LAGRANGIAN DYNAMICS FOR A PASSIVE TRACER IN A CLASS OF GAUSSIAN MARKOVIAN FLOWS

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ABSTRACT. We formulate a stochastic differential equation describing the Lagrangian environment process of a passive tracer in Ornstein–Uhlenbeck velocity fields. We prove local existence and uniqueness when the velocity field is regular. When the Ornstein–Uhlenbeck velocity field is only spatially Hölder continuous, we construct and identify the probability law for the Lagranging process under a condition on the time correlation function and the Hölder exponent.

1. INTRODUCTION.

A major problem in fluid turbulence is the transport of passive scalar whose displacement $\boldsymbol{x}$ in a d-dimensional flow V satisfies the ordinary differential equation

$$d\boldsymbol{x}(t)/dt = V(t, \boldsymbol{x}(t)), \quad \boldsymbol{x}(0) = \mathbf{0}. \tag{1.1}$$

When the velocity field V is temporally and spatially continuous, Peano's theorem guarantees the existence of solutions of Eq. (1.1) and when V is also spatially Lipschitz, the solution is unique. If V is not spatially Lipschitz, then the equation typically has many trajectories for each given initial point $\boldsymbol{x}(0)$.

The inertial-scale (see below) turbulent velocity fluctuation is usually modelled by an irregular and time-space stationary random field. The correlation functions $R = [R_{ij}]$ of the stationary process V naturally has the spectral representation

$$R(t, \mathbf{x}) := \mathbb{E}\left[V(t, \mathbf{x}) \otimes V(0, \mathbf{0})\right] = \int_{\mathbb{R}^d} \cos(\mathbf{k} \cdot \mathbf{x}) r(t, \mathbf{k}) \frac{\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\, d\mathbf{k}, \tag{1.2}$$

where r is the time correlation function, $\mathcal{E}$ the matrix-valued spectral (shell) density and $\mathbb{E}$ denotes the average w.r.t. the ensemble of the velocity field. According to Kolmogorov's similarity hypothesis, an incompressible turbulent velocity has the following power-law spectral density

$$\mathcal{E}(\mathbf{k}) = c_0 |\mathbf{k}|^{-5/3} (\mathbf{I} - \mathbf{k} \otimes \mathbf{k}/|\mathbf{k}|^2), \quad \ell_0^{-1} < |\mathbf{k}| < \ell_K^{-1} \tag{1.3}$$

in the so called inertial range of scales (ℓ_K, ℓ_0). Here ℓ_0 is commonly called the integral scale, ℓ_K the Kolmogorov dissipation scale and c_0 the Kolmogorov constant. Eq. (1.3) is called Kolmogorov's 5/3-law. In the large Reynolds number limit Re $\gg 1$, the dissipation length

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$\ell_K \asymp \ell_0 \mathrm{Re}^{-3/4}$ tends to zero while the integral length ℓ_0 stays fixed. If V is further assumed to be Gaussian, as the first approximation, then in the large Reynolds number limit Eq. (1.3) gives rise to a Hölder continuous velocity field with exponent less than $1/3$.

The question then is how to make sense of the trajectories of the passive scalar in turbulent velocities. Our approach is based on the notion of the *Lagrangian environment process*, which roughly speaking describes the medium from the point of view of the moving particle and is defined formally as

$$\eta(t,\cdot) := V(t, \boldsymbol{x}(t)+\cdot), \quad t \geq 0.$$

The particle trajectory is then recovered as an additive functional of η;

$$\boldsymbol{x}(t) = \int_0^t \eta(s,\mathbf{0})\, ds, \qquad t \geq 0. \tag{1.4}$$

The representation of the trajectory in the form given by (1.4) is quite useful in the study of long time behavior of the particle displacement (see [14, 16]) and was found particularly suitable for the case of transport in incompressible Markovian flows, see [9]. We will not, however, be restricted to divergence-free velocity fields, so in the present article the spectral density $\mathcal{E}$ is a general positive definite symmetric real $d \times d$ matrix satisfying $\mathcal{E}(-\mathbf{k}) = \mathcal{E}(\mathbf{k})$. As we will see in Section 2.2, the Eulerian velocity process V can be thought of as a stationary infinite dimensional Ornstein–Uhlenbeck process in an appropriate Hilbert space and described by the stationary solution of an infinite dimensional linear stochastic differential equation (SDE)

$$dV(t) = -AV(t)\, dt + B\, dW(t), \tag{1.5}$$

where A, B are linear operators related to the spectral density $\mathcal{E}$ and the time correlation function r, and W is a cylindrical Wiener process (see Section 2.2), and the law $V(0)$ is a stationary measure for (1.5). We show that for regular velocity V, see (2.2), the Lagrangian process η satisfies a nonlinear SDE

$$d\eta(t) = \left(-A\eta(t) + \eta(t,\mathbf{0}) \cdot \nabla_{\mathbf{x}}\eta(t)\right)\, dt + B d\tilde{W}(t), \tag{1.6}$$

for a suitable initial condition and the cylindrical Wiener process $\tilde{W}$ obtained from W by a translation of $\boldsymbol{x}$ (see Theorem 1 for a precise statement).

One may therefore ask first whether the equation

$$d\,\eta_f(t) = \left(-A\eta_f(t) + \eta_f(t,\mathbf{0}) \cdot \nabla_{\mathbf{x}}\eta_f(t)\right)\, dt + B\, dW(t), \qquad \eta_f(0) = f, \tag{1.7}$$

with given a cylindrical Wiener process W and an initial condition f possesses a solution. For regular f and the velocity field the answer to this question is affirmative. We prove the local existence and uniqueness of strong solutions in an appropriate Hilbert space in Theorems 2 and 3. We also show that for almost sure initial condition in the support of the stationary measure for (1.5) the existence of solution can, in fact, be guaranteed globally (see Corollary 1).

In Section 6 we turn to irregular fields with the power-law type spectral density

$$|\mathcal{E}(\mathbf{k})| \sim |\mathbf{k}|^{1-2\alpha}, \quad 1 < \alpha < 2, \quad \text{for } |\mathbf{k}| \gg 1$$

and the power-law correlation time

$$r(t,\mathbf{k}) = c_1 \exp\{-c_1|\mathbf{k}|^{2\beta}t\}, \quad \beta > 0. \tag{1.8}$$

Without loss of generality we set $c_1 = 1$ in the analysis. These Ornstein–Uhlenbeck fields are spatially Hölder continuous with exponent $H < \alpha - 1$. The Kolmogorov spectrum (1.3) corresponds to $\alpha = 4/3$. Instead of the sample-wise construction we obtain the probability law of the Lagrangian process η by studying Eq. (1.7) via a limiting procedure: consider a sequence of velocity fields V_K, $K \geq 1$ obtained by truncation of the spectrum of V for large wave numbers i.e.

$$R_K(t,\mathbf{x}) := \mathbb{E}\left[V_K(t,\mathbf{x}) \otimes V_K(0,\mathbf{0})\right] = \int\limits_{\mathbb{R}^d} \cos(\mathbf{k}\cdot\mathbf{x}) \mathrm{e}^{-|\mathbf{x}|^{2\beta}t}\, \frac{\mathbf{1}_{[0,K]}(|\mathbf{k}|)\mathcal{E}(\mathbf{k})}{|\mathbf{k}|^{d-1}}\, d\mathbf{k}. \tag{1.9}$$

This truncation corresponds to replacing in (1.5) the diffusion operator B and the initial invariant measure μ by some other operator B_K and measure μ_K, respectively. Such fields V_K are smooth, in fact analytic, therefore we can define their corresponding Lagrangian processes η_K, $K \geq 1$ satisfying (1.6). The key observation is that the law of η_K on $C([0,+\infty); C(\mathbb{R}^d;\mathbb{R}^d))$ coincides with that of the solution $\overline{\eta}_K$ of the modified equation (1.7) with B_K in place of B. With the Wiener process W fixed it is possible to prove almost sure convergence of $\overline{\eta}_K$, as $K \uparrow +\infty$ when $\alpha + \beta > 2$ and $\alpha + 3\beta > 3$. Thus, we can conclude that the laws of η_K, $K \geq 1$ on $C([0,+\infty); C(\mathbb{R}^d;\mathbb{R}^d))$ converge weakly to a certain law, which then is naturally called the probability law of the Lagrangian environment process for the irregular velocity field V, see Theorem 4 of Section 6 for more details.

In the case of temporally white-noise, spatially Hölderian velocity fields the Lagrangian semigroup has been constructed in [15] in the presence of molecular diffusion (see also [7, 11]). Roughly speaking, white-noise-in-time corresponds to $\beta = 0$ and $c_1 \to \infty$ in (1.8).

2. Preliminaries.

2.1. **Functional spaces and operators.** For brevity we set $\mathcal{S}_d := \mathcal{S}(\mathbb{R}^d,\mathbb{R}^d)$. Given $\rho > 0$ and a positive integer m, we define $\vartheta_\rho(\mathbf{x}) = (1+|\mathbf{x}|^2)^{-\rho}$, $\mathbf{x} \in \mathbb{R}^d$, and $\mathbb{W}^{p,m}_\rho$ as the completion of $\mathcal{S}_d$ with respect to the norm

$$||\psi||^p_{\mathbb{W}^{p,m}_\rho} := \int\limits_{\mathbb{R}^d} (|\psi(\mathbf{x})|^p + |\nabla_{\mathbf{x}}\psi(\mathbf{x})|^p + \cdots + |\nabla^m_{\mathbf{x}}\psi(\mathbf{x})|^p)\vartheta_\rho(\mathbf{x})\, d\mathbf{x}.$$

For $p = 2$, we write $\mathbb{H}^m_\rho := \mathbb{W}^{2,m}_\rho$ or $\mathbb{L}^2_\rho$ depending on whether $m \neq 0$ or $m = 0$. We shall omit writing the subscript ρ in case when $\rho = 0$. Using Sobolev embedding, see e.g. [12] Theorem 7.10, p. 154, it can easily be observed that for $m > d/2$ there exists a constant $C > 0$ such that $|\psi(\mathbf{0})| \leq C\|\psi\|_{\mathbb{H}^m_\rho}$, $\psi \in \mathbb{H}^m_\rho$. Notice also that $J_{\rho,p} : \mathbb{W}^{p,m}_\rho \to \mathbb{W}^{p,m}$ given by $J_{\rho,p}\psi(\mathbf{x}) := \psi(\mathbf{x})\vartheta_{\rho/p}(\mathbf{x})$ is a linear isomorphism satisfying $J_{\rho,p}(\mathcal{S}_d) = \mathcal{S}_d$, and that in the special case $m = 0$, $p = 2$ $J_{\rho,2} : \mathbb{L}^2 \to \mathbb{L}^2$ is unitary. In the sequel we shall consider spaces $C([0,T];\mathbb{H}^m_\rho)$, $T > 0$ consisting of continuous functions $f : [0,T] \to \mathbb{H}^m_\rho$ equipped with the norm $\|f\|_{\mathbb{H}^m_\rho,T} := \sup\limits_{0\leq t\leq T} \|f(t)\|_{\mathbb{H}^m_\rho}$.

The following proposition is an immediate consequence of the Rellich–Kondrachev compact embedding theorem (cf. e.g. Theorem 7.22, p. 163 of [12]).

Proposition 1. *For any $m' > m$ and $\rho' < \rho$ the embedding $\mathbb{H}^{m'}_{\rho'} \subseteq \mathbb{H}^m_\rho$ is compact.*

Let

$$\mathcal{H}_\rho := \bigcap_{m\geq 1} \mathbb{H}^m_\rho, \quad \mathcal{C}_\rho := \bigcap_{p,m\geq 1} \mathbb{W}^{p,m}.$$

Obviously $\mathcal{H}_\rho$ and $\mathcal{C}_\rho$ are dense in any $\mathbb{H}^m_\rho$.

Let $S(d)$ denotes the space of all real, symmetric positive definite $d\times d$ matrices. Given $\beta \geq 0$ and $\mathcal{E} : \mathbb{R}^d \to S(d)$ we set

$$\widehat{B_\beta\psi}(\mathbf{k}) = \sqrt{2|\mathbf{k}|^{2\beta}\overline{\mathcal{E}}\,(\mathbf{k})}\hat{\psi}(\mathbf{k}) \qquad \text{and} \qquad \widehat{S_\beta(t)\psi}(\mathbf{k}) = \mathrm{e}^{-|\mathbf{k}|^{2\beta}t}\hat{\psi}(\mathbf{k}),\ \psi \in \mathcal{S}_d, \tag{2.1}$$

with $\overline{\mathcal{E}}(\mathbf{k}) := \mathcal{E}(\mathbf{k})|\mathbf{k}|^{1-d}$. Throughout Sections 2 and 5 we assume that

$$\int_{\mathbb{R}^d} \left(1+|\mathbf{k}|^2\right)^m \operatorname{Tr}\ \overline{\mathcal{E}}(\mathbf{k})\,d\mathbf{k} < \infty \qquad \text{for any } m\in\mathbb{N}. \tag{2.2}$$

In what follows $\|\cdot\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)}$ denotes the respective Hilbert–Schmidt operator norm, see [4]. Let $\gamma(\beta) = \beta$ if $\beta \notin \mathbb{Z}$ and $+\infty$ otherwise.

Proposition 2. *(i) B_β extends to a Hilbert–Schmidt operator $B_\beta : \mathbb{L}^2 \to \mathbb{H}^m_\rho$.*
Suppose that $\rho \in (d/2, d/2+\gamma(\beta))$. Then the following hold.

- *(ii) $S_\beta(t)$ extends to a bounded linear operator $S_\beta(t) : \mathbb{W}^{p,m}_\rho \to \mathbb{W}^{p,m}_\rho$. In addition $S_\beta(t)$, $t\geq 0$ form a C_0-semigroup on $\mathbb{W}^{p,m}_\rho$, for any $p\geq 1$, $m\geq 0$, $t\geq 0$. In what follows we denote by $-A_\beta$, where $A_\beta : D(A_\beta) \to \mathbb{H}^m_\rho$ the generator of the semigroup on $\mathbb{H}^m_\rho$, or $\mathbb{W}^{p,m}_\rho$.*
- *(iii) $\mathcal{C}_\rho$, $\mathcal{H}_\rho$, and $\mathcal{S}_d$ are cores of any A_β, $\beta\geq 0$. Moreover, $\widehat{A_\beta\psi}(\mathbf{k}) = |\mathbf{k}|^{2\beta}\hat{\psi}(\mathbf{k})$ when $\psi\in\mathcal{S}_d$, and $\mathbb{H}^{m+[2\beta]+1}_\rho \subseteq D(A_\beta)$.*
- *(iv) Suppose that $\beta,\beta' \geq 0$ and $t>0$ and $\rho \in (d/2, d/2+\gamma(\beta)\wedge\gamma(\beta'))$. Then for any $f\in\mathbb{H}^m_\rho$ we have $S_\beta(t)f \in D(A_{\beta'})$. Moreover, there exists C possibly depending on β,β',d but independent of t such that $\|A_{\beta'}\,S_\beta(t)\,B_\beta\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} \leq C$ and $\|A_{\beta'}\,S_\beta(t)\psi\|_{\mathbb{H}^m_\rho} \leq Ct^{-\beta'/\beta}\|\psi\|_{\mathbb{H}^m_\rho}$.*

Due to technical nature of the proposition we postpone its proof till the appendix. In what follows we shall suppress the index β of the operators when there is no danger of confusion.

Let $\mathbf{v}\in\mathbb{R}^d$. We define

$$R_{\mathbf{v}}(t)f(\mathbf{x}) := f(\mathbf{x}+t\mathbf{v}), \quad t\geq 0,\ f\in\mathbb{H}^m_\rho,\ x\in\mathbb{R}^d.$$

Then $R_{\mathbf{v}}$ forms a C_0-group of operators on $\mathbb{H}^m_\rho$.

Proposition 3. *$\|R_{\mathbf{v}}(t)f\|_{\mathbb{H}^m_\rho} \leq (1+|\mathbf{v}\,t|)^\rho\|f\|_{\mathbb{H}^m_\rho}$ for all $f\in\mathbb{H}^m_\rho$, $t\geq 0$, $v\in\mathbb{R}^d$.*

Proof. The conclusion of the proposition can be reached by a direct calculation with the help of change of variables and an elementary inequality

$$\frac{1}{1+|\mathbf{x}-\mathbf{v}\,t|^2} \leq \frac{(1+|\mathbf{v}\,t|)^2}{1+|\mathbf{x}|^2} \qquad \text{for all } t\in\mathbb{R},\ \mathbf{x},\mathbf{v}\in\mathbb{R}^d. \quad \blacksquare$$

Let

$$C_{\mathbf{v}}(t) := R_{\mathbf{v}}(t)S(t),\ t\geq 0. \tag{2.3}$$

The following proposition holds.

Proposition 4. *(i)* $[S(t), R_{\mathbf{v}}(s)] := S(t)R_{\mathbf{v}}(s) - R_{\mathbf{v}}(s)S(t) = 0$ *for all* $t,s\geq 0$.
(ii) $C_{\mathbf{v}}(t)$, $t\geq 0$ *is a* C_0*-semigroup of operators on* $\mathbb{H}^m_\rho$ *for any* $\mathbf{v}\in\mathbb{R}^d$.
(iii) $\sup\limits_{0\leq t\leq T}\|C_{\mathbf{v}}(t)B\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} < \infty$ *for all* $T>0$, $\mathbf{v}\in\mathbb{R}^d$, *and* $\rho\in(d/2, d/2+\gamma(\beta))$.

Proof. Notice that $\widehat{R_{\mathbf{v}}(s)\psi}(\mathbf{k}) = \mathrm{e}^{s\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{v}}\hat{\psi}(\mathbf{k})$ for any $\psi\in\mathcal{S}_d$, hence *(i)* holds trivially in light of Proposition 2. Part *(ii)* is a consequence of *(i)*. Part *(iii)* follows from the following computation

$$\sup_{0\leq t\leq T}\|C_{\mathbf{v}}(t)B\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} \leq \sup_{0\leq t\leq T}\|R_{\mathbf{v}}(t)\|_{L(\mathbb{H}^m_\rho)}\sup_{0\leq t\leq T}\|S(t)B\|_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} < \infty. \quad \blacksquare$$

Denote by $D_{\mathbf{v}}$ the generator of $C_{\mathbf{v}}(t)$, $t\geq 0$. A direct computation shows that

$$D_{\mathbf{v}}f = -Af + \mathbf{v}\cdot\nabla f, \qquad f\in\mathcal{C}_\rho.$$

Additionally, $\mathcal{C}_\rho$ is a core of $D_{\mathbf{v}}$. This follows from the fact that $\mathcal{C}_\rho$ is invariant both under $R_{\mathbf{v}}(t)$, $t\geq 0$ and $S(t)$, $t\geq 0$, thus it is also invariant under $C_{\mathbf{v}}(t)$, $t\geq 0$.

2.2. **The Ornstein–Uhlenbeck process.** Let $(\Omega,\mathcal{V},\mathbb{P})$ be a probability space. By $W(t) = \{W_j(t);\ j=1,\dots,d\}$, $t\geq 0$ we denote a cylindrical Wiener process on $\mathbb{L}^2 = L^2(\mathbb{R}^d;\mathbb{R}^d)$ over the given probability space, that is an $\mathcal{S}'_d := \mathcal{S}'(\mathbb{R}^d;\mathbb{R}^d)$-valued Gaussian process satisfying

(A.1) for every $\psi\in\mathcal{S}_d$, $\{\langle W(t),\psi\rangle\}_{t\in[0,\infty)}$ is a one dimensional Wiener process. Here $\langle\cdot\,,\cdot\rangle$ denotes the pairing between $\mathcal{S}'_d$ and $\mathcal{S}_d$ spaces.

(A.2) for all $\psi,\varphi\in\mathcal{S}_d$ and $t,s\in[0,\infty)$ one has

$$\mathbb{E}\,\langle W(t),\psi\rangle\langle W(s),\varphi\rangle = t\wedge s\sum_{l=1}^d\int_{\mathbb{R}^d}\psi_l(\mathbf{x})\cdot\varphi_l(\mathbf{x})\,d\mathbf{x}.$$

We shall denote by $\mathcal{W}_t$, $t\geq 0$ the natural filtration corresponding to W. Assume that $\rho\in(d/2, d/2+\gamma(\beta))$. By virtue of Proposition 2*(iv)* and the argument used in the proof of Proposition 4.15, p. 104 of [4] we conclude that for any $f\in\mathbb{H}^m_\rho$ there exists a unique $\mathbb{H}^m_\rho$-valued strong solution V of (1.5) satisfying $V(0)=f$, that is $V(t)\in D(A)$ for $t>0$ and

$$V(t) = f - \int_0^t AV(s)\,ds + BW(t), \quad t\geq 0.$$

We denote this solution by V_f. From Theorem 5.4 of [4] we have $V_f(t) = S(t)f + W_A(t)$, $t \geq 0$, with

$$W_A(t) := \int_0^t S(t-s)\, B\, dW(s).$$

When $m > d/2$ we can define a Gaussian random field $W_A(t,x) := W_A(t)(x)$ over $(\Omega, \mathcal{V}, \mathbb{P})$ whose co-variance matrix is given by

$$\begin{aligned} R(s,t,\mathbf{x}-\mathbf{y}) : &= \mathbb{E}[W_A(t,\mathbf{x}) \otimes W_A(s,\mathbf{y})] \\ &= \int_{\mathbb{R}^d} \cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}))\, \mathrm{e}^{-|\mathbf{k}|^{2\beta}|t-s|}[1-\mathrm{e}^{-2|\mathbf{k}|^{2\beta}(s\wedge t)}]\overline{\mathcal{E}}(\mathbf{k})\, d\mathbf{k}. \end{aligned}$$

Standard regularity results for Gaussian processes (see corollary to Theorem 3.4.1 of [1]) allow us to conclude that $W_A(t,\mathbf{x})$, $t > 0$, $\mathbf{x} \in \mathbb{R}^d$ is jointly continuous and, thanks to (2.2), C^∞ in $\mathbf{x}$ for a fixed t.

Let $T, \kappa > 0$ be arbitrary. The entropy number $N(\epsilon)$, $\epsilon > 0$ of $[0,T] \times \mathbb{R}^d$ with respect to the random field $Y(t,\mathbf{x}) := W_A(t,\mathbf{x})(1+|\mathbf{x}|)^{-\kappa}$ is the smallest number of balls (in the pseudo-metric $d(t,\mathbf{x};s,\mathbf{y}) := \left[\mathbb{E}\,|Y(t,\mathbf{x}) - Y(s,\mathbf{y})|^2\right]^{1/2}$ on $[0,T]\times\mathbb{R}^d$) with radius at most ϵ needed to cover $\mathbb{R}^d$. It can be seen that $N(\epsilon) \leq K\epsilon^{-d-1}$ for some constant $K > 0$ independent of $\epsilon > 0$. Hence by virtue of Theorem 6.9.2, p. 161 of [2],

$$-\log \mathbb{P}\left(\sup_{0\leq t\leq T, \mathbf{x}\in\mathbb{R}^d} |Y(t,\mathbf{x})| > u\right) \sim u^2/(2\sigma^2),$$

where

$$\sigma^2 := \sup_{0\leq t\leq T, \mathbf{x}\in\mathbb{R}^d} \mathbb{E}\,|Y(t,\mathbf{x})|^2,$$

and in consequence for any $T > 0$ there exists a random constant $C_T > 0$ such that

$$\sup_{0\leq t\leq T} |W_A(t,\mathbf{x})| \leq C_T(1+|\mathbf{x}|)^\kappa, \qquad \forall\, \mathbf{x}\in\mathbb{R}^d.$$

2.3. **Invariant measure.** In this section we assume that $\rho \in (d/2, d/2+\gamma(\beta))$. From Proposition 2 we conclude that

$$\int_0^\infty ||S(t)\, B||^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)}\, dt \leq C \int_{\mathbb{R}^d} \left(1+|\mathbf{k}'|^2\right)^m \,\mathrm{Tr}\,\overline{\mathcal{E}}(\mathbf{k}')\, d\mathbf{k}' < \infty.$$

Let μ be a Gaussian measure on $\mathbb{H}^m_\rho$ that is of zero mean and with the covariance given by $Q\,Q^*$, where

$$Q := \int_0^\infty S(t)\, B\, dt \in L_{(HS)}(\mathbb{L}^2, \mathbb{H}^m_\rho).$$

By Theorem 11.7, p. 308 of [4] it is invariant for (1.5).

$G(\mathbf{x}; f) := f(\mathbf{x})$, $f \in \mathbb{H}^m_\rho$, $\mathbf{x} \in \mathbb{R}^d$ is a Gaussian, homogeneous random field over the probability space $(\mu, \mathbb{H}^m_\rho, \mathcal{B}(\mathbb{H}^m_\rho))$ with the co-variance matrix

$$R_G(\mathbf{x}) := \int f(\mathbf{x}) \otimes f(\mathbf{0})\, \mu(df) = \int_{\mathbb{R}^d} \cos(\mathbf{x} \cdot \mathbf{k})\, \overline{\mathcal{E}}(\mathbf{k})\, d\mathbf{k}.$$

Proposition 5. $\mu(\mathcal{C}_\rho) = 1$.

Proof. With no loss of generality we can assume that $m > d/2$. Since R_G is C^∞ by virtue of (2.2) the realizations of the field are C^∞ μ-a.s.. The conclusion of the proposition follows from Theorem 3.1, p. 168 of [19] that implies

$$|f(\mathbf{x})| + \cdots + |\nabla^m_{\mathbf{x}} f(\mathbf{x})| \leq C(1 + |\log \mathbf{x}|)^{1/2}$$

and, in consequence, $\|f\|_{\mathbb{H}^m_\rho} < \infty$ for all $m \geq 1$ and μ-a.s. f. ∎

Note that $V := V_f(t)$, $t \geq 0$ and $f \in \mathbb{H}^m_\rho$ is a stationary solution of (1.5) over the probability space $(\Omega \times \mathbb{H}^m_\rho, \mathcal{W} \otimes \mathcal{B}(\mathbb{H}^m_\rho), \mathbb{P} \otimes \mu)$, that is adapted with respect to the filtration $\mathcal{V}_t := \mathcal{W}_t \otimes \mathcal{B}(\mathbb{H}^m_\rho)$, $t \geq 0$. For $m > d/2$ we can define a stationary random field $V(t, \mathbf{x}) := V(t)(\mathbf{x})$ with the co-variance matrix R given by (1.2).

3. SDE for the Lagrangian process.

Let us assume that $\rho \in (d/2, d/2 + \gamma(\beta))$. Suppose that $m > d/2 + 1$ so that $\mathbb{H}^m_\rho \subseteq C^1(\mathbb{R}^d, \mathbb{R}^d)$. We can define a d dimensional random field $V_f(t, \mathbf{x}) := V_f(t)(\mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$. Its mean and co-variance matrix equal correspondingly $F(t, \mathbf{x}) := S(t) f(\mathbf{x})$ and

$$R_f(t, s, \mathbf{x} - \mathbf{y}) = \int_{R^d} \cos((\mathbf{x} - \mathbf{y}) \cdot \mathbf{k})\, e^{-|\mathbf{k}|^{2\beta}|t-s|}[1 - e^{-|\mathbf{k}|^{2\beta}(t \wedge s)}]\, \overline{\mathcal{E}}(\mathbf{k})\; d\mathbf{k}.$$

Let $f \in \mathcal{C}_\rho$. Then $f \in \mathbb{W}^{p,m}_\rho$ for arbitrary p, m. In particular when $p > 2\rho > d$ one can find $C_f > 0$ such that $|f(\mathbf{x})| \leq C_f(1 + |\mathbf{x}|^2)^{\rho/p}$ and in consequence the ordinary differential equation (1.1) with V_f standing in place of V on its right hand side has a unique global solution $\boldsymbol{x}_f$. Let us set $\eta_f(t, \cdot) := V_f(t, \boldsymbol{x}_f(t) + \cdot)$, $t \geq 0$.

Theorem 1. *Let $\rho \in (d/2, d/2 + \gamma(\beta))$. Suppose that $f \in \mathcal{C}_\rho$. Then there exists an $(\mathcal{W}_t)$-adapted cylindrical Wiener process W_f on $\mathbb{L}^2$, defined on $(\Omega, \mathcal{V}, \mathbb{P})$ such that η_f is a strong solution* (*in the sense of* [4], *p.* 118) *of*

$$d\eta_f(t) = (-A\eta_f(t) + \eta_f(t, \mathbf{0}) \cdot \nabla_{\mathbf{x}} \eta_f(t))\, dt + B dW_f(t), \qquad \eta_f(0) = f. \tag{3.1}$$

Proof. Let $\psi \in \mathcal{S}_d$. Since V_f is a strong solution to (1.5), using Itô's formula we obtain

$$
\begin{aligned}
\langle \eta_f(t), \psi \rangle &= \int\limits_{\mathbb{R}^d} V_f(t,\mathbf{x}) \cdot \psi(\mathbf{x} - \boldsymbol{x}_f(t))\, d\mathbf{x} = \int\limits_{\mathbb{R}^d} V_f(0,\mathbf{x}) \cdot \psi(\mathbf{x})\, d\mathbf{x} \\
&+ \int\limits_{\mathbb{R}^d} \int\limits_0^t \left[-AV_f(s)(\mathbf{x}) \cdot \psi(\mathbf{x} - \boldsymbol{x}_f(s)) + V_f(s,\mathbf{x}) \cdot \frac{d}{ds} \psi(\mathbf{x} - \boldsymbol{x}_f(s)) \right] ds\, d\mathbf{x} \\
&+ \int\limits_0^t \langle\, B\, dW(s), \psi(\cdot - \boldsymbol{x}_f(s)) \rangle.
\end{aligned} \tag{3.2}
$$

The first two terms on the utmost right hand side of (3.2) can be rewritten as

$$
\begin{aligned}
& \langle \eta_f(0), \psi \rangle + \int\limits_0^t \left[\langle -A\eta_f(s), \psi \rangle - \langle \eta_f(s), \eta_f(s, \mathbf{0}) \cdot \nabla \psi \rangle \right] ds \\
&= \quad \langle \eta_f(0), \psi \rangle + \int\limits_0^t \langle -A\eta_f(s) + \eta_f(t, \mathbf{0}) \cdot \nabla \eta_f(s), \psi \rangle\, ds.
\end{aligned}
$$

Thus the proof will be complete as soon as we can show that

$$
\langle W_f(t), \psi \rangle := \int\limits_0^t \langle\, dW(s), \psi(\cdot - \boldsymbol{x}_f(s)) \rangle, \qquad t \geq 0,\ \psi \in \mathcal{S}_d,
$$

defines a cylindrical Wiener process W_f on $\mathbb{L}^2$. This follows from the fact that for any ψ, $\langle W_f(t), \psi \rangle$, $t \geq 0$ is an $(\mathcal{W}_t)$-adapted square integrable martingale, and for all ψ, φ, the quadratic variation $\langle \langle W_f(\cdot), \psi \rangle \langle W_f(\cdot), \varphi \rangle \rangle_t$, $t \geq 0$ of $\langle W_f(t), \psi \rangle$, $\langle W_f(t), \varphi \rangle$, $t \geq 0$ is equal to

$$
\int_0^t \int_{\mathbb{R}^d} \psi(\mathbf{x} - \boldsymbol{x}_f(s)) \cdot \varphi(\mathbf{x} - \boldsymbol{x}_f(s))\, d\mathbf{x}\, ds = \int_0^t \int_{\mathbb{R}^d} \psi(\mathbf{x}) \cdot \varphi(\mathbf{x})\, d\mathbf{x}\, ds = t\, \langle \psi, \varphi \rangle.
$$

∎

4. Existence of strong solutions.

The main result of the present section is the following theorem on the existence of a solution to Eq. (1.7).

Theorem 2. *Suppose that $m > d/2$ and $\rho \in (d/2, d/2 + \gamma(\beta))$. Let W be a cylindrical Wiener process on $\mathbb{L}^2$, and let $f \in \mathbb{H}_\rho^{m+[2\beta]+1}$. Then there exists a stopping time $\tau_f > 0$ w.r.t the standard filtration corresponding to the Wiener process and $\eta_f(t)$, $0 \leq t < \tau_f$ a non-anticipative process such that $\eta_f(t) \in D(A) \cap D(\nabla)$, $0 \leq t < \tau_f$ and*

$$
\eta_f(t) = f + \int\limits_0^t \left(-A\eta_f(s) + \eta_f(s, \mathbf{0}) \cdot \nabla \eta_f(s) \right) ds + B\, W(t),\ 0 \leq t < \tau_f. \tag{4.1}
$$

Additionally, one can take $\tau_f = +\infty$, $\mathbb{P}$-a.s. for any $f \in \mathcal{C}_\rho$.

Let $m > d/2$ and $\rho \in (d/2, d/2 + \gamma(\beta))$. Take $T < \infty$ and $f \in \mathbb{H}^m_\rho$. For a given positive integer $n \geq 1$ we take a partition of $[0,T]$ given by points k/n, $k = 0, \cdots, [T]n$. We construct an approximating sequence of solutions $\eta^{(n)}_f(t)$, $t \in [0,T]$ in the following way. First, we solve the linear equation obtained from (1.7) by putting the coefficient by the gradient term equal to $f(0)$ and thus obtain

$$\eta^{(n)}_f(t) := C_{f(0)}(t)f + \int\limits_0^t C_{f(0)}(t-s)\, B\, dW(s) \quad \text{for } 0 \leq t \leq 1/n.$$

Here $C_{\mathbf{v}}$ is given by (2.3). We can continue the construction of the approximate solution on a given partition interval feeding in for the coefficient by the gradient term the final value of the solution (taken at $\mathbf{0}$) at the preceding partition interval and solving obtained that way linear equation with the initial condition given by the final value of the approximate solution in the previous interval. More precisely, the process $\eta^{(n)}_f(t)$ is defined for $t \in [k/n, (k+1)/n]$, where $k = 0, \cdots, [T]n$ by

$$\eta^{(n)}_f(t) = C_{v^{(n)}_k}\left(t - \frac{k}{n}\right)\left[\eta^{(n)}_f\left(\frac{k}{n}\right)\right] + \int\limits_{k/n}^t C_{v^{(n)}_k}(t-s)\, B\, dW(s). \tag{4.2}$$

Here $v^{(n)}_k := \eta^{(n)}_f(k/n, \mathbf{0})$. We show in Lemmas 2 and 3 that if $f \in \mathbb{H}^{m+[2\beta]+1}_\rho$, then there exists a random variable $T_f > 0$ such that the sequence $\eta^{(n)}_f$ is compact $\mathbb{P}$-a.s. in the space $C([0,T_f]; \mathbb{H}^m_\rho)$. Choosing a converging subsequence we construct a local solution.

First note that as $B\,W(t) \in \mathcal{H}_\rho$, $t \geq 0$ we have

$$\begin{aligned}\eta^{(n)}_f(t) &= C_{v^{(n)}_k}\left(t - \frac{k}{n}\right)\left[\eta^{(n)}_f\left(\frac{k}{n}\right)\right] \\ &+ B\left[W(t) - W\left(\frac{k}{n}\right)\right] + \int\limits_{k/n}^t C_{v^{(n)}_k}(t-s)\,(-A + v^{(n)}_k \cdot \nabla) B\left[W(s) - W\left(\frac{k}{n}\right)\right] ds.\end{aligned}$$

We can write therefore that

$$F^{(n)}_{k+1} = C_{v^{(n)}_k}\left(\frac{1}{n}\right) F^{(n)}_k + R^{(n)}_k, \tag{4.3}$$

where $F^{(n)}_k := \eta^{(n)}_f(k/n) - W^{(n)}_k$, $W^{(n)}_k := B\,W(k/n)$ and

$$\begin{aligned}R^{(n)}_k &:= C_{v^{(n)}_k}\left(\frac{1}{n}\right) W^{(n)}_k - W^{(n)}_k \\ &+ \int\limits_{k/n}^{(k+1)/n} C_{v^{(n)}_k}\left(\frac{k+1}{n} - s\right) D_{v^{(n)}_k}\, B\left[W(s) - W\left(\frac{k}{n}\right)\right] ds.\end{aligned} \tag{4.4}$$

Iterating (4.3) we get

$$F_{k+1}^{(n)} = C_{s_{k,0}^{(n)}}\left(\frac{1}{n}\right) f + \sum_{p=1}^{k+1} S\left(\frac{k+1-p}{n}\right) R_{s_{k,p}^{(n)}}\left(\frac{1}{n}\right) R_{p-1}^{(n)}, \tag{4.5}$$

with

$$s_{k,p}^{(n)} := \begin{cases} v_k^{(n)} + \cdots + v_p^{(n)}, & p \le k \\ \mathbf{0}, & p > k. \end{cases}$$

Write $f_k^{(n)} := \|F_k^{(n)}\|_{\mathbb{H}_\rho^m}$. We shall need the following fact.

Lemma 1. *Let $r_k^{(n)} := \|R_k^{(n)}\|_{\mathbb{H}_\rho^m}$, $a := \sup\limits_{0\le t\le T} [\|A\,B\,W(t)\|_{\mathbb{H}_\rho^m} + \|\nabla\,B\,W(t)\|_{\mathbb{H}_\rho^m}]$, and let $M := \sup\limits_{0\le t\le T} \|B\,W(t)\|_{\mathbb{H}_\rho^m} + 1$. Then there exists a deterministic constant $c > 0$ independent of n such that*

$$r_k^{(n)} \le \frac{c\,a}{n}\left[1 + f_k^{(n)} + M)\right]\left[1 + \frac{f_k^{(n)} + M}{n}\right]^\rho \quad \textit{for } k = 0, \cdots, n[T], \qquad \mathbb{P} - a.s.. \tag{4.6}$$

Proof. We have

$$\begin{aligned} &\left\| C_{v_k^{(n)}}\left(\frac{1}{n}\right) W_k^{(n)} - W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} \\ &\le \left\| R_{v_k^{(n)}}\left(\frac{1}{n}\right) S\left(\frac{1}{n}\right) W_k^{(n)} - S\left(\frac{1}{n}\right) W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} + \left\| S\left(\frac{1}{n}\right) W_k^{(n)} - W_k^{(n)} \right\|_{\mathbb{H}_\rho^m} \\ &\le \left\| \int_0^{1/n} S\left(\frac{1}{n}\right) R_{v_k^{(n)}}(s)\left(v_k^{(n)}\cdot\nabla W_k^{(n)}\right) ds \right\|_{\mathbb{H}_\rho^m} + \left\| \int_0^{1/n} S(s)\, A W_k^{(n)}\, ds \right\|_{\mathbb{H}_\rho^m}. \end{aligned} \tag{4.7}$$

The Sobolev inequality stating that $F_k^{(n)}(\mathbf{0}) \le c_1 f_k^{(n)}$ yields

$$|v_k^{(n)}| = |F_k^{(n)}(0) + W_k^{(n)}(0)| \le c_2(1 + f_k^{(n)} + M). \tag{4.8}$$

Thus from Proposition 3 we conclude that the utmost left hand side of (4.7) is less than or equal to

$$\frac{c_3\,a}{n}\left(1 + f_k^{(n)} + M\right)\left(1 + \frac{f_k^{(n)} + M}{n}\right)^\rho. \tag{4.9}$$

An analogous estimate can be carried out for the norm of the second term on the right hand side of (4.4) which yields that it is less than or equal to

$$\frac{c_4\,a}{n}\left(1 + f_k^{(n)} + M\right)\left(1 + \frac{f_k^{(n)} + M}{n}\right)^\rho. \tag{4.10}$$

Combining both (4.9) and (4.10) we conclude (4.6). ∎

Lemma 2. *There exists a random variable $T_f > 0$ such that $\sup_{n\geq 1} \|\eta_f^{(n)}\|_{\mathbb{H}_\rho^m, T_f} < \infty$, $\mathbb{P}$-a.s..*

Proof. Obviously one has

$$\left(1+\frac{1}{n}\sum_{p=0}^{k}\left(f_p^{(n)}+M\right)\right)^{\rho} \leq \prod_{p=0}^{k}\left(1+\frac{f_p^{(n)}+M}{n}\right)^{\rho}.$$

Hence using (4.8), Proposition 3, Lemma 1, and identity (4.5) we obtain that for some constant $C > 0$ one has $\mathbb{P}$-a.s.,

$$\begin{aligned} f_{k+1}^{(n)} &\leq C\,\|f\|_{\mathbb{H}_\rho^m} \prod_{p=0}^{k}\left(1+\frac{f_p^{(n)}+M}{n}\right)^{\rho} \\ &+\frac{C\,a}{n}\sum_{p=1}^{k-1}\left(1+f_p^{(n)}+M\right)\left(1+\frac{f_p^{(n)}+M}{n}\right)^{\rho}\prod_{q=p+1}^{k}\left(1+\frac{f_q^{(n)}+M}{n}\right)^{\rho} \\ &+\frac{C\,a}{n}\left(1+f_k^{(n)}+M\right)\left(1+\frac{f_k^{(n)}+M}{n}\right)^{\rho}. \end{aligned}$$

Thus we have

$$f_{k+1}^{(n)} \leq C\left(\|f\|_{\mathbb{H}_\rho^m}+\frac{ak(1+M)}{n}+\frac{a}{n}\sum_{p=1}^{k}f_p^{(n)}\right)\prod_{p=0}^{k}\left(1+\frac{f_p^{(n)}+M}{n}\right)^{2\rho}.$$

Now let us choose $K > (C+1)\|f\|_{\mathbb{H}_\rho^m}$. Obviously there is a strictly positive random variable $T_f > 0$ sufficiently small so that

$$K \;>\; \mathrm{e}^{2\rho(K+M)T_f}\{\,C\,\|f\|_{\mathbb{H}_\rho^m}+C\,a\,(1+K+M)\,T_f\}, \qquad \mathbb{P}-\text{a.s..}$$

Then $f_k^{(n)} \leq K$ for all k-s satisfying $k/n \leq T_f$. For $f_0^{(n)} = \|f\|_{\mathbb{H}_\rho^m} \leq K$, and assuming that $f_p^{(n)} \leq K$ for all $p \leq k$, k such that $k/n \leq T_f$, then provided $(k+1)/n \leq T_f$ we obtain

$$f_{k+1}^{(n)} \leq \left(1+\frac{K+M}{n}\right)^{2\rho(k+1)}\left(\,C\,\|f\|_{\mathbb{H}_\rho^m}+C\,a\,(1+M+K)\,T_f\right) \leq K, \qquad \mathbb{P}-\text{a.s..}$$

Hence

$$\sup_{k/n\leq T_f}\|\eta_f^{(n)}(k/n)\|_{\mathbb{H}_\rho^m} \leq (K+M), \qquad \mathbb{P}-\text{a.s..}$$

Combining this with (4.2) one obtains the desired conclusion. ∎

Lemma 3. *Suppose $f \in \mathbb{H}_{\rho'}^{m'}$ where $m' > m+[2\beta]+1$, $d/2 < \rho' < \rho$. Let T_f be the random variable constructed in Lemma 2. Then the sequence $\eta_f^{(n)}$, $n \geq 1$ is pre-compact in $C([0,T_f];\mathbb{H}_\rho^m)$, P-a.s..*

Proof. From (4.3), see also the proof of (4.5), we get

$$\|F^{(n)}_{k+1} - F^{(n)}_k\|_{\mathbb{H}^m_\rho} \le \left\| C_{v^{(n)}_k} F^{(n)}_k - F^{(n)}_k \right\|_{\mathbb{H}^m_\rho} + \left\| R^{(n)}_k \right\|_{\mathbb{H}^m_\rho}$$
$$\le r^{(n)}_k + + \left\| \int_0^{1/n} S\left(\frac{1}{n}\right) R_{v^{(n)}_k}(s) \left(v^{(n)}_k \cdot \nabla F^{(n)}_k \right) ds \right\|_{\mathbb{H}^m_\rho} + \left\| \int_0^{1/n} S(s)\, A F^{(n)}_k \, ds \right\|_{\mathbb{H}^m_\rho}.$$

Thus using arguments from the proof of Lemma 1 one can easily obtain

$$\|F^{(n)}_{k+1} - F^{(n)}_k\|_{\mathbb{H}^m_\rho} \le C \frac{a}{n} \left[1 + f^{(n)}_k + M \right] \left[1 + \frac{f^{(n)}_k + M}{n} \right]^\rho \left(\|F^{(n)}_k\|_{\mathbb{H}^{m'}_{\rho'}} + 1 \right), \qquad \mathbb{P} - \text{a.s.}.$$

From Lemma 2 we know that

$$R := \sup_{n \ge 1} \sup_{k \le n\,[T_f]} \left(f^{(n)}_k + \|F^{(n)}_k\|_{\mathbb{H}^{m'}_{\rho'}} \right) < \infty.$$

Hence $\eta^{(n)}_f - B\,W$, $n \ge 1$ are uniformly continuous in $C([0,T_f];\mathbb{H}^m_\rho)$. From Proposition 1, the ball $B(R) := [\psi \in \mathbb{H}^{m'}_{\rho'} : \|\psi\|_{\mathbb{H}^{m'}_{\rho'}} \le R]$ is compact in $\mathbb{H}^m_\rho$. By virtue of the infinite dimensional version of Arzela–Ascoli lemma (see e.g. remark on p. 382 of [6]) we conclude the result. ∎

Proof of Theorem 2. Let η_f, $0 \le t \le T_f$ be the limiting function of the sequence $(\eta^{(n)}_f)$. In view of Proposition 2(iii) we have $\eta_f(t) \in D(A) \cap D(\nabla)$, $0 \le t \le T_f$. With the convention $\inf \emptyset = T$ we define $\tau_f = \inf\{t \in [0,T] : \|\eta(t)\|_{\mathbb{H}^m_\rho} = \infty\}$. Clearly, τ_f is a Markov stopping time, and $\mathbb{P}(\tau_f \ge T_f > 0) = 1$. Using (4.2) and passing to the limit with $n \uparrow \infty$ we obtain that $\eta_f(t)$, $0 \le t \le T_f$ satisfies (4.1).

We prove now the global existence for any $f \in \mathcal{C}_\rho$. Let τ be a possible blow-up time of a solution. It is obviously a stopping time. Let us set

$$\boldsymbol{x}_f(t) := \int_0^t \eta_f(s, \mathbf{0}) \; ds \text{ and } \; V_f(t, \mathbf{x}) := \eta_f(t, \mathbf{x} - \boldsymbol{x}_f(t)), \;\; 0 \le t < \tau.$$

Clearly, $\boldsymbol{x}_f(t)$, $0 \le t < \tau$ solves the initial value problem

$$d\boldsymbol{x}_f(t)/dt = V_f(t, \boldsymbol{x}_f(t)) \text{ and } \boldsymbol{x}_f(0) = \mathbf{0}.$$

Using Itô formula, exactly in the same fashion as we did to derive (1.6), we check that

$$V_f(t) = f - \int_0^t A\, V_f(s)\, ds + B\, \tilde{W}_f(t), \qquad 0 \le t < \tau,$$

with $\tilde{W}_f(t)$, $t \ge 0$ some cylindrical Brownian motion on $\mathbb{L}^2$. Notice also that by virtue of Proposition 2(iii) and the arguments made at the and of Section 2.2,

$$\sup_{0 \le t \le T} |V_f(t, x)|$$

has sub-linear growth in $\mathbf{x}$, $\mathbb{P}$-a.s. for any $T > 0$. In consequence

$$C(\omega) := \sup_{0\le t<\tau} |\boldsymbol{x}_f(t)| < \infty.$$

Thus, by Proposition 3,

$$\begin{aligned}\sup_{0\le t<\tau} \|\eta_f(t)\|_{\mathbb{H}^m_\rho} &= \sup_{0\le t<\tau} \|V_f(t, \boldsymbol{x}_f(t) + \cdot)\|_{\mathbb{H}^m_\rho} \\ &\le (1 + C(\omega))^\rho \sup_{0\le t<\tau} \|V_f(t)\|_{\mathbb{H}^m_\rho} < \infty,\end{aligned}$$

which contradicts the fact that τ should be the blow-up time. ∎

5. Uniqueness.

In this section we prove the following.

Theorem 3. *Suppose that $m > d/2$ and $\rho \in (d/2, d/2 + \gamma(\beta))$. Then for any $f \in \mathbb{H}^m_\rho$, Eq. (1.7) possesses at most one (maximal in time) strong solution.*

In the first step of the proof we will show that any solution of (1.7) satisfies a certain integral equation that we call its *mild formulation*. This is the content of the following.

Lemma 4. *Suppose that $\eta(t) \in \mathbb{H}^m_\rho$, $t \in [0, \tau)$ is a local solution of (1.7) with the given cylindrical Brownian motion W on $\mathbb{L}^2$, m, ρ are as in Theorem 3. Then*

$$\eta(t, \mathbf{x}) = S(t)f\left(\mathbf{x} + \int_0^t \eta(s, \mathbf{0})\, ds\right) + \overline{W}_A\left(t, \mathbf{x} + \int_0^t \eta(s, \mathbf{0})\, ds\right), \tag{5.1}$$

where $\overline{W}_A(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ is a Gaussian field given by

$$\overline{W}_A(t) := \int_0^t S(t-s)\, B\, d\overline{W}(s) \tag{5.2}$$

and $\overline{W}(t)$, $t \ge 0$ is a cylindrical Wiener process on $\mathbb{L}^2$ given by

$$\langle \overline{W}(t), \psi\rangle = \int_0^t \left\langle dW(s), \psi\left(\cdot + \int_0^s \eta(u, \mathbf{0})du\right)\right\rangle, \qquad t \ge 0,\ \psi \in \mathcal{S}_d.$$

Proof. Note that for any predictable Itô's integrable process ξ one has

$$\int_0^t \langle d\overline{W}(s), \xi(s)\rangle = \int_0^t \left\langle dW(s), \xi\left(\cdot + \int_0^s \eta(u, \mathbf{0})du\right)\right\rangle, \qquad t \ge 0.$$

Thus using the arguments from the proof of Theorem 1 one can show that the right hand side $\overline{\eta}$ of (5.1) satisfies

$$d\overline{\eta}(t) = (-A\,\overline{\eta}(t) + \eta(t, \mathbf{0}) \cdot \nabla_{\mathbf{x}}\overline{\eta}(t))dt + B\, dW(t).$$

Hence $v(t) := \overline{\eta}(t) - \eta(t)$ satisfies

$$dv(t)/dt = -A\,v(t) + \eta(t,\mathbf{0})\cdot\nabla_{\mathbf{x}}\,v(t), \qquad v(0) = \mathbf{0}.$$

In consequence

$$v(t) = \int_0^t \eta(s,\mathbf{0})\cdot\nabla_{\mathbf{x}}\left[\,S(t-s)\,v(s)\right]\,ds. \tag{5.3}$$

Let ψ be such that $\hat{\psi} \in C_0^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\};\mathbb{R}^d)$. From (5.3) we know that $\hat{v}(t)$ - the Fourier transform of $v(t,\cdot)$ is a distribution satisfying

$$\hat{v}(t)(\hat{\psi}) = \int_0^t \hat{v}(s)(g(s,\cdot))ds, \tag{5.4}$$

where $g(s,\mathbf{k}) := \mathrm{i}\,\mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}(\eta(s,\mathbf{0})\cdot\mathbf{k})\hat{\psi}(\mathbf{k})$. Iterating we obtain that the left hand side of (5.4) equals $\int_0^t \hat{v}(s)(g_N(s,\cdot))ds$ with

$$g_N(s,\mathbf{k}) := \int\cdots\int_{t\ge s_1\ge\cdots\ge s_N\ge s} \mathrm{e}^{-|\mathbf{k}|^{2\beta}(t-s)}(\eta(s,\mathbf{0})\cdot\mathbf{k})(\eta(s_1,\mathbf{0})\cdot\mathbf{k})\cdots(\eta(s_N,\mathbf{0})\cdot\mathbf{k})\hat{\psi}(\mathbf{k})$$

for an arbitrary integer $N \ge 1$. A direct calculation shows that

$$\sup_{0\le s\le t,\,\mathbf{k}\in\mathbb{R}^d} |\nabla_{\mathbf{k}}^m g_N(s,\mathbf{k})| \le \frac{C^N}{N!}$$

with the constant C independent of N. Hence $g_N(s,\cdot)$ tends to 0 in the space $\mathcal{D}(\mathbb{R}^d;\mathbb{R}^d)$ when $N \uparrow +\infty$. Hence $\hat{v}(t)(\hat{\psi}) = 0$ for all $\hat{\psi} \in C_0^\infty(\mathbb{R}^d \setminus \{\mathbf{0}\};\mathbb{R}^d)$. When β is an integer (5.4) extends by the foregoing argument to any ψ such that $\hat{\psi} \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$. That implies $v(t,\cdot) \equiv \mathbf{0}$.

Suppose that β is a non-integer. The above argument shows

$$\mathrm{supp}\ \hat{v}(t,\cdot) = \{\mathbf{0}\} \qquad \text{and} \qquad v(t,\mathbf{x}) = \sum_p c_p(t)\mathbf{x}^p,$$

where the summation extends over a certain finite set of multi-indices $p = (p_1,\cdots,p_d) \in \mathbb{Z}^d$. Suppose now that $p_* = (p_{*1},\cdots,p_{*d})$ is the multi-index with the largest norm $|p_*| := \sum p_{*i}$. We apply $\partial_{x_1}^{p_{*1}}\cdots\partial_{x_d}^{p_{*d}}$ to both sides of (5.3) and obtain that

$$p_*!c_{p_*}(t) = \int_0^t \eta(s,\mathbf{0})\cdot S(t-s)[\nabla_{\mathbf{x}}\partial_{x_1}^{p_{*1}}\cdots\partial_{x_d}^{p_{*d}}v(s)] = 0,$$

so that $c_p(t) \equiv \mathbf{0}$ for all p and $t \ge 0$. ∎

Proof of Theorem 3. When $\mathbf{x} = \mathbf{0}$ we have the following equation for $\eta(t, \mathbf{0})$

$$\eta(t,\mathbf{0}) = F\left(t, \int_0^t \eta(s,\mathbf{0})ds\right) + B\,W(t,\mathbf{0}) + \int_0^t G\left(t, s, \int_s^t \eta(u,\mathbf{0})\,du\right) ds, \tag{5.5}$$

with $F(t, \mathbf{x}) := S(t)f(\mathbf{x})$ a certain Lipschitz function and $G(t, s, \mathbf{x}) := A\,S(t-s)\,B\,W(s, \mathbf{x})$ - a Gaussian random field with the compactly supported spectral measure. It is therefore elementary to conclude that the random field is Lipschitz in $\mathbf{x}$, $\mathbb{P}$-a.s.. The equation (5.5) and thus also (5.1) admits a unique local solution. Hence the desired conclusion follows from Lemma 4. ∎

Directly from the above result and the global existence part of Theorem 2 we conclude the following.

Corollary 1. *Suppose that m, ρ are as in the statement of Theorem 3. Then for any $f \in \mathcal{C}_\rho$ there exists a unique strong global solution η_f of (1.7) satisfying $\eta_f(0) = f$.*

6. Lagrangian process for irregular fields.

In this section we suppose that a Gaussian random field $V(t, \mathbf{x})$, $(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^d$ whose co-variance matrix is given by (1.2) satisfies

C) $k_0 :=$ dist$(\mathbf{0}$, supp $\mathcal{E}) > 0$ and there exists $C > 0$ such that Tr $\mathcal{E}(\mathbf{k}) \le C|\mathbf{k}|^{1-2\alpha}$ for all $|\mathbf{k}| \ge 1$ and $\alpha \in (1, 2)$.

The class of fields satisfying Condition C includes fields which are Hölder but not necessarily Lipschitz in the spatial variable. As we have pointed out in the introduction it may be impossible therefore to solve uniquely the equation (1.1). This difficulty prevents us from defining the Lagrangian dynamics directly as it was done in [13]. Below we construct such a dynamics as the limit of Lagrangian processes $\eta_K(t) := V_K(t, \boldsymbol{x}_K(t) + \cdot)$, $t \ge 0$, $K \ge 1$ corresponding to the fields V_K, $K \ge 1$ whose spectra are given by (1.9). Smoothness of V_K, $K \ge 1$ guarantees the existence and uniqueness of

$$d\boldsymbol{x}_K(t)/dt = V_K(t, \boldsymbol{x}_K(t)), \qquad \boldsymbol{x}_K(0) = \mathbf{0}. \tag{6.1}$$

By $\eta_K(t) := V_K(t, \boldsymbol{x}_K(t) + \cdot)$, $t \ge 0$ we denote the corresponding Lagrangian processes and by Q_K their respective laws on $\mathcal{M} := C([0, +\infty); \mathcal{X})$. Throughout this section we shall denote $\mathcal{X} := C(\mathbb{R}^d; \mathbb{R}^d))$. Suppose now that $V_K(t)$ is the stationary solution of the Ornstein–Uhlenbeck equation (1.5) with A defined by (2.1). B_K is also defined with the help of (2.1) but with $\mathcal{E}_K$ used in place of $\mathcal{E}$. Let μ_K be the law of $V_K(0)$ on $\mathcal{X}$. It is Gaussian with the co-variance matrix

$$\int f(\mathbf{x}) \otimes f(\mathbf{y})\,\mu_K(df) = \Gamma_K(\mathbf{x} - \mathbf{y})$$

with

$$\Gamma_K(\mathbf{x}) := \int_{\mathbb{R}^d} \cos(\mathbf{x} \cdot \mathbf{k})\,\frac{\mathcal{E}(\mathbf{k})\mathbf{1}_{[0,K]}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\,d\mathbf{k}.$$

As we already know Q_K is identical with the law on $\mathcal{M}$ of the solution $\overline{\eta}_K(t)$, $t \geq 0$ of equation (4.1) with operators A, B_K described above. According to Lemma 4, $\overline{\eta}_K$ satisfies

$$\overline{\eta}_K(t,\mathbf{x}) = S(t)f\left(\mathbf{x} + \int_0^t \overline{\eta}_K(s,\mathbf{0})\,ds\right) + \overline{W}_{A,K}\left(t, \mathbf{x} + \int_0^t \overline{\eta}_K(s,\mathbf{0})\,ds\right), \tag{6.2}$$

where

$$\overline{W}_{A,K}(t) = \int_0^t S(t-s)\,B_K\,d\overline{W}(s)$$

and

$$\langle \overline{W}(t), \psi\rangle = \int_0^t \left\langle dW(s), \psi\left(\cdot + \int_0^s \overline{\eta}_K(u,\mathbf{0})du\right)\right\rangle, \qquad t \geq 0,\ \psi \in \mathcal{S}_d.$$

Define $U(t,\mathbf{x}) := W(t, \mathbf{x} - \int_0^t \overline{\eta}_K(u,\mathbf{0})du)$. Then, an integration by parts formula yields

$$\begin{aligned} \int_0^t \langle\, dU(s), \psi\rangle &= \int_0^t \left\langle dW(s), \psi\left(\cdot + \int_0^s \overline{\eta}_K(u,\mathbf{0})du\right)\right\rangle \\ &+ \int_0^t \left\langle W(s), \overline{\eta}_K(s,\mathbf{0})\cdot\nabla_{\mathbf{x}}\psi\left(\cdot + \int_0^s \overline{\eta}_K(u,\mathbf{0})du\right)\right\rangle ds, \qquad t \geq 0,\ \psi \in \mathcal{S}_d. \end{aligned} \tag{6.3}$$

Using (6.3) we can rewrite (6.2) in the form

$$\begin{aligned} \overline{\eta}_K(t,\mathbf{x}) &= S(t)f\left(\mathbf{x} + \int_0^t \overline{\eta}_K(u,\mathbf{0})\,du\right) + U_{A,K}\left(t, \mathbf{x} + \int_0^t \overline{\eta}_K(u,\mathbf{0})\,du\right) \\ &+ \int_0^t \eta(s,\mathbf{0})\cdot\nabla_{\mathbf{x}}\,S(t-s)\,B_K\,W\left(s, \mathbf{x} - \int_0^s \overline{\eta}_K(u,\mathbf{0})du\right) ds, \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} U_{A,K}(t) &= \int_0^t S(t-s)\,B_K\,dU(s) \\ &= S(t)\,B_K U(t) + \int_0^t A\,S(t-s)[B_K U(t) - B_K U(s)]\,ds. \end{aligned}$$

Denoting

$$F(t,\mathbf{x}) := S(t)f(\mathbf{x}) \tag{6.5}$$

a Gaussian random field over $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu_K)$, $G_K(t, \mathbf{x}) := S(t)B_K W(t, \mathbf{x})$ and $H_K(t, s, \mathbf{x}) := AS(t-s)B_K W(t, \mathbf{x})$, $I_K(t, s, \mathbf{x}) := AS(t-s)B_K W(s, \mathbf{x})$, $J_K(t, s, \mathbf{x}) := \nabla_{\mathbf{x}} S(t-s)\, B_K\, W\,(s, \mathbf{x})$ - Gaussian fields over $(\Omega, \mathcal{W}, \mathbb{P})$ we can rewrite (6.4) in the form

$$
\begin{aligned}
\overline{\eta}_K(t,\mathbf{x}) &= F\left(t, \mathbf{x} + \int_0^t \overline{\eta}_K(s,\mathbf{0})ds\right) + G_K(t,\mathbf{x}) \\
&\quad + \int_0^t \left[H_K(t,s,\mathbf{x}) - I_K\left(t,s,\mathbf{x}+\int_s^t \overline{\eta}_K(u,\mathbf{0})\,du\right)\right] ds \\
&\quad + \int_0^t \overline{\eta}_K(s,\mathbf{0}) \cdot J_K\left(t,s,\mathbf{x}+\int_s^t \overline{\eta}_K(u,\mathbf{0})\,du\right) ds.
\end{aligned} \tag{6.6}
$$

A direct calculation shows that

$$
\begin{aligned}
&\int |F_K(t,\mathbf{x})|^2\, d\mu_K + \mathbb{E}\,|G_K(t,\mathbf{x})|^2 \\
&\quad = \int_{\mathbb{R}^d} \mathrm{e}^{-2|\mathbf{k}|^{2\beta}t}(1+|\mathbf{k}|^{2\beta}t)\frac{\mathrm{Tr}\mathcal{E}(\mathbf{k})\mathbf{1}_{[0,K]}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\, d\mathbf{k} \le C, \quad \forall\, K \ge 1
\end{aligned} \tag{6.7}
$$

and

$$
\mathbb{E}\left[|H_K(t,s,\mathbf{x})|^2 + |I_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2 + |J_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2\right] \le C(t-s)^{-r}
$$

for some $r > 0$ and all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. Skorochod's embedding theorem (see e.g. Theorem 2.4, p. 33 of [4]) implies that there exist random fields

$$
\begin{aligned}
\mathcal{Z}_K(t,s,\mathbf{x},\mathbf{y}) &:= (\hat{F}_K(t,\mathbf{x}), \hat{G}_K(t,\mathbf{x}), \hat{H}_K(t,s,\mathbf{x}), \hat{I}_K(t,s,\mathbf{x}+(t-s)\mathbf{y}), \\
&\qquad J_K\,(t,s,\mathbf{x}+(t-s)\mathbf{y})), \\
(t,s,\mathbf{x},\mathbf{y}) \in D &:= [(t,s,\mathbf{x},\mathbf{y}) : t > s \ge 0,\ \mathbf{x},\mathbf{y} \in \mathbb{R}^d
\end{aligned}
$$

defined over certain probability space, that with no loss of generality will be assumed to coincide with $(\Omega, \mathcal{W}, \mathbb{P})$, whose laws coincide with those of respective (F, G_K, H_K, I_K, J_K) over $(C(\mathbb{R}^d;\mathbb{R}^d) \times \Omega, \mathcal{B}(C(\mathbb{R}^d;\mathbb{R}^d)) \otimes \mathcal{W}, \mu_K \otimes \mathbb{P})$. These fields converge uniformly on compact subsets of D, as $K \uparrow +\infty$, to a certain Gaussian field on D,

$$
\begin{aligned}
&\mathcal{Z}(t,s,\mathbf{x},\mathbf{y}) \\
&\quad := \left(\hat{F}(t,\mathbf{x}), \hat{G}(t,\mathbf{x}), \hat{H}(t,s,\mathbf{x}), \hat{I}(t,s,\mathbf{x}+(t-s)\mathbf{y}), \hat{J}(t,s,\mathbf{x}+(t-s)\mathbf{y})\right),
\end{aligned}
$$

whose law coincides with that of

$$
(F(t,\mathbf{x}), G(t,\mathbf{x}), H(t,s,\mathbf{x}), I(t,s,\mathbf{x}+(t-s)\mathbf{y}), J(t,s,\mathbf{x}+(t-s)\mathbf{y}))
$$

F being given by (6.5), a Gaussian random field over $(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$, $G(t, \mathbf{x}) := S(t)BW(t, \mathbf{x})$ and

$$
H(t,s,\mathbf{x}) := AS(t-s)BW(t,\mathbf{x}), \qquad I(t,s,\mathbf{x}) = AS(t-s)BW(s,\mathbf{x})
$$

and

$$J(t,s,\mathbf{x}) := \nabla_{\mathbf{x}}\, S(t-s)\, B_K\, W\,(s,\mathbf{x})\,.$$

In what follows we suppress writing the dash sign over the relevant random fields.
In what follows we consider the equation

$$\overline{\eta}(t,\mathbf{x}) = F\left(t,\mathbf{x}+\int\limits_0^t \overline{\eta}(s,\mathbf{0})ds\right)+G(t,\mathbf{x}) + \int\limits_0^t \left[H(t,s,\mathbf{x}) - I\left(t,s,\mathbf{x}+\int\limits_s^t \overline{\eta}(u,\mathbf{0})\,du\right)\right] ds + \int\limits_0^t \eta(s,\mathbf{0})\cdot J\left(t,s,\mathbf{x}+\int\limits_s^t \overline{\eta}(u,\mathbf{0})\,du\right) ds. \tag{6.8}$$

We say that $\overline{\eta}(\cdot,\cdot)$ is a solution of the equation if $\overline{\eta}(\cdot,\cdot;\omega)\in C([0,\infty);\mathcal{X})$, $\mathbb{P}$-a.s. and (6.8) holds for all $\mathbf{x}\in\mathbb{R}^d$.

Theorem 4. *Suppose $\alpha\in(1,2)$, $\beta\ge 0$ are such that $\alpha+\beta>2$, $\alpha+3\beta>3$. In addition, assume that condition C) holds. Then (6.8) has a unique global solution $\overline{\eta}$. Additionally, if $\{\overline{\eta}_K\}$ are the solutions of (6.6) corresponding to the approximating fields constructed by means of the Skorochod embedding theorem, then for all T, $R>0$,*

$$\lim_{K\uparrow+\infty}\ \sup_{0\le t\le T,|\mathbf{x}|\le R} |\overline{\eta}_K(t,\mathbf{x})-\overline{\eta}(t,\mathbf{x})| = 0, \qquad \mathbb{P}\text{-}a.s.. \tag{6.9}$$

From the above theorem we conclude immediately the following.

Corollary 2. *The measures Q_K, $K\ge 1$ are weakly convergent over $(\mathcal{M},\mathcal{B}(\mathcal{M}))$ as $K\uparrow+\infty$.*

The proof of Theorem 4 we start with the following lemma.

Lemma 5. *Under the assumptions of Theorem 4 the equation*

$$\boldsymbol{x}'(t) = F\left(t,\boldsymbol{x}(t)\right)+G(t,\mathbf{0})+\int\limits_0^t \left[H(t,s,\mathbf{0}) - I(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s)) + \boldsymbol{x}'(s)\cdot J\left(t,s,\boldsymbol{x}(t)-\boldsymbol{x}(s)\right)\right] ds \quad t>0, \quad \boldsymbol{x}(0)=\mathbf{0}, \tag{6.10}$$

has a unique global solution $\mathbb{P}$-a.s..

Proof. *Uniqueness.* Suppose that $\boldsymbol{x}$ and $\overline{\boldsymbol{x}}$ are two solution of (6.10) and $T > 0$ is such that $\sup \lim_{0 \le t \le T} [|\boldsymbol{x}(t)| + |\overline{\boldsymbol{x}}(t)|] = R < \infty$. Hence,

$$
\begin{aligned}
\boldsymbol{x}'(t) - \overline{\boldsymbol{x}}'(t) &= F(t, \boldsymbol{x}(t)) - F(t, \overline{\boldsymbol{x}}(t)) \\
&\quad + \int_0^t [I(t, s, \overline{\boldsymbol{x}}(t) - \overline{\boldsymbol{x}}(s)) - I(t, s, \boldsymbol{x}(t) - \boldsymbol{x}(s))] \, ds \\
&\quad + \int_0^t \left[\boldsymbol{x}'(s) \cdot J(t, s, \boldsymbol{x}(t) - \boldsymbol{x}(s)) - \overline{\boldsymbol{x}}'(s) \cdot J(t, s, \overline{\boldsymbol{x}}(t) - \overline{\boldsymbol{x}}(s))\right] \, ds.
\end{aligned} \tag{6.11}
$$

Notice that

$$
\mathbb{E}\,|\nabla_{\mathbf{x}} F(t, \mathbf{x})|^2 \le C \int_{k_0}^{+\infty} \mathrm{e}^{-2k^{2\beta} t} k^{3-2\alpha} \, dk. \tag{6.12}
$$

Substituting $\xi := k\, t^{1/(2\beta)}$ we obtain that the right hand side of (6.12) equals

$$
C\, t^{(\alpha-2)/\beta} \int_{k_0 t^{1/(2\beta)}}^{+\infty} \mathrm{e}^{-2\xi^{2\beta} t} \xi^{3-2\alpha} \, d\xi \le C\, t^{(\alpha-2)/(2\beta)}.
$$

We have also

$$
\mathbb{E}\,|\nabla_{\mathbf{x}} I(t, s, \mathbf{x})|^2 \le C \int_{k_0}^{+\infty} \mathrm{e}^{-2k^{2\beta}(t-s)} k^{3+6\beta-2\alpha} \, dk \tag{6.13}
$$

and after substitution $\xi := k\,(t-s)^{1/(2\beta)}$ we obtain that the right hand side of (6.13) is less than or equal to $C(t-s)^{(\alpha-3\beta-2)/\beta}$.

Finally, we obtain

$$
\mathbb{E}\,|J(t, s, \mathbf{x})|^2 \le C \int_{k_0}^{+\infty} \mathrm{e}^{-2k^{2\beta}(t-s)} k^{3+2\beta-2\alpha} \, dk \tag{6.14}
$$

and

$$
\mathbb{E}\,|\nabla_{\mathbf{x}} J(t, s, \mathbf{x})|^2 \le C \int_{k_0}^{+\infty} \mathrm{e}^{-2k^{2\beta}(t-s)} k^{5+2\beta-2\alpha} \, dk, \tag{6.15}
$$

which after substitution $\xi := k\,(t-s)^{1/(2\beta)}$ yield that the right hand sides of (6.14) and (6.15) are less than or equal to $C(t-s)^{(\alpha-\beta-2)/\beta}$ and $C(t-s)^{(\alpha-\beta-3)/\beta}$ correspondingly. Hence by virtue of Theorem 6.9.2, p. 161 of [2], for any $\gamma > 1$ one can find a random variable $C_{R,\gamma}$

such that

$$\begin{aligned}
\sup_{|\mathbf{x}|\le R}|\nabla_{\mathbf{x}}F(t,\mathbf{x})| &\le C_{R,\gamma}\, t^{\gamma(\alpha-2)/(2\beta)},\\
\sup_{|\mathbf{x}|\le R}|\nabla_{\mathbf{x}}I(t,s,\mathbf{x})| &\le C_{R,\gamma}\,(t-s)^{\gamma(\alpha-3\beta-2)/(2\beta)},\\
\sup_{|\mathbf{x}|\le R}|J(t,s,\mathbf{x})| &\le C_{R,\gamma}\,(t-s)^{\gamma(\alpha-\beta-2)/(2\beta)},\\
\sup_{|\mathbf{x}|\le R}|\nabla_{\mathbf{x}}J(t,s,\mathbf{x})| &\le C_{R,\gamma}(\omega)\,(t-s)^{\gamma(\alpha-\beta-3)/(2\beta)}.
\end{aligned}$$

Denoting $v(t) := |\boldsymbol{x}'(t)-\overline{\boldsymbol{x}}'(t)|$ we obtain from (6.11),

$$v(t)\le C_{R,\gamma}\left\{\int_0^t s^{\gamma(\alpha-2)/(2\beta)}\,v(s)\,ds+\int_0^t (t-s)^{\gamma(\alpha-3\beta-2)/(2\beta)}\,ds\int_s^t v(u)\,du\right.$$
$$\left.+\int_0^t (t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}\,v(s)\,ds+\int_0^t (t-s)^{\gamma(\alpha-\beta-3)/(2\beta)}\,ds\int_s^t v(u)\,du\right\}. \tag{6.16}$$

Interchanging the order of integration in the second and fourth terms on the right hand side of (6.16) we obtain that

$$v(t)\le C_{R,\gamma}\left\{\int_0^t s^{\gamma(\alpha-2)/(2\beta)}\,v(s)\,ds+\int_0^t (t-s)^{[\gamma(\alpha-3\beta-2)+2\beta]/(2\beta)}\,v(s)\,ds\right.$$
$$\left.+\int_0^t (t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}\,v(s)\,ds+\int_0^t (t-s)^{[\gamma(\alpha-\beta-3)+2\beta]/(2\beta)}\,v(s)\,ds.\right\}.$$

We can apply Gronwall's inequality, provided the kernels in the above integrals are integrable. Choosing sufficiently small $\gamma>1$ we obtain therefore $v(t)\equiv 0$, $0\le t\le T$, $\forall T>0$, provided that $\alpha+\beta>2$ and $\alpha+3\beta>3$.

Global existence. We start with certain a'priori estimate for $\boldsymbol{y}:=\boldsymbol{x}'$, where $\boldsymbol{x}$ is a solution of (6.10). Note that for any $r\in(0,1)$ there is a constant C independent of K such that

$$\mathbb{E}\,[|H_K(t,s,\mathbf{x})-I_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2] \tag{6.17}$$
$$\begin{aligned}
&\le C\int_{\mathbb{R}^d} \mathrm{e}^{-2\,|\mathbf{k}|^{2\beta}\,(t-s)}|\mathbf{k}|^{6\beta}\left\{t-s+s[1-\cos(\mathbf{y}\cdot\mathbf{k}(t-s))]\right\}\frac{\mathrm{Tr}\;\mathcal{E}(\mathbf{k})\mathbf{1}_{[0,K]}(|\mathbf{k}|)}{|\mathbf{k}|^{d-1}}\,d\mathbf{k}\\
&\le C\int_{k_0}^{+\infty} \mathrm{e}^{-2\,k^{2\beta}\,(t-s)}\left[t-s+s|\mathbf{y}|^{1+r}k^{1+\gamma}(t-s)^{1+r}\right]k^{6\beta+1-2\alpha}\,dk.
\end{aligned}$$

The last inequality follows from an elementary bound $|1-\cos\phi| \le 2|\phi|^{1+r}$ for all ϕ and $r\in(0,1)$, and integration over $d-1$ dimensional shells. Substituting $\xi := k(t-s)^{1/(2\beta)}$ we conclude that the utmost left hand side of (6.17) is less than or equal to

$$C[(t-s)^{-2+(\alpha-1)/\beta} + s|\mathbf{y}|^{1+r}(t-s)^{-2+(\alpha+r\beta-1)/\beta}]. \tag{6.18}$$

Likewise, cf. (6.14),

$$\mathbb{E}\left[|J_K(t,s,\mathbf{x}+(t-s)\mathbf{y})|^2\right] \le C(t-s)^{(\alpha-\beta-2)/\beta}. \tag{6.19}$$

These estimates, according to [2], yield

$$|H(t,s,\mathbf{0}) - I(t,s,(t-s)\mathbf{y})| \le C_{r,\gamma}\left(1+|\mathbf{y}|^{\gamma(1+r)/2}\right)(t-s)^{\gamma[-2+(\alpha+r\beta-1)]/2\beta} \tag{6.20}$$

and

$$|J(t,s,(t-s)\mathbf{y})| \le C_{r,\gamma}(t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}(1+|\mathbf{y}|^r) \tag{6.21}$$

for all $t>s$, $\mathbf{y}\in\mathbb{R}^d$ and $\gamma>1$, $r\in(0,1)$. From (6.7) we also have $|F(t,\mathbf{x})|+|G(t,\mathbf{x})| \le C_r(1+|\mathbf{x}|^r)$ for an arbitrary $r\in(0,1)$. Hence, from (6.10), we get, for a suitable choice of r and γ,

$$\begin{aligned} |\boldsymbol{y}(t)| \le C\Bigg\{ &\int_0^t |\boldsymbol{y}(s)|\,ds + \sup_{s\in[0,t]}|G(s,\mathbf{0})| \\ &+ \int_0^t (t-s)^{\gamma[-2+(\alpha+r\beta-1)]/2\beta}\left(\int_s^t\int_0^u |\boldsymbol{y}(u_1)|\,du\,du_1\right)ds \\ &+ \int_0^t (t-s)^{\gamma(\alpha-\beta-2)/(2\beta)}\left(\int_s^t\int_0^u |\boldsymbol{y}(u_1)|\,du\,du_1\right)ds\Bigg\}. \end{aligned} \tag{6.22}$$

From (6.22) we conclude that

$$\sup_{t\in[0,T]}|\boldsymbol{y}(t)| \le \tilde{C} < \infty, \tag{6.23}$$

where $\tilde{C}$ depends only on parameters α,β and the constant C appearing in (6.22).

Denote $\boldsymbol{x}_K(t;\omega) := \int_0^t \overline{\eta}_K(s,\mathbf{0};\omega)\,ds$, $\boldsymbol{y}_K(t,\omega) := \overline{\eta}_K(t,\mathbf{0};\omega)$.

Lemma 6. *Under the assumptions of Theorem* 4, $\boldsymbol{x}_K$ *is the solution of* (6.1) *and satisfies the equation*

$$\begin{aligned} \boldsymbol{x}_K'(t) = F_K(t,\boldsymbol{x}_K(t)) + G_K(t,\mathbf{0}) + \int_0^t \big[H_K(t,s,\boldsymbol{x}_K(t)-\boldsymbol{x}_K(s))\,ds \\ - I_K(t,s,\boldsymbol{x}_K(t)-\boldsymbol{x}_K(s)) + \boldsymbol{x}_K'(s)\cdot J_K(t,s,\boldsymbol{x}_K(t)-\boldsymbol{x}_K(s))\big]\,ds, \quad \boldsymbol{x}_K(0)=\mathbf{0} \end{aligned} \tag{6.24}$$

for any $K \geq 1$. Moreover the sequence $\boldsymbol{x}_K$ converges as $K \uparrow \infty$ in $\mathcal{M}$, $\mathbb{P}$-a.s. to a solution of (6.10).

Proof. Let $T > 0$ be an arbitrary. With the choice of γ, r as before we obtain in exactly the same fashion as in (6.17) - (6.22) a'priori estimates $\sup_{t\in[0,T]} |\boldsymbol{y}_K(t)| \leq \tilde{C}$, with $\tilde{C}$ the same as in (6.23). A standard compactness argument involving the Arzela–Ascoli theorem yields the conclusion of the lemma. ∎

Returning to the proof of the theorem we notice that having $\boldsymbol{x}$ as in Lemma 5 we can define

$$\overline{\eta}(t,\mathbf{x}) := F\left(t, \mathbf{x} + \boldsymbol{x}(t)\right) + G(t,\mathbf{x}) + \int_0^t \left[H(t,s,\mathbf{x}) - I\left(t,s,\mathbf{x} + \boldsymbol{x}(t) - \boldsymbol{x}(s)\right)\right] ds.$$

It can be easily verified that $\eta \in \mathcal{M}$, $\mathbb{P}$-a.s., and that (6.9) is a conclusion from Lemma 6. ∎

Appendix A. Proof of Proposition 2.

Proof of (i). We denote by $\mathcal{F}f$ or $\hat{f}$ the Fourier transform of a given function f. Let $\{e_k\}$, $k \in \mathbb{N}$ be an orthonormal basis of $\mathbb{L}^2$. Recall, see Section 2, that $J_{\rho,2}\psi(x) = \psi(x)\vartheta_{\rho/2}(x)$ is an isomorphism between $\mathbb{H}^m_\rho$ and $\mathbb{H}^m$. Hence, there is a constant c_1 such that

$$\begin{aligned} ||S_\beta(t)\, B_\beta||^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} &= \sum_k ||S_\beta(t)\, B_\beta e_k||^2_{\mathbb{H}^m_\rho} \leq c_1 \sum_k ||J_{\rho,2} S_\beta(t)\, B_\beta e_k||^2_{\mathbb{H}^m} \\ &\leq c_1 \sum_k \int_{\mathbb{R}^d} \left(1 + |\mathbf{k}|^2\right)^m \left|\mathcal{F}\left(\vartheta_\rho^{1/2} S_\beta(t) B_\beta e_k\right)(\mathbf{k})\right|^2_{\mathbb{R}^d} d\mathbf{k}. \end{aligned}$$

Note that

$$\mathcal{F}\left(S_\beta(t)\, B_\beta\, e_k\right)(\mathbf{k}) = \mathrm{e}^{-|\mathbf{k}|^{2\beta} t} \sqrt{2|\mathbf{k}|^{2\beta}\overline{\mathcal{E}}(\mathbf{k})}\,\hat{e}_k(\mathbf{k}).$$

Thus

$$\begin{aligned} &\sum_k \left|\mathcal{F}\left(\vartheta_\rho^{1/2} S_\beta(t)\, B_\beta e_k\right)(\mathbf{k})\right|^2_{\mathbb{R}^d} \\ &= \sum_k \left|\int_{\mathbb{R}^d} \widehat{\vartheta_\rho^{1/2}}(\mathbf{k}-\mathbf{k}')\mathrm{e}^{-|\mathbf{k}'|^{2\beta} t} \sqrt{2|\mathbf{k}'|^{2\beta}\mathcal{E}(\mathbf{k}')}\,\hat{e}_k(\mathbf{k}')\, d\mathbf{k}'\right|^2_{\mathbb{R}^d} \\ &\leq 2 \int_{\mathbb{R}^d} \left|\widehat{\vartheta_\rho^{1/2}}(\mathbf{k}-\mathbf{k}')\right|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta} t}\, |\mathbf{k}'|^{2\beta}\, \mathrm{Tr}\, \mathcal{E}(\mathbf{k}')\, d\mathbf{k}'. \end{aligned}$$

Hence

$$
\begin{aligned}
\text{(A.1)} \qquad & ||S_\beta(t)\, B_\beta||^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} \\
\le\ & 2c_1 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \left(1+|\mathbf{k}|^2\right)^m \left|\widehat{\vartheta^{1/2}_\rho}(\mathbf{k}-\mathbf{k}')\right|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}\, |\mathbf{k}'|^{2\beta}\, \mathrm{Tr}\, \mathcal{E}(\mathbf{k}')\, d\mathbf{k}'\, d\mathbf{k} \\
=\ & 2c_1 \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \left(1+|\mathbf{k}+\mathbf{k}'|^2\right)^m \left|\widehat{\vartheta^{1/2}_\rho}(\mathbf{k})\right|^2 \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}\, |\mathbf{k}'|^{2\beta}\, \mathrm{Tr}\, \mathcal{E}(\mathbf{k}')\, d\mathbf{k}'\, d\mathbf{k}.
\end{aligned}
$$

Since $1+|\mathbf{k}+\mathbf{k}'|^2 \le (1+|\mathbf{k}|^2)(1+|\mathbf{k}'|^2)$ we conclude that the utmost left hand side of (A.1) is less than or equal to

$$
c_2 \int\limits_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m \left|\widehat{\vartheta^{1/2}_\rho}(\mathbf{k})\right|^2 \, d\mathbf{k} \int\limits_{\mathbb{R}^d} \left(1+|\mathbf{k}'|^2\right)^m \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}\, |\mathbf{k}'|^{2\beta}\, \mathrm{Tr}\, \mathcal{E}(\mathbf{k}')\, d\mathbf{k}'
$$

and, because

$$
\int\limits_{\mathbb{R}^d} (1+|\mathbf{k}|^2)^m \left|\widehat{\vartheta^{1/2}_\rho}(\mathbf{k})\right|^2 \, d\mathbf{k} < \infty,
$$

we have

$$
\text{(A.2)} \qquad ||S_\beta(t)\, B_\beta||^2_{L_{(HS)}(\mathbb{L}^2,\mathbb{H}^m_\rho)} \le c \int\limits_{\mathbb{R}^d} \left(1+|\mathbf{k}'|^2\right)^m \mathrm{e}^{-2|\mathbf{k}'|^{2\beta}t}\, |\mathbf{k}'|^{2\beta}\, \mathrm{Tr}\, \mathcal{E}(\mathbf{k}')\, d\mathbf{k}'.
$$

In particular, taking $t=0$ we obtain the desired conclusion from (2.2). ∎

Proof of (ii). We will tread the case of a non-integer β. The case of integer β is treated in [18]. Let

$$
p_\beta(\mathbf{x}) := \int\limits_{\mathbb{R}^d} \mathrm{e}^{-|\mathbf{k}|^{2\beta}} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.
$$

Then, see e.g. [3], Theorem 2.1, p. 263, there exists a constant $C>0$ such that

$$
|p_\beta(\mathbf{x})| \le \frac{C}{(1+|\mathbf{x}|^2)^{d/2+\beta}}.
$$

We show boundedness of

$$
S_\beta(t) f(\mathbf{x}) = \frac{1}{t^{d/(2\beta)}} \int\limits_{\mathbb{R}^d} p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\, d\mathbf{y}
$$

using an interpolation argument. First consider the operator on $\mathbb{L}^\infty$. Then obviously

$$
\|S_\beta(t) f\|_{\mathbb{L}^\infty} \le \sup_{\mathbf{x}\in\mathbb{R}^d} \left[\frac{1}{t^{d/(2\beta)}} \int\limits_{\mathbb{R}^d} \left| p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right)\right| \, d\mathbf{y} \right] \|f\|_{\mathbb{L}^\infty} = \|p_\beta\|_{L^1} \|f\|_{\mathbb{L}^\infty},
$$

and the boundedness follows from $p_\beta \in L^1$. Next, consider the operator on $\mathbb{L}^1_\rho$. The boundedness of $S_\beta(t)$ on this space is equivalent with boundedness of

$$T_{\beta,\rho}(t)f(\mathbf{x}) := \frac{1}{t^{d/(2\beta)}}\int_{\mathbb{R}^d}\left(\frac{1+|\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho p_\beta\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\,d\mathbf{y}$$

on $\mathbb{L}^1$. Using an elementary inequality $(1+|\mathbf{y}|^2)^\rho \le C[(1+|\mathbf{x}|^2)^\rho + |\mathbf{x}-\mathbf{y}|^{2\rho}]$ we conclude that

$$\|T_{\beta,\rho}(t)f\|_{\mathbb{L}^1} \le I_1(f) + I_2(f),$$

where

$$\begin{aligned} I_1(f) &= \frac{C}{t^{d/(2\beta)}}\int_{\mathbb{R}^d} d\mathbf{x}\int_{\mathbb{R}^d}\left|p\left(\frac{|\mathbf{x}-\mathbf{y}|}{t^{1/(2\beta)}}\right) f(\mathbf{y})\right| d\mathbf{y},\\ I_2(f) &= \frac{C}{t^{d/(2\beta)}}\int_{\mathbb{R}^d} \frac{d\mathbf{x}}{(1+|\mathbf{x}|^2)^\rho}\int_{\mathbb{R}^d}|\mathbf{x}-\mathbf{y}|^{2\rho}\left|p\left(\frac{\mathbf{x}-\mathbf{y}}{t^{1/(2\beta)}}\right) f(\mathbf{y})\right| d\mathbf{y}. \end{aligned}$$

Then $I_1(f) = C\,\|p_\beta\|_{L^1}\|f\|_{\mathbb{L}^1}$ while

$$I_2(f) \le Ct^{(2\rho-d)/(2\beta)}\sup_{u>0}\left(u^{2\rho}|p_\beta(u)|\right)\int_{\mathbb{R}^d}\frac{d\mathbf{x}}{(1+|\mathbf{x}|^2)^\rho}\,\|f\|_{\mathbb{L}^1}, \tag{A.3}$$

where the supremum appearing in (A.3) remains finite as long as $2\rho \le d+2\beta$. In conclusion we obtain that $S_\beta(t) : \mathbb{L}^p_\rho \to \mathbb{L}^p_\rho$ is a bounded operator when $1 \le p \le \infty$. This together with the continuity of the semigroup on $\mathcal{S}_d$ proves that S_β is a C_0-semigroup on any $\mathbb{L}^p_\rho$. In fact since $S_\beta(t)$ commutes with $\nabla_{\mathbf{x}}$ we have proved that the semigroup in C_0 on any $\mathbb{W}^{p,m}_\rho$. ∎

In the proof of the third part of the proposition we need two lemmas. Write

$$\widehat{K_{\beta,\lambda}\psi}(\mathbf{k}) := |\mathbf{k}|^{2\beta}(|\mathbf{k}|^2+\lambda)^{-\beta}\hat\psi(\mathbf{k}) \quad \text{for any } \lambda>0,\ \psi\in\mathcal{S}_d. \tag{A.4}$$

Lemma 7. *$K_{\beta,\lambda}$ extends to a bounded operator $K_{\beta,\lambda} : \mathbb{H}^m_\rho \to \mathbb{H}^m_\rho$ for all $\lambda > 0$, $m \ge 1$, and $\rho \in (d/2, d/2+\gamma(\beta))$.*

Proof. By [20], Lemma 3.2.2, p. 133 the Fourier transform $\mu_{2\beta} := \mathcal{F}(|\mathbf{k}|^{2\beta}(|\mathbf{k}|^2+\lambda)^{-\beta})$ is a signed measure with a finite total variation $\|\mu_{2\beta}\|_{TV}$. Hence the operator $K_{\beta,\lambda} : \mathbb{L}^\infty \to \mathbb{L}^\infty$ is bounded and

$$|K_{\beta,\lambda}f(\mathbf{x})| \le \int_{\mathbb{R}^d}|f(\mathbf{x}-\mathbf{y})||\mu_{2\beta}|(d\mathbf{y}) \le \|f\|_{\mathbb{L}^\infty}\|\mu_{2\beta}\|_{TV},\ \mathbf{x}\in\mathbb{R}^d.$$

Using a general interpolation argument as in the proof of (ii) we note that in order to finish the proof of the lemma we only need to show that $K_{\beta,\lambda}$ is bounded on $\mathbb{L}^1_\rho$. This is equivalent with proving that

$$R_{\beta,\lambda}f(\mathbf{x}) := \int_{\mathbb{R}^d}\left(\frac{1+|\mathbf{x}-\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho f(\mathbf{x}-\mathbf{y})\mu_{2\beta}(d\mathbf{y})$$

extends to a bounded operator in $\mathbb{L}^1$. It is known, see [3], p. 134, that

$$\left(\frac{|\mathbf{k}|^2}{|\mathbf{k}|^2+\lambda}\right)^\beta = 1 + \sum_{m=1}^{[d/2]} c_m(\lambda+|\mathbf{k}|^2)^{-m} + \hat{r}(\mathbf{k}),$$

where $\hat{r}(k) \in L^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ so its inverse Fourier transform r satisfies $r(\mathbf{x}) \ll |\mathbf{x}|^{-n}$, $|\mathbf{x}| \gg 1$ for an arbitrary $n \geq 1$. Hence

$$\mu_{2\beta}(d\mathbf{x}) = \delta_0(d\mathbf{x}) + G(\mathbf{x})\,d\mathbf{x} + r(\mathbf{x})\,d\mathbf{x}$$

with

$$G(\mathbf{x}) := \sum_{m=1}^{[d/2]} c_m G_{2m}(\mathbf{x})$$

and

$$G_{2m}(\mathbf{x}) = 2^{d-2\beta}(\sqrt{\lambda}\pi)^{d-\beta}\frac{1}{\Gamma(\beta)}\int_0^{+\infty} \mathrm{e}^{-\pi|\mathbf{x}|^2/\delta}\mathrm{e}^{-\delta/(4\pi)}\delta^{(-d+2\beta)/2}\,\frac{d\delta}{\delta}.$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^d} |R_{\beta,\lambda}f(\mathbf{x})|\,d\mathbf{x}\\ &\qquad\leq \int_{\mathbb{R}^d} |f(\mathbf{x})|\,d\mathbf{x} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{1+|\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho [|G(\mathbf{x}-\mathbf{y})+r(\mathbf{x}-\mathbf{y})|]\,|f(\mathbf{y})|\,d\mathbf{x}\,d\mathbf{y}.\end{aligned}$$

The second term on the right hand side is less than or equal to

$$\|G+r\|_{\mathbb{L}^1}\|f\|_{\mathbb{L}^1} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\frac{|\mathbf{x}-\mathbf{y}|^2}{1+|\mathbf{x}|^2}\right)^\rho [|G(\mathbf{x}-\mathbf{y})|+|r(\mathbf{x}-\mathbf{y})|]\,|f(\mathbf{y})|\,d\mathbf{x}\,d\mathbf{y}. \tag{A.5}$$

(29) and (30) from p. 132 of [3] imply that $G_{2m}(\mathbf{x})|\mathbf{x}|^{2\rho} \leq C$, $\mathbf{x} \in \mathbb{R}^d$ for some constant $C > 0$, provided $\rho > d/2$. Repeating estimates performed in the proof of (ii) we bound the second term of (A.5) by

$$C\sup_{\mathbf{x}\in\mathbb{R}^d} |\mathbf{x}|^{2\rho}(|G(\mathbf{x})|+|r(\mathbf{x})|)\int_{\mathbb{R}^d}\frac{d\mathbf{x}}{(1+|\mathbf{x}|^{2\rho})}\,\|f\|_{\mathbb{L}^1} \leq C\|f\|_{\mathbb{L}^1},$$

which proves that $R_{\beta,\lambda}$ is a bounded operator on $\mathbb{L}^1$. ∎

Lemma 8. *Suppose that β, m and ρ are as in Lemma 7. Then there exists $C > 0$ such that $\|A_\beta f\|_{\mathbb{H}^m_\rho} \leq C\|f\|_{\mathbb{H}^{m+[2\beta]+1}_\rho}$ for any $f \in \mathcal{S}_d$.*

Proof. For a sufficiently large λ the operator

$$L_{\rho,\lambda} := J_{\rho,2}G_\lambda J_{-\rho,2} \qquad \text{with} \qquad \widehat{G_\lambda\psi}(\mathbf{k}) := (|\mathbf{k}|^2+\lambda)\hat{\psi}(\mathbf{k})$$

extends to a positive definite elliptic operator $L_{\rho,\lambda} : D(L_{\rho,\lambda}) \to \mathbb{L}^2$. It is known, see e.g. [17], Theorem 6.10, p. 73, that $\|L^{\beta}_{\rho,\lambda} f\|_{\mathbb{H}^m} \leq C\|f\|_{\mathbb{H}^{m+[2\beta]+1}}$. Hence, by Lemma 7,

$$\begin{aligned}\|A_\beta f\|_{\mathbb{H}^m_\rho} &= \|K_{\beta,\lambda} G_\lambda f\|_{\mathbb{H}^m_\rho} \leq C\|G_\lambda f\|_{\mathbb{H}^m_\rho} \leq C\|L^{\beta}_{\rho,\lambda} J_{\rho,2} f\|_{\mathbb{H}^m} \leq C\|J_{\rho,2} f\|_{\mathbb{H}^{m+[2\beta]+1}} \\ &\leq C\|f\|_{\mathbb{H}^{m+[2\beta]+1}_\rho}. \quad \blacksquare\end{aligned}$$

Proof of (iii). From (ii) we conclude that $S_\beta(t)(\mathcal{C}_\rho) \subseteq \mathcal{C}_\rho$ and $S_\beta(t)(\mathcal{H}_\rho) \subseteq \mathcal{H}_\rho$ for all $t \geq 0$. Hence, in a consequence of Proposition 3.3 of [8], $\mathcal{C}_\rho$ and $\mathcal{H}_\rho$ are cores of A_β. We prove now that $\mathcal{S}_d$ is also a core of A_β. Let us consider first the case when β is a positive integer. Then, obviously $S_\beta(t)(\mathcal{S}_d) \subseteq \mathcal{S}_d$, $t \geq 0$ and our claim follows from Proposition 3.3 of [8]. Suppose that $\beta \notin \mathbb{Z}$. Since $\mathcal{S}_d$ is a core of $A_{[\beta]+m+1}$ for any $f \in \mathcal{H}_\rho$ there exists a sequence $f_n \in \mathcal{S}_d$, $n \geq 1$ such that $f_n \to f$, $A_{[\beta]+m+1} f_n \to A_{[\beta]+m+1} f$, in $\mathbb{H}^m_\rho$, as $n \uparrow +\infty$. A computation analogous to that made in (i) shows that also sequence $A_\beta f_n$, $n \geq 1$ is convergent in $\mathbb{H}^m_\rho$. Hence from closedness of A_β we conclude that $A_\beta f_n \to A_\beta f$, as $n \uparrow +\infty$, in $\mathbb{H}^m_\rho$. Our claim holds thanks to the fact that $\mathcal{H}_\rho$ is a core of A_β.

We prove now that $\mathbb{H}^{m+[2\beta]+1}_\rho \subseteq D(A_\beta)$. Let $f \in \mathbb{H}^{m+[2\beta]+1}_\rho$. Thanks to the fact that $\mathcal{S}_d$ is the core of A_β we can approximate f, in the graph norm of the operator, by elements $\psi_n \in \mathcal{S}_d$, $n \geq 1$. Then $f \in D(A_\beta)$ as a consequence of Lemma 8. $\blacksquare$

Proof of (iv). We only consider the case of non-integer β, β'. Let

$$q_{\beta,\beta'}(|\mathbf{x}|, t) := \int\limits_{\mathbb{R}^d} |\mathbf{k}|^{2\beta'} \mathrm{e}^{-|\mathbf{k}|^{2\beta} t} \mathrm{e}^{\mathrm{i}\,\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{k}.$$

Then

$$A_{\beta'} S_\beta(t)\psi(\mathbf{x}) = \int q_{\beta,\beta'}(|\mathbf{x}-\mathbf{y}|, t)\psi(\mathbf{y})\, d\mathbf{y}$$

for any $\psi \in \mathcal{S}_d$. For any integer $M \geq 1$ one can choose N sufficiently large and $c_{m,n} \in \mathbb{R}$, $m, n = 0, \cdots, N$ such that $c_{0,0} = 0$ and

$$H(\mathbf{k}) := (|\mathbf{k}|^{2\beta'} + 1)\mathrm{e}^{-|\mathbf{k}|^{2\beta}} - \sum_{m,n=0}^{N} c_{m,n} \mathrm{e}^{-|\mathbf{k}|^{2(m\beta'+n\beta)}}$$

is M-times differentiable and $|H(\mathbf{k})| \leq C\mathrm{e}^{-|\mathbf{k}|^{2\gamma}}$, where $\gamma = \beta \wedge \beta'$. Hence

$$q_{\beta,\beta'}(|\mathbf{x}|, 1) \leq \frac{C}{(1 + |\mathbf{x}|^2)^{d/2+\gamma}}.$$

Repeating now the argument leading to estimate of the $\mathbb{H}^m_\rho$-norm of $S_\beta(t)f$, $f \in \mathbb{H}^m_\rho$, see the calculations made in the proof of (ii), we obtain the first estimate. The second inequality follows from calculations identical with those made in the proof of (i). $\blacksquare$

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