follows from a simple L^2-bound on $\nabla \phi$ appearing in Lemma 3.3 below). However, that means that solution should have, roughly speaking, the same profile in each cell. This is incompatible with the preservation of the L^1-norm of ϕ unless this function is very small in each of the cells which means that solution has to be less than θ_0 everywhere. The proof follows this heuristic outline – the technical difficulty is that we are able to control the uniformity of the solution along the streamlines only in a space-time averaged sense. Additional ingredients are required to obtain the point-wise control.

3.1 The Nash inequality lemma

We will need throughout the proof an $L^1 - L^\infty$ decay estimate for solutions of the linear diffusion-advection

$$\frac{\partial T}{\partial t} + v \cdot \nabla T = \Delta T \tag{19}$$

that is independent of the advection strength. Equation (19) is considered in the infinite strip $D = \mathbb{R} \times [-\pi l, \pi l]$ with the $2\pi l$-periodic boundary conditions in y direction.

Lemma 3.1 *There exists a constant $C > 0$ so that the solution of*

$$\frac{\partial \psi}{\partial t} + v \cdot \nabla \psi = \Delta \psi \tag{20}$$
$$\psi(0, \mathbf{x}) = \psi_0(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^2$$

with the $2\pi l$-periodic boundary condition in y and a flow v that is $2\pi l$-periodic, sufficiently regular and divergence-free: $\nabla \cdot v = 0$, satisfies

$$\|\psi(t)\|_{L^\infty(D)} \leq Cn^2(t) \|\psi_0\|_{L^1(D)}, \tag{21}$$

where $D = \mathbb{R}_x \times [-\pi l, \pi l]_y$. Here $n(t)$ is the unique solution of

$$\frac{4n^4(t)}{1+4n^3(t)l^3} = \frac{C_1}{l^2 t,} \tag{22}$$

and the constants C, C_1 do not depend on v.

Remark. Note that (22) implies that for $t \geq l^2$, we have $n(t)^2 \sim \frac{C}{l\sqrt{t}}$. Hence, solution decays as the solution of the one-dimensional problem after the time it takes the diffusion to feel the boundary.

Proof. We multiply (20) by ψ and integrate over the domain D to obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 = -\|\nabla\psi\|_2^2. \tag{23}$$

Here and below in the proof of this Lemma, $\|\cdot\|_p$ denotes the norm in $L^p(D)$.

We now prove the following version of the Nash inequality [27] for a strip of width l:

$$\|\nabla\psi\|_2^2 \geq C\frac{l^2\|\psi\|_2^6}{\|\psi\|_1^4 + l^3\|\psi\|_1\|\psi\|_2^3} \tag{24}$$

The proof of (24) is similar to that of the usual Nash inequality. We represent ψ in terms of its Fourier series-integral:

$$\psi(x,y) = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} e^{iny/l+ikx}\hat{\psi}_n(k)\frac{dk}{(2\pi)^2},$$

where

$$\hat{\psi}_n(k) = \frac{1}{l}\int_D e^{-ikx-iny/l}\psi(x,y)dxdy.$$

Therefore we have $|\hat{\psi}_n(k)| \leq \dfrac{1}{l}\|\psi\|_{L^1}$. The Plancherel formula becomes

$$\begin{aligned}\int_D |\psi(x,y)|^2 dxdy &= \sum_{n,m\in\mathbb{Z}}\int_{\mathbb{R}^2} e^{iny/l-imy/l+ikx-ipx}\hat{\psi}_n(k)\bar{\hat{\psi}}_m(p)\frac{dkdpdxdy}{(2\pi)^4}\\ &= l\sum_{n\in\mathbb{Z}}\int|\hat{\psi}_n(k)|^2\frac{dk}{(2\pi)^2}\end{aligned}$$

and similarly

$$\int_D |\nabla\psi(x,y)|^2 dxdy = l\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\left(k^2+\frac{n^2}{l^2}\right)|\hat{\psi}_n(k)|^2\frac{dk}{(2\pi)^2}.$$

Let $\rho > 0$ be a positive number to be chosen later. Then using the Plancherel formula we may write

$$\|\psi\|_2^2 = I + II,$$

where

$$I = l\sum_{|n|\leq\rho l}\int_{|k|\leq\rho}|\hat{\psi}_n(k)|^2\frac{dk}{2\pi} \leq \frac{Cl\rho([l\rho]+1)}{l^2}\|\psi\|_1^2 \leq \frac{C\rho(l\rho+1)}{l}\|\psi\|_1^2.$$

The rest may be bounded by

$$II \le \frac{l}{\rho^2} \sum_{n \in \mathbb{Z}} \int_{k \in \mathbb{R}} \left(k^2 + \frac{n^2}{l^2}\right) |\hat{\psi}_n(k)|^2 \frac{dk}{(2\pi)^2} \le \frac{C}{\rho^2} \|\nabla \psi\|_2^2.$$

Therefore we have for all $\rho > 0$:

$$\|\psi\|_2^2 \le \frac{C\rho(l\rho + 1)}{l} \|\psi\|_1^2 + \frac{C}{\rho^2} \|\nabla \psi\|_2^2.$$

We choose ρ so that

$$\rho^3 = \frac{l\|\nabla\psi\|_2^2}{\|\psi\|_1^2}$$

and obtain

$$\begin{aligned} \|\psi\|_2^2 &\le \frac{C\|\nabla\psi\|_2^{2/3}}{l^{2/3}\|\psi\|_1^{2/3}} \left(\frac{l^{4/3}\|\nabla\psi\|_2^{2/3}}{\|\psi\|_1^{2/3}} + 1 \right) \|\psi\|_1^2 + \frac{C\|\nabla\psi\|_2^2 \|\psi\|_1^{4/3}}{l^{2/3}\|\nabla\psi\|_2^{4/3}} \\ &= \frac{2C}{l^{2/3}} \|\psi\|_1^{4/3} \|\nabla\psi\|_2^{2/3} + Cl^{2/3} \|\nabla\psi\|_2^{4/3} \|\psi\|_1^{2/3}. \end{aligned}$$

This is a quadratic inequality $ax^2+bx-c \ge 0$ with $x = \|\nabla\psi\|_2^{2/3}$, $a = l^{2/3}\|\psi\|_1^{2/3}$, $b = \frac{2}{l^{2/3}}\|\psi\|_1^{4/3}$, and $c = \|\psi\|_2^2/C$ and hence

$$x \ge \frac{-b + \sqrt{b^2 + 4ac}}{2a} = \frac{2c}{b + \sqrt{b^2 + 4ac}} \ge \frac{c}{\sqrt{b^2 + 4ac}}.$$

This implies that

$$\|\nabla\psi\|_2^{2/3} \ge C\|\psi\|_2^2 \left(\frac{4\|\psi\|_1^{8/3}}{l^{4/3}} + 4l^{2/3}\|\psi\|_1^{2/3}\|\psi\|_2^2 \right)^{-1/2}$$

and therefore

$$\begin{aligned} \|\nabla\psi\|_2^2 &\ge C\|\psi\|_2^6 \left(\frac{4\|\psi\|_1^{8/3}}{l^{4/3}} + 4l^{2/3}\|\psi\|_1^{2/3}\|\psi\|_2^2 \right)^{-3/2} \ge C\|\psi\|_2^6 \left(\frac{\|\psi\|_1^4}{l^2} + l\|\psi\|_1\|\psi\|_2^3 \right)^{-1} \\ &\ge \frac{Cl^2\|\psi\|_2^6}{\|\psi\|_1^4 + l^3\|\psi\|_1\|\psi\|_2^3}. \end{aligned}$$

Hence (24) indeed holds.

We insert (24) into the inequality (23) and using the conservation of the L^1-norm of ψ (recall that the initial data is non-negative) obtain

$$\frac{d\|\psi\|_2}{dt} \le -\frac{Cl^2\|\psi\|_2^5}{\|\psi_0\|_1^4 + l^3\|\psi_0\|_1\|\psi\|_2^3}. \tag{25}$$

Integrating (25) in time we have

$$Cl^2 t \le \frac{\|\psi_0\|_1^4}{4\|\psi\|_2^4} + \frac{l^3\|\psi_0\|_1}{\|\psi\|_2} \le \frac{1}{z(t)}\left[l^3 + \frac{1}{4z^3(t)}\right],$$

where $z(t) = \|\psi(t)\|_2/\|\psi_0\|_1$, and thus

$$\frac{4z^4(t)}{1+4l^3z^3(t)} \le \frac{1}{Cl^2t}. \tag{26}$$

The function on the left side of (26) is monotonically increasing and hence we have

$$\|\psi(t)\|_2 \le n(t)\|\psi_0\|_1, \tag{27}$$

where $n(t)$ is the solution of (22).

Let us denote by $\mathcal{P}_t$ the solution operator for (20): $\psi(t) = \mathcal{P}_t\psi_0$. Then (27) implies that $\|\mathcal{P}_t\|_{L^1\to L^2} \le n(t)$. The adjoint operator $\mathcal{P}_t^*$ is the solution operator for

$$\begin{aligned} &\frac{\partial\tilde\psi}{\partial t} - v\cdot\nabla\tilde\psi = \Delta\tilde\psi \\ &\tilde\psi(0,x) = \tilde\psi_0(x), \quad x\in\mathbb{R}^d \end{aligned} \tag{28}$$

Note that the preceding estimates rely only on the skew adjointness of the convection operator $v\cdot\nabla$. Therefore we have the bound $\|\mathcal{P}_t^*\|_{L^1\to L^2} \le n(t)$ and hence $\|\mathcal{P}_t\|_{L^2\to L^\infty} \le n(t)$ so that

$$\|\psi(t)\|_{L^\infty} \le n(t/2)\|\psi(t/2)\|_{L^2} \le n^2(t/2)\|\psi_0\|_{L^1} \tag{29}$$

and the proof of Lemma 3.1 is complete. □

A very similar argument leads to an estimate for the solution of (19) with the $2\pi l$-periodic or zero Dirichlet boundary conditions in both x and y. We state this variant which we will need.

Lemma 3.2 *Consider equation (20) with the $2\pi l$-periodic boundary conditions in x and y. Assume that the initial data $\psi_0(\mathbf{x})$ is mean zero: $\int\psi_0(\mathbf{x})\,d\mathbf{x} = 0$. Then there exists a constant $C>0$ such that*

$$\|\psi(t)\|_{L^\infty(D)} \le Cn^2(t)\|\psi_0\|_{L^1(D)},$$

where $D = [0,2\pi l]_x \times [0,2\pi l]_y$. Here $n(t)$ is the unique solution of

$$\frac{4n^4(t)}{1+4n^3(t)l^3} = \frac{C_1}{l^2t},$$

and the constants C, C_1 do not depend on v. The same result holds for the zero Dirichlet boundary conditions, without the assumption that the initial data is mean zero.

3.2 Variation of temperature on streamlines and gradient bounds

The next step of the proof is to estimate the time-space averages of the oscillations of the solution along the streamlines of u and show that they are small if the advection amplitude is sufficiently strong. This does not have to be true in general, but a bound of this type holds if the initial data is uniform on the streamlines – then one has only to show that no large oscillations appear at later times. First, we reduce the problem to such initial data with the help of the Nash inequality in Lemma 3.1. The maximum principle implies that it suffices to prove Theorem 1.2 for the initial data of the form $T_0 = 1$ for $|x| \le L_0$ and $T_0 = 0$ elsewhere. We split the initial data as

$$T_0(\mathbf{x}) = T_0(\mathbf{x})\eta(h(\mathbf{x})/\delta_0) + T_0(\mathbf{x})(1 - \eta(h(\mathbf{x})/\delta_0)) = \phi_{01} + \phi_{02}. \tag{30}$$

The small parameter δ_0 will be specified later. The cutoff function $\eta(h)$ satisfies

$$0 \le \eta(h) \le 1 \text{ for all } h \in \mathbb{R},\ \eta(h) = 0 \text{ for } |h| \le 1,\ \eta(h) = 1 \text{ for } |h| \ge 2.$$

We split the solution $\phi(t, \mathbf{x})$ of (18) as a sum $\phi = \phi_1 + \phi_2$. The functions $\phi_{1,2}$ satisfy (18) with the initial data ϕ_{01} and ϕ_{02}, respectively. The function ϕ_{01} is smooth and is constant (1 or 0) on the streamlines of the flow. In particular it is equal to zero in the whole "water-pipe" system of boundary layers around the separatrices. The function ϕ_{02} satisfies a bound

$$\|\phi_{02}\|_{L^1(D)} \le |\{\mathbf{x} \in D = \mathbb{R} \times [-\pi l, \pi l] : \ |x| \le L_0,\ |h(\mathbf{x})| \le 2\delta_0\}| \le C\delta_0 L_0 \ln(l/\delta_0). \tag{31}$$

Therefore, Lemma 3.1 implies that if L_0 is sufficiently large and we require that

$$\delta_0 \le \frac{C\theta_0 l^2}{L_0(\ln(L_0/l))^2}, \tag{32}$$

with an appropriate constant C, then the function ϕ_2 satisfies a uniform upper bound

$$\|\phi_2(t = l^2)\|_{L^\infty(D)} \le \frac{\theta_0'}{10}. \tag{33}$$

Hence we choose δ_0 as in (32) and concentrate on the function ϕ_1 that is initially uniform along the streamlines of u. We drop the subscript one to simplify the notation wherever this causes no confusion.

Next, we obtain some uniform estimates for solutions of (18) that are initially constant on the streamlines.

Lemma 3.3 *For any time $t > 0$ we have*

$$\int_0^t \int_D |\nabla\phi|^2 d\mathbf{x}ds \le \int_D |\phi_0(x)|^2 d\mathbf{x}. \tag{34}$$

Assume in addition that the initial data $\phi_0(x)$ for the equation (18) are constant on streamlines. Then

$$\int_0^t \int_D |u \cdot \nabla\phi|^2 d\mathbf{x}ds \le CA^{-2}t \int_D |\Delta\phi_0|^2 d\mathbf{x} + CA^{-1} \int_D |\phi_0|^2 d\mathbf{x}. \tag{35}$$

Proof. Multiplying (18) by $\phi(t,\mathbf{x})$ and integrating we trivially obtain

$$\int_D |\phi(t,\mathbf{x})|^2 d\mathbf{x} + \int_0^t \int_D |\nabla\phi(s,\mathbf{x})|^2 d\mathbf{x}ds = \int_D |\phi_0(\mathbf{x})|^2 d\mathbf{x},$$

which implies (34). Next, we multiply (18) by $u\cdot\nabla\phi$ and integrate over D to get

$$\begin{aligned}\int_D |u\cdot\nabla\phi|^2 d\mathbf{x} &= A^{-1}\int_D u\cdot\nabla\phi(\Delta\phi - \phi_t)d\mathbf{x}\\ &\le \frac{1}{2A^2}\int_D \phi_t^2 d\mathbf{x} + \frac{1}{2}\int_D |u\cdot\nabla\phi|^2 d\mathbf{x} - A^{-1}\int_D \nabla\phi^T\nabla u\nabla\phi d\mathbf{x}.\end{aligned}$$

Thus we get

$$\int_D |u\cdot\nabla\phi|^2\, d\mathbf{x} \le CA^{-2}\int_D \phi_t^2 d\mathbf{x} + CA^{-1}\int_D |\nabla\phi|^2 d\mathbf{x}. \tag{36}$$

From (36) and (34) we obtain that

$$\int_0^t \int_D |u\cdot\nabla\phi(\mathbf{x},s)|^2\, d\mathbf{x}ds \le CA^{-2}\int_0^t\int_D \phi_s^2 d\mathbf{x}ds + CA^{-1}\int_D |\phi_0|^2 d\mathbf{x}. \tag{37}$$

Recall that initially $\phi_0(x,y)$ is constant on streamlines of u, so that $u\cdot\nabla\phi_0 = 0$. Since the function ϕ_t satisfies the same equation (18) as ϕ, this implies that

$$\int_D |\phi_t(t,\mathbf{x})|^2 d\mathbf{x} + \int_0^t\int_D |\nabla\phi_t(s,\mathbf{x})|^2 d\mathbf{x}ds = \int_D |\phi_t(0,\mathbf{x})|^2 d\mathbf{x} = \int_D |\Delta\phi_0|^2 d\mathbf{x}.$$

Therefore, bounding from above the time derivative term in (37), we arrive at (35). $\square$

Note that the initial data for the function ϕ_1 in (30) obeys an upper bound

$$\begin{aligned}&\int_D |\Delta\phi_{01}|^2 d\mathbf{x} = \int_{-L_0}^{L_0}\int_0^l \left|\Delta\left[\eta\left(\frac{h(\mathbf{x})}{\delta_0}\right)\right]\right|^2 dydx\\ &= \int_{-L_0}^{L_0}\int_0^l \left|\frac{\Delta h(\mathbf{x})}{\delta_0}\eta'\left(\frac{h(\mathbf{x})}{\delta_0}\right) + \frac{|\nabla h(\mathbf{x})|^2}{\delta_0^2}\eta''\left(\frac{h(\mathbf{x})}{\delta_0}\right)\right|^2 dxdy \le \frac{CL_0}{\delta_0^3}\ln\left(\frac{l}{\delta_0}\right).\end{aligned}$$

Hence, according to (35), the total oscillation on streamlines is bounded by

$$\begin{aligned}\int_0^t\int_D |u\cdot\nabla\phi(s,\mathbf{x})|^2\, d\mathbf{x}ds &\le CA^{-2}t\int_D |\Delta\phi_{01}|^2 d\mathbf{x} + CA^{-1}\int_D |\phi_{01}|^2 d\mathbf{x}\\ &\le \frac{CA^{-2}\tau L_0}{\delta_0^3}\ln\left(\frac{l}{\delta_0}\right) + CA^{-1}L_0\end{aligned} \tag{38}$$

for $t \le \tau$.

3.3 Boundary layer and cell-to-cell heat conduction

Lemma 3.3 suggests that for most times, there should be very little temperature variation along the streamlines. Our next goal is make this statement more precise. Let us fix a time τ to be chosen later. We will denote by h and θ the coordinates inside each cell, with h being our usual stream function and θ the orthogonal coordinate normalized by the condition $|\nabla\theta| = |\nabla h|$ along the cell boundary and increasing in the direction of the flow. Let $\delta > 0$ be an arbitrary, small number to be chosen later.

Lemma 3.4 *There exists h_0 satisfying $2\delta > h_0 > \delta$ such that*

$$\int_0^\tau \sum_{cells} \sup_{\theta_1,\theta_2} |\phi(h_0,\theta_1,t) - \phi(h_0,\theta_2,t)|^2\, dt \le C\delta^{-1}A^{-1}L_0 l \ln\left(\frac{l}{\delta}\right)(A^{-1}\tau\delta_0^{-3}\ln\left(\frac{l}{\delta_0}\right) + 1) \quad (39)$$

where δ_0 satisfies (32). As a consequence, given a small number $\gamma > 0$, for all times except for a set S_γ of Lebesgue measure at most $\gamma\tau$ we have

$$\sum_{cells} \sup_{\theta_1,\theta_2} |\phi(h_0,\theta_1,t) - \phi(h_0,\theta_2,t)|^2 \le C\delta^{-1}A^{-1}\gamma^{-1}L_0 l \ln\left(\frac{l}{\delta}\right)(A^{-1}\delta_0^{-3}\ln\left(\frac{l}{\delta_0}\right) + \tau^{-1}). \quad (40)$$

Proof. It is easy to check that

$$\frac{\partial\phi}{\partial\theta} = |\nabla h|^{-1}|\nabla\theta|^{-1} u\cdot\nabla\phi.$$

Let us denote by S_a the streamline $h = a \in [\delta, 2\delta]$ in a given cell. Then we have

$$\sup_{\theta_1,\theta_2}|\phi(h,\theta_1) - \phi(h,\theta_2)|^2 \le \left(\int_{S_h} |u\cdot\nabla\phi| \frac{d\theta}{|\nabla\theta||\nabla h|}\right)^2$$
$$\le C\int_{S_h} |u\cdot\nabla\phi|^2 \frac{d\theta}{|\nabla\theta||\nabla h|}\int_{S_h}\frac{d\theta}{|\nabla\theta||\nabla h|} \le Cl\ln\frac{l}{\delta}\int_{S_h}|u\cdot\nabla\phi|^2\frac{d\theta}{|\nabla\theta||\nabla h|}.$$

The last step follows from the estimate

$$\int_{S_h}\frac{d\theta}{|\nabla\theta||\nabla h|} \le Cl\ln\left(\frac{l}{\delta}\right)$$

in the tube $\delta < h < 2\delta$. Integrating in h and in time and summing over all cells, we obtain

$$\int_0^\tau\int_\delta^{2\delta}\sum_{cells}\sup_{\theta_1,\theta_2}|\phi(h,\theta_1,t) - \phi(h,\theta_2,t)|^2\, dhdt \le Cl\ln\left(\frac{l}{\delta}\right)\int_0^\tau\int_D |u\cdot\nabla\phi|^2\, d\mathbf{x}dt. \quad (41)$$

Now (39) and (40) follow from (38) by an application of the mean value theorem. □

We see that for all times but a small set of "exceptional" times, the value of the temperature on the $h = h_0$ streamline is close to some constant in any cell. Our next goal is to establish a control on how different these constants can be for two neighboring cells. Consider two such cells, $\mathcal{C}_-$ and $\mathcal{C}_+$, and let B be their common boundary. Let us choose the coordinates on these two cells so that $h = 0$ on B, $h > 0$ on the right cell $\mathcal{C}_+$, $h < 0$ on the left cell $\mathcal{C}_-$, and the angular coordinates θ_- and θ_+ are equal to zero in the mid-point of B. Denote by $D_\pm$ the region bounded by curves $\theta_{-,+} = \pm y$ and streamlines $h = \pm h_0$, where $y \sim l$ is chosen so that $D_\pm$ is sufficiently far away from the corners of the cells (so that $|\nabla h|$, $|\nabla \theta_\pm| \geq C$ on $D_\pm$). Also denote by $S_{-,+}$ the pieces of streamlines bounding $D_\pm$. Let us define the temperature drop between $\mathcal{C}_-$ and $\mathcal{C}_+$ as follows:

$$|\phi_+ - \phi_-| = \max\left\{0, \min_{S_+}\phi - \max_{S_-}\phi, \min_{S_-}\phi - \max_{S_+}\phi\right\}.$$

Note that if the time is not exceptional, Lemma 3.4 implies that maximum and minimum of ϕ along $h = h_0$ streamline in any cell differ by at most $C\delta^{-1}A^{-1}l\ln(l/\delta)\gamma^{-1}L_0(A^{-1}\delta_0^{-3}\ln(l/\delta_0) + \tau^{-1})$. Now we are ready to state

Lemma 3.5 *For any $\tau > 0$, we have for $D = \mathbb{R} \times [-\pi l, \pi l]$*

$$\int_0^\tau \sum_{cells} |\phi_+ - \phi_-|^2\, dt \leq C\delta l^{-1} \int_0^\tau \int_D |\nabla \phi|^2\, d\mathbf{x}dt \leq CL_0\delta. \tag{42}$$

Therefore, given $\gamma > 0$, for all times with an exception of a set of Lebesgue measure at most $\gamma\tau$, we have

$$\sum_{cells} |\phi_+ - \phi_-|^2 \leq C\delta L_0\gamma^{-1}\tau^{-1}. \tag{43}$$

Proof. The proof is straightforward and has already appeared in [20]. We sketch it here for the sake of completeness. Clearly,

$$\left|\int_{-h_0}^{h_0} \frac{\partial \phi}{\partial h}(h, \theta)\, dh\right| \geq |\phi_+ - \phi_-|,$$

for any $\theta_\pm = \theta$. Since in the region $D_\pm$ we have $|\nabla h|, |\nabla \theta_\pm| \sim 1$, integrating in the curvilinear coordinates we obtain

$$\int_D |\nabla\phi|^2 d\mathbf{x} \geq \int_{-y}^{y}\int_{-h_0}^{h_0} \left|\frac{\partial \phi}{\partial h}\right|^2\, d\theta dh \geq C\delta^{-1}l|\phi_+ - \phi_-|^2,$$

finishing the proof. $\Box$

Lemmas 3.4 and 3.5 allow to control how much the temperature changes from cell to cell. Our next lemma summarizes this in a way that will prove useful.

Lemma 3.6 *Assume that at a certain time t, the estimates (40) and (43) hold. Suppose that there exists a cell $\mathcal{C}$ such that $\phi(h_0, \theta) \geq \beta > 0$ at some point $(h_0, \theta) \in \mathcal{C}$. Then for at least N cells, we have $\phi(h_0, \theta) \geq \beta/2$ for any θ inside these cells, where the number N can be estimated from below as follows:*

$$N \geq C\beta^2\gamma\tau\left(\delta L_0 + \delta^{-1}A^{-2}L_0\delta_0^{-3}l\ln(l/\delta)\tau\ln(l/\delta_0) + \delta^{-1}A^{-1}L_0 l\ln(l/\delta)\right)^{-1}. \tag{44}$$

Proof Consider a cell $\mathcal{C}_1$ which is the closest to $\mathcal{C}$ and such that there exists a point $(h_0, \theta) \in \mathcal{C}_1$ with $\phi(h_0, \theta) < \beta/2$. Then we must have

$$\sum_{cells} |\phi_+ - \phi_-| + \sum_{cells} \mathrm{sup}_{\theta_1,\theta_2} |\phi(h_0, \theta_1) - \phi(h_0, \theta_2)| > \beta/2, \tag{45}$$

where the sum is over all cells between $\mathcal{C}$ and $\mathcal{C}_1$. On the other hand, Lemmas 3.4 and 3.5 imply that

$$\begin{aligned} &\sum_{cells} |\phi_+ - \phi_-|^2 + \sum_{cells} \mathrm{sup}_{\theta_1,\theta_2} |\phi(h_0, \theta_1) - \phi(h_0, \theta_2)|^2 \\ &\leq C\gamma^{-1}\tau^{-1}(\delta L_0 + \delta^{-1}A^{-1}L_0 l \ln(l/\delta)(A^{-1}\delta_0^{-3}\tau \ln(l/\delta_0) + 1)). \end{aligned} \tag{46}$$

Since $N\sum_{m=1}^N a_n^2 \geq \left(\sum_{m=1}^M a_n\right)^2$, a combination of (45) and (46) yields (44). $\square$

Now we can explain the strategy of the proof of Theorem 1.2 in more detail. Assume that at a certain "good" (not exceptional in the sense of Lemmas 3.4, 3.5) time we have a sufficiently high temperature $\phi = \beta$ in some cell on a streamline $h = h_0$. Then an appropriate choice of δ (hence h_0) and an application of Lemma 3.6 ensure that $\phi \geq \beta/2$ in many cells. Assume that for a sufficiently large portion of times $\leq \tau$, the temperature at $h = h_0$ streamlines is high in many cells. Clearly, the interiors of the cells will heat up too, giving (for a suitable choice of δ ensuring N is large enough) a contradiction with the preservation of the L^1 norm of ϕ. On the other hand, if the temperature on the streamline $h = h_0$ is low most of the time, we expect the solution inside the cells to be small and quenching to happen if the cells are sufficiently small – the last condition ensures that the interaction between the boundary and the interior of the cell happens on a short time scale. To make the above plan work, we need a good understanding of an auxiliary "cell heating" problem. The next section is devoted to this goal.

3.4 An auxiliary cell heating problem

Consider a domain Ω which is a sub-domain of a cell, defined by the condition $|h| \geq |h_0|$, that is, Ω is the *interior* of the closed streamline $\{h = h_0\}$. We consider the following initial-boundary value problem on Ω :

$$\begin{aligned} w_t + Au\cdot\nabla w - \Delta w &= 0 \\ w(t, \mathbf{x}) &= \sigma(t), \;\; \mathbf{x} \in \partial\Omega \\ w(\mathbf{x}, 0) &= g(\mathbf{x}). \end{aligned} \tag{47}$$

Here $\sigma(t)$ is some (smooth) function of time, independent of $\mathbf{x}$. Let us derive a compact formula for the solution of (47). Set $v = w - \sigma(t)$. Then v satisfies the zero Dirichlet boundary conditions, and we have

$$v_t + Au\cdot\nabla v - \Delta v = -\sigma'(t), \;\; v(\mathbf{x}, 0) = g(\mathbf{x}) - \sigma(0).$$

Let us denote by $\mathcal{H}$ the differential operator $-\Delta + Au\cdot\nabla$ with the zero Dirichlet boundary conditions on $\partial\Omega$ and by $e^{-t\mathcal{H}}$ the positive semigroup generated by $\mathcal{H}$. Using the Duhamel formula, we get

$$v(t, \mathbf{x}) = e^{-t\mathcal{H}}(g - \sigma(0)\mathbf{1}) - \int_0^t e^{-(t-s)\mathcal{H}}\mathbf{1}\sigma'(s)\,ds. \tag{48}$$

Here the semigroup is applied to a function equal identically to one in all of Ω, denoted by $\mathbf{1}$. Integrating by parts in (48), and recalling that $w = v + \sigma(t)$, we obtain

$$w(t, \mathbf{x}) = e^{-t\mathcal{H}} g(\mathbf{x}) + \int_0^t \sigma(s) \mathcal{H} e^{-(t-s)\mathcal{H}} \mathbf{1}\, ds. \tag{49}$$

It will be convenient to use probabilistic interpretation of (49). Recall that we can represent the semigroup action in the following way (see, e.g. [12]):

$$e^{-t\mathcal{H}} f(\mathbf{x}) = \mathbb{E}^{\mathbf{x}} \left[f(X_t^{\mathbf{x}}(\omega)) \right],$$

where $X_t^{\mathbf{x}}$ is the diffusion process corresponding to $\mathcal{H}$ starting at $\mathbf{x}$, and the expectation $\mathbb{E}^{\mathbf{x}}$ is taken over all paths that start at $\mathbf{x}$ and never leave Ω up to time t. The latter restriction corresponds to the zero Dirichlet boundary condition on $\partial\Omega$. Note that in the case $f \equiv 1$, the function $Q(t, \mathbf{x}) = e^{-t\mathcal{H}} \mathbf{1}$ coincides with probability that the stochastic process $X_t^{\mathbf{x}}(\omega)$ never leaves Ω before time t :

$$e^{-t\mathcal{H}} \mathbf{1} = \mathbb{E}^{\mathbf{x}} \left[\mathbf{1}(X_t^{\mathbf{x}}(\omega)) \right] = \mathbb{P}(X_s^{\mathbf{x}}(\omega) \in \Omega, \ \ \forall\, s \le t).$$

In terms of the function $Q(t, \mathbf{x})$ expression (49) becomes

$$w(t, \mathbf{x}) = e^{-t\mathcal{H}} g(\mathbf{x}) - \int_0^t \sigma(s) \partial_t Q(t - s, \mathbf{x})\, ds \tag{50}$$

In order to control the solution of an auxiliary cell problem, we need to control the properties of $Q(t, \mathbf{x})$, uniformly in the flow amplitude A – this is akin to Lemma 3.1. The first result we need in this direction is the following lemma, showing that the exit probability is bounded from above by a constant independent of A.

Lemma 3.7 *For any t satisfying $0 \le t \le C_1 l^2$ we have*

$$\int_\Omega Q(t, \mathbf{x})\, dx \ge C_2 l^2 > 0, \tag{51}$$

where C_2 is a universal constant depending only on C_1 but not on A or l.

Proof. Let us rewrite the generator $\mathcal{H}$ in the natural coordinates h, θ :

$$\mathcal{H} = |\nabla\theta|^2 \frac{\partial^2}{\partial\theta^2} + |\nabla h|^2 \frac{\partial^2}{\partial h^2} + \Delta h \frac{\partial}{\partial h} + (\Delta\theta - A|\nabla\theta||\nabla h|) \frac{\partial}{\partial\theta}.$$

Note that $\Delta h = -2l^{-2} h$ – this property is by no means crucial for our analysis but it does simplify some computations. The diffusion process $X_t^{\mathbf{x}}$ corresponding to $\mathcal{H}$, written in the (h, θ)-coordinates, is given by

$$\begin{aligned} dX_t^h &= \sqrt{2} |\nabla h| dB_t^{(1)} - 2l^{-2} X_t^h dt \\ dX_t^\theta &= \sqrt{2} |\nabla\theta| dB_t^{(2)} + (\Delta\theta - A|\nabla h||\nabla\theta|) dt, \end{aligned} \tag{52}$$

where the values of all functions are taken at a point (X_t^h, X_t^θ), while $B^{(1)}$ and $B_t^{(2)}$ are independent one-dimensional Brownian motions. Clearly, $Q(t,\mathbf{x}) \geq P(X_s^{h(\mathbf{x})} \in [0,l], \ \forall\, s \leq t)$. It is not difficult to see that the exit probabilities of X_t^h are majorized by the exit probabilities of the Ornstein-Uhlenbeck process where the factor $|\nabla h|$ in (52) is dropped. Indeed, let us introduce

$$\alpha(t,\omega) = \sqrt{2}|\nabla h(X_t^h(\omega), X_t^\theta(\omega))|.$$

Multiplying (52) with $e^{2l^{-2}t}$ and integrating leads to

$$e^{2l^{-2}t} X_t^{h(\mathbf{x})} = h(\mathbf{x}) + \int_0^t e^{2l^{-2}s} \alpha(s,\omega) dB_s^{(1)}.$$

Making a random time change (see, e.g., [29]), we find that the integral above has the same distribution as the Brownian motion $B^0_{\rho(t,\omega)}$, where

$$\rho(t,\omega) = \int_0^t e^{4l^{-2}s} |\alpha(s,\omega)|^2\, ds.$$

Since $|\alpha(s,\omega)|^2 \leq 2$, we have

$$\rho(t,\omega) \leq \rho(t) \equiv \frac{1}{2} l^2 (e^{4l^{-2}t} - 1),$$

and therefore

$$Q(t,\mathbf{x}) \geq P(B_{\rho(s)}^{h(\mathbf{x})} e^{-2l^{-2}s} \in [0,l], \ \forall\, s \leq t). \tag{53}$$

Let us remark that the expression on the right hand side is exactly the exit probability for the Ornstein-Uhlenbeck process mentioned above. The claim of the lemma now follows from a simple rescaling $\tau = t/l^2$, $Y_\tau = l^{-1} X_{l^2\tau}$. $\Box$

Next, we need some information on the behavior of $\partial_t Q(t,\mathbf{x})$.

Lemma 3.8 *For any t satisfying $C_1 l^2 \geq t \geq 0$, we have*

$$\int_\Omega \partial_t Q(t,\mathbf{x}) d\mathbf{x} \leq -C_3,$$

with a constant $C_3 > 0$ that depends on C_1 but is independent of A or l. Moreover, $\int_\Omega \partial_t Q(t,\mathbf{x}) d\mathbf{x}$ is monotonically increasing in time.

Proof. According to the previous lemma,

$$\int_\Omega Q(t,\mathbf{x})\, d\mathbf{x} \equiv \|Q(t,\mathbf{x})\|_{L^1} \geq C_2 l^2 > 0$$

for any $t \leq C_1 l^2$. The Cauchy-Schwartz inequality then implies a lower bound for the L^2 norm: $\|Q(t,\mathbf{x})\|_{L^2} \geq C_2 l$. Recall that the function $Q(t,\mathbf{x})$ solves

$$\frac{\partial Q}{\partial t} = \mathcal{H} Q(t,\mathbf{x})$$

with the zero Dirichlet boundary conditions and the initial condition $Q(0,\mathbf{x}) = 1$. Then

$$\frac{1}{2}\partial_t \|Q\|_{L^2}^2 = -\|\nabla Q\|_{L^2}^2 \leq -l^{-2}\|Q\|_{L^2}^2 \tag{54}$$

by the Poincare inequality. Thus we have $\int QQ_t d\mathbf{x} \leq -C_2^2$ where C_2 is independent of A and l. It is clear that $0 \leq Q(t,\mathbf{x}) \leq 1$, and also that $\partial_t Q(t,\mathbf{x}) \leq 0$ since $Q(t,\mathbf{x})$ is the probability that the diffusion process starting at $\mathbf{x}$ does not exit Ω before time t. This implies the first statement of the lemma.

To prove the second statement, we use again the monotonic decay of $Q(t,\mathbf{x})$ in time. Fixing some time $t > 0$ and integrating by parts (using the fact that the flow u is tangent to $\partial\Omega$) we find that

$$\int_\Omega \partial_t Q(t,\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega} \frac{\partial Q}{\partial n}\, ds.$$

Since the function Q vanishes on the boundary and decays (in time) inside, we see that $\int_\Omega \partial_t Q\, d\mathbf{x}$ is monotonically increasing in time. □

Corollary 3.9 *The solution $w(t,\mathbf{x})$ of (47) with any nonnegative boundary and initial data satisfies*

$$\int_\Omega w(t,\mathbf{x}) d\mathbf{x} \geq C \int_0^t \sigma(s)\, ds, \tag{55}$$

where C is a positive constant depending on t but not on A. If $t \leq C_1 l^2$, the constant C can be chosen independent of l.

Proof. The corollary follows immediately from the previous lemma and the representation (50). □

Since we have no control over what happens at a small set of exceptional times, we need an estimate from above on how much the L^1 norm of the solution can change if the boundary data is close to one for a short time. The second part of Lemma 3.8 implies that it is sufficient to look at the situation when this hot period occurs at the end of the interval $[0,t]$. More precisely, if we replace $\sigma(t)$ by its monotonically increasing rearrangement then the L^1-norm of the solution increases. The following lemma will be useful in such scenario.

Lemma 3.10 *For a time t satisfying $0 < t < l^2$, we have*

$$\int_\Omega (1 - Q(t,\mathbf{x}))\, d\mathbf{x} \leq Cl^2 (tl^{-2})^{1/2} \ln\left(\frac{l^2}{t}\right). \tag{56}$$

Proof. Note that

$$\int_\Omega (1 - Q(t,\mathbf{x}))\, d\mathbf{x} = \int_\Omega \mathbb{P}(\exists\ s < t:\ \ X_s^{\mathbf{x}}(\omega) \notin \Omega)\, d\mathbf{x}.$$

It follows from the proof of Lemma 3.7 that the probability on the right hand side is majorized by

$$\mathbb{P}(\exists\ s < t : B_{\rho(s)}^{h(\mathbf{x})} e^{-2l^{-2}s} \notin [0,l]) \leq \mathbb{P}(\exists\ s < t : B_{\rho(s)}^{h(\mathbf{x})} \notin [0,l]), \tag{57}$$

where $\rho(t) = \frac{1}{2}l^2(e^{4l^{-2}t} - 1)$. Take a constant C so that $\rho(t) \le Ct$ for any t as in the statement of the lemma. Let $\tilde{t} = tl^{-2}$ be the rescaled time. Then the probability on the right hand side of (57) does not exceed the probability that B_s^0 leaves the interval $C^{-1/2}[-h/l, 1-h/l]$ before the time $\tilde{t}$. Using the reflection principle, we can estimate this probability from above by one, if h/l or $1-h/l$ are less than $\sqrt{\tilde{t}}$, and by $C_n(\tilde{t}l/h)^n$ otherwise (where $n > 0$ is arbitrary). Integrating and taking into account the Jacobian of transformation between h, θ and $\mathbf{x}$, we arrive at

$$\int_\Omega (1 - Q(t,\mathbf{x}))\,d\mathbf{x} \le Cl(\tilde{t}l^2)^{1/2}\ln\left(\frac{Cl}{(\tilde{t}l^2)^{1/2}}\right) = Clt^{1/2}\ln\left(\frac{Cl^2}{t^{1/2}}\right),$$

which is nothing but (56). □

3.5 Proof of Theorem 1.2

Now we are ready to complete the proof of Theorem 1.2. In this proof, we will work with the time scale $\tau = C_0 l^2$ for a sufficiently large universal constant C_0. It means that in all estimates of the previous sections that we are going to use, we take $\tau = C_0 l^2$. This ensures, in particular, applicability of Lemma 3.8 and Corollary 3.9. If we show that $\phi(t,\mathbf{x}) \le \theta_0'/2 = \theta_0/(2e)$ for some $t \le \tau$, this will be sufficient for quenching since we assume that $\tau \le \tau_c = M^{-1}$ (small cell size).

Proof. We have a set S_γ of exceptional times of size at most $\gamma\tau$ such that on the complement S_γ^c of S_γ the estimates (40), (43) are valid. The value of the constant γ will be chosen later and will be independent of A and l. Assume first that for all $t \in S_\gamma^c$ except for a set S_b (of "bad" times) of size at most $\gamma\tau$ we have $\phi(h_0,\theta) \le \gamma$ for all cells and all θ. We are going to show that if γ is chosen sufficiently small (with the choice being uniform in A, l), then we must have quenching. Let Ω_n be the subset of a cell $\mathcal{C}_n$ enclosed by $|h| = |h_0|$. Then for any $\mathbf{x} \in \Omega_n$ we have $\phi(t,\mathbf{x}) \le \tilde{\phi}(t,\mathbf{x})$ where $\tilde{\phi}(t,\mathbf{x})$ satisfies

$$\tilde{\phi}_t - \Delta\tilde{\phi} + Au\cdot\nabla\tilde{\phi} = 0$$

with the initial data $\tilde{\phi}(\mathbf{x},0) = 1$ in Ω_n, and the boundary data given by

$$\tilde{\phi}(s,\mathbf{x})|_{h=h_0} = \begin{cases} 1, & s \in S_\gamma \cup S_b; \\ \gamma, & \text{otherwise} \end{cases}.$$

This bound on ϕ follows from the representation formula (50) and the above assumption on the behavior of $\phi(h_0,\theta)$. By linearity, inside each region Ω_n, we have $\tilde{\phi}(t,\mathbf{x}) = \tilde{\phi}_1(t,\mathbf{x}) + \tilde{\phi}_2(t,\mathbf{x})$, where $\tilde{\phi}_1$ satisfies the zero Dirichlet boundary conditions, while $\tilde{\phi}_2$ has zero initial data. Let $t_1 = C_1 l^2$, where C_1 is a universal constant which will be chosen below. A simple argument using (54) shows that

$$l^{-2}\|\tilde{\phi}_1(t_1,\cdot)\|_{L^1(\Omega_n)} \le l^{-1}\|\tilde{\Phi}_1(t_1,\cdot)\|_{L^2(\Omega_n)} \to 0 \tag{58}$$

uniformly in A, l as $C_1 \to \infty$.

The function $\tilde{\phi}_2$ can be estimated in the L^1-norm using (cf. (50))

$$\tilde{\phi}_2(t,\mathbf{x}) = -\int_0^t \sigma(s)\partial_t Q(t-s,\mathbf{x})ds$$

and the second statement in Lemma 3.8 as

$$\begin{aligned}
\|\tilde{\phi}_2(t_1,\cdot)\|_{L^1(\Omega_n)} &\le -\int_0^{t_1} \sigma(s) \int_{\Omega_n} \partial_{t_1} Q(t_1-s,\mathbf{x}) d\mathbf{x} ds \\
&\le -\int_{t_1-|S_\gamma\cup S_b|}^{t_1} \int_{\Omega_n} \partial_{t_1} Q(t_1-s,\mathbf{x}) d\mathbf{x} ds - \gamma \int_0^{t_1-|S_\gamma\cup S_b|} \int_{\Omega_n} \partial_t Q(t_1-s,\mathbf{x}) d\mathbf{x} ds \\
&\le \int_{\Omega_n} \left[1 - Q(|S_\gamma \cup S_b|,\mathbf{x})\right] d\mathbf{x} + \gamma \int_{\Omega_n} \left[1-Q(t_1,\mathbf{x})\right] d\mathbf{x} \\
&\le Cl^2 \left[(C_1\gamma)^{1/2} \ln\left(\frac{1}{C_1\gamma}\right) + C_1\gamma \right] \qquad (59)
\end{aligned}$$

by Lemma 3.10.

It follows now from the parabolic maximum principle that for all times $t \ge t_1$, the solution $\phi(t,\mathbf{x})$ of the original linear advection-diffusion problem (18) in any cell $\mathcal{C}_n$ satisfies

$$\phi(t,\mathbf{x}) \le \phi^*(t,\mathbf{x}) + \phi_2(t,\mathbf{x}).$$

The function ϕ_2 has been estimated in (31) and (33). The function $\phi^*(t,\mathbf{x})$ solves (47) for $t \ge t_1$ with the $2\pi l$-periodic boundary conditions in x, y and at time $t = t_1 = C_1 l^2$ is given by

$$\phi^*(t_1,\mathbf{x}) = \begin{cases} \tilde{\phi}(t_1,\mathbf{x}), & \mathbf{x} \in \Omega_n \\ 1 & \mathbf{x} \in \mathcal{C}_n \setminus \Omega_n. \end{cases}$$

Lemma 3.2 allows us to choose a positive number $\tilde{\gamma}$ so that if

$$\|\phi^*(t_1,\mathbf{x})\|_{L^1(\mathcal{C}_n)} \le \tilde{\gamma} l^2,$$

then

$$\|\phi^*(t_1+l^2,\mathbf{x})\|_{L^\infty(\mathcal{C}_n)} \le Cn^2(l^2)\|\phi^*(t_1,\mathbf{x})\|_{L^1(\mathcal{C}_n)} \le \theta_0'/4.$$

This is possible, as $n^2(l^2) \sim l^{-2}$ (see Remark after Lemma 3.1). Note that

$$\|\phi^*(t_1,\mathbf{x})\|_{L^1(\mathcal{C}_n)} \le \|\tilde{\phi}_1(t_1,\mathbf{x})\|_{L^1(\Omega_n)} + \|\tilde{\phi}_2(t_1,\mathbf{x})\|_{L^1(\Omega_n)} + C\delta l \ln(l/\delta).$$

Choosing C_1 large enough we can use (58) to estimate the first term on the right hand side and make sure it does not exceed $\tilde{\gamma} l^2/3$. Next we choose γ small enough so that (59) gives similar control of the second term. Finally, choose δ small enough so that the last term is also sufficiently small. With this choice of C_1, γ and δ (uniform in A, l) we have

$$\|\phi(t_1+l^2,\mathbf{x})\|_{L^\infty} \le \|\phi^*(t_1+l^2,\mathbf{x})\|_{L^\infty(\mathcal{C}_n)} + \theta_0'/10 \le \theta_0'/2$$

and thus quenching.

It remains to consider the case when there exists a set of bad times $S_b \in [0,t_1]$ of size at least $\gamma\tau$ (with γ universal constant determined above) such that for any $t \in S_b$, there exists a cell $\mathcal{C}_n$ such that for some point $(h_0,\theta) \in \mathcal{C}_n$, we have $\phi(h_0,\theta) \ge \gamma$. We are going to show that this cannot be true if A is large enough, thus forcing the scenario which is considered above and leads to quenching. Indeed, Lemma 3.6 implies that there are at least

$$N = C\gamma^3\tau \left(\delta L_0 + \delta^{-1}A^{-2}L_0\delta_0^{-3}\tau l \ln(l/\delta)\ln(l/\delta_0) + \delta^{-1}A^{-1}L_0 l \ln(l/\delta)\right)^{-1} \qquad (60)$$

cells such that $\phi(h_0,\theta) > \gamma/2$ for any θ in these cells. For each cell, let $\sigma(t) = \min_\theta \phi(h_0,\theta,t)$. On each cell, solve the initial-boundary value problem (47) with $g = 0$, and denote the solution $\bar{\phi}(t,\mathbf{x})$. By the parabolic maximum principle, we have that $\phi(t,\mathbf{x}) \geq \bar{\phi}(t,\mathbf{x})$ for $|h| \geq |h_0|$. Applying Corollary 3.9, we obtain

$$\int_D \phi(\tau,\mathbf{x})\,d\mathbf{x} \geq C \sum_{cells} \int_0^\tau \sigma(s)\,ds = C\int_0^\tau \sum_{cells} \sigma(s)\,ds \geq C\gamma^2 \tau N, \tag{61}$$

where N is given by (60). We claim that if $\delta = \delta(L_0)$ is chosen to be sufficiently small and $A = A(L_0)$ sufficiently large then (61) leads to

$$\int_D \phi(\tau,\mathbf{x})\,d\mathbf{x} \gg lL_0, \tag{62}$$

obtaining a contradiction since the L^1 norm of ϕ is preserved by evolution. Indeed, since γ and τ/l^2 are fixed constants, it suffices to choose δ so that

$$\delta \ll \frac{\tau^2}{L_0^2 l}, \tag{63}$$

and then choose A so that

$$A \gg \max\left\{ \frac{C(l)L_0}{\delta_0^{3/2}\delta^{1/2}} \left(\ln\frac{l}{\delta_0}\ln\frac{l}{\delta}\right)^{1/2}, \frac{C(l)L_0^2}{\delta}\ln\left(\frac{l}{\delta}\right)\right\}. \tag{64}$$

Recalling the formula (32) for δ_0, we discover that $A > C(l)L_0^4 \ln L_0$ is sufficient to satisfy (64), (63), completing the proof of Theorem 1.2. $\square$

References

[1] M. Abel, A. Celani, D. Vergni and A. Vulpiani, Front propagation in laminar flows, Physical Review E, **64**, 2001, 046307.

[2] M. Abel, M. Cencini, D. Vergni and A. Vulpiani, Front speed enhancement in cellular flows, Chaos, **12**, 2003, 481.

[3] B. Audoly, H. Berestycki and Y. Pomeau, Réaction-diffusion en écoulement stationnaire rapide, C. R. Acad. Sci. Paris, **328 II**, 2000, 255–262.

[4] H. Berestycki, The influence of advection on the propagation of fronts in reaction-diffusion equations, in *Nonlinear PDEs in Condensed Matter and Reactive Flows*, NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[5] H. Berestycki and F. Hamel, Front propagation in periodic excitable media, Comm. Pure Appl. Math., **55**, 2002, 949–1032.

[6] H. Berestycki, B. Larrouturou and P.-L. Lions, Multi-dimensional traveling wave solutions of a flame propagation model, Arch. Rational Mech. Anal., **111**, 1990, 33–49.

[7] H. Berestycki and L. Nirenberg, Traveling fronts in cylinders, Annales de l'IHP, Analyse non linéare, **9**, 1992, 497–572.

[8] S. Childress, Alpha-effect in flux ropes and sheets, Phys. Earth Planet Inter., **20**, 1979, 172-180.

[9] P. Constantin, A. Kiselev, A. Oberman and L. Ryzhik, Bulk burning rate in passive-reactive diffusion, Arch. Rat. Mech. Anal., **154**, 2000, 53–91.

[10] P. Constantin, A. Kiselev and L. Ryzhik, Quenching of flames by fluid advection, Comm. Pure Appl. Math **54**, 2001, 1320–1342.

[11] A. Fannjiang and G. Papanicolau, Convection enhanced diffusion for periodic flows, SIAM Jour. Appl. Math., **54**, 1994, 333–408.

[12] M. Freidlin, Functional Integration and Partial Differential Equations, Princeton University Press, Princeton 1985.

[13] M. Freidlin, Reaction-diffusion in incompressible fluid: asymptotic problems, J. Diff. Eq., **179** (2002), 44-96

[14] M. Freidlin and A. Wentzell, *Random perturbations of Hamiltonian systems*, Memoir AMS, 1994.

[15] M. Freidlin and A. Wentzell, Random Perturbations of Dynamical Systems, 2nd ed., Springer-Verlag, New York/Berlin, 1998.

[16] Ya. Kanel, Stabilization of solutions of the Cauchy problem for equations encountered in combustion theory, Mat. Sbornik, **59**, 1962, 245–288.

[17] L. Kagan and G. Sivashinsky, Flame propagation and extinction in large-scale vortical flows, Combust. Flame, **120**, 2000, 222–232.

[18] L. Kagan, P.D. Ronney and G. Sivashinsky, Activation energy effect on flame propagation in large-scale vortical flows, Combust. Theory Modelling **6**, 2002, 479–485.

[19] B. Khoudier, A. Bourlioux and A. Majda, Parametrizing the burning rate speed enhancement by small scale periodic flows: I. Unsteady shears, flame residence time and bending," Combust. Theory Model., **5**, 2001, 295–318.

[20] A. Kiselev and L. Ryzhik, Enhancement of the traveling front speeds in reaction-diffusion equations with advection, Ann. de l'Inst. Henri Poincaré, C. Analyse non linéaire, **18**, 2001, 309–358.

[21] A. Kiselev and A. Zlatos, Quenching of combustion by shear flows, Preprint, 2004.

[22] I. Kiss, J. Merkin and Z. Neufeld, Combustion initiation and extinction in a 2D chaotic flow, Phys. D., **183**, 2003, 175–189.

[23] I. Kiss, J. Merkin, S. Scott, P. Simon, S. Kalliadasis, and Z. Neufeld, The structure of flame filaments in chaotic flows, Phys. D., **176**, 2003, 67–81.

[24] L. Koralov, Random perturbations of two-dimensional Hamiltonian flows, Prob. Theor. Rel. Fields, **129**, 37–62, 2004.

[25] A. Majda and P. Kramer, Simplified models for turbulent diffusion: theory, numerical modelling, and physical phenomena. Phys. Rep., **314**, 1999, 237–574.

[26] J.-F. Mallordy and J.-M. Roquejoffre, A parabolic equation of the KPP type in higher dimensions, SIAM J. Math. Anal., **26**, 1995, 1-20.

[27] J. Nash, Continuity of solutions of parabolic and elliptic equations, Amer. Jour. Math., **80**, 1958, 931–954.

[28] A. Novikov, G. Papanicolaou and L. Ryzhik, Boundary layers for cellular flows at high Péclet numbers, to appear in Comm. Pure Appl. Math., 2004.

[29] B. Oksendal, *Stochastic Differential Equations,* Springer-Verlag, Berlin, 1995.

[30] N. Peters, *Turbulent combustion*, Cambridge University Press, 2000.

[31] Roquejoffre, J.-M., Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders, Annal. Inst. Poincare, Analyse Nonlinéare, **14**, 1997, 499-552.

[32] N. Vladimirova, P. Constantin, A. Kiselev, O. Ruchayskiy and L. Ryzhik, Flame Enhancement and Quenching in Fluid Flows, Combust. Theory Model., **7**, 2003, 487–508

[33] V.A. Volpert and A.I. Volpert, Existence and stability of multidimensional travelling waves in the monostable case, Israel Jour. Math., **110**, 1999, 269–292.

[34] J. Xin, Existence of planar flame fronts in convective-diffusive periodic media, Arch. Rat. Mech. Anal., **121**, 1992, 205–233.

[35] J. Xin, Existence and nonexistence of travelling waves and reaction-diffusion front propagation in periodic media, Jour. Stat. Phys., **73**, 1993, 893–926.

[36] J. Xin, Analysis and modelling of front propagation in heterogeneous media, SIAM Rev., **42**, 2000, 161–230.

[37] J. Xin, KPP front speeds in random shears and the parabolic Anderson model, Meth. Appl. Anal., **10**, 2003, 191–198.