ELSEVIER

Physica D 136 (2000) 145–174

PHYSICA D

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Phase diagram for turbulent transport: sampling drift, eddy diffusivity and variational principles

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Received 12 March 1998; received in revised form 1 February 1999; accepted 4 June 1999
Communicated by U. Frisch

Abstract

We study the long-time, large scale transport in a three-parameter family of isotropic, incompressible Gaussian velocity fields with power-law spectra. Scaling law for transport is characterized by the scaling exponent q and the Hurst exponent H, as functions of the parameters. The parameter space is divided into regimes of scaling laws of different *functional forms* of the scaling exponent and the Hurst exponent. We present the full three-dimensional phase diagram. The limiting process is one of three kinds: Brownian motion ($H = 1/2$), persistent fractional Brownian motions ($1/2 < H < 1$) and regular (or smooth) motion ($H = 1$). We discover that a critical wave number divides the infrared cut-offs into three categories, critical, subcritical and supercritical; they give rise to different scaling laws. We introduce the notions of sampling drift and eddy diffusivity, and formulate variational principles to estimate the eddy diffusivity. We show that the fractional Brownian motions result from a dominant sampling drift. ©2000 Elsevier Science B.V. All rights reserved.

MSC: 76M35; 76R50; 76F05; 60G60.

Keywords: Sampling drift; Critical wave number; Eddy diffusivity, Variational principles; Fractional Brownian motion

1. Introduction

The movement of a passive scalar in a turbulent flow is described by the stochastic differential equation

$$\mathrm{d}\boldsymbol{x}(t) = \mathbf{V}(\mathbf{x}(t), t, \omega)\,\mathrm{d}t + \sqrt{2\kappa}\,\mathrm{d}\boldsymbol{w}(t), \qquad \mathbf{x}(0) = 0,$$

where $\mathbf{x}(t)$ is the position of the particle at time t, $\kappa \geq 0$ the molecular diffusivity, $\boldsymbol{w}(t)$ the standard Brownian motion and $\mathbf{V}(\mathbf{x}, t, \omega)$ a time-stationary, space-homogeneous, incompressible velocity field. Here ω denotes an element of an ensemble of random flows.

We are concerned with the long-time, large scale behavior of the displacement $\mathbf{x}(t)$. To this end, we study the scaling limit

$$\mathbf{x}^{\varepsilon}(t) = \varepsilon\mathbf{x}(t/\varepsilon^{2q}), \quad \text{as } \varepsilon \to 0, \tag{1}$$

0167-2789/00/$ – see front matter
PII: S0167-2789(99)00150-5

with suitable $q > 0$. The scaling exponent q characterizes the timescale associated with transport on the spatial observation scale $1/\varepsilon$. The equation for the rescaled displacement (1) becomes

$$\mathrm{d}\mathbf{x}^{\varepsilon}(t) = \varepsilon^{1-2q}\mathbf{V}(\mathbf{x}^{\varepsilon}(t)/\varepsilon, t/\varepsilon^{2q})\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\kappa}\,\mathrm{d}\boldsymbol{w}(t). \tag{2}$$

When molecular diffusion is evidently negligible, we set $\kappa = 0$ to simplify the equation

$$\mathrm{d}\mathbf{x}^{\varepsilon}(t) = \varepsilon^{1-2q}\mathbf{V}(\boldsymbol{x}^{\varepsilon}(t)/\varepsilon, t/\varepsilon^{2q})\,\mathrm{d}t. \tag{3}$$

The effect of molecular diffusion is discussed where the issue arises and in the concluding remarks.

Motivated by existing diffusion limit theorems for steady flows with finite-range spatial correlations [14,19], on one hand, and those for temporally mixing flows with long-range, spatial correlations [9,15,31], on the other hand, we consider turbulent transport in a class of random flows with power-law spectra parameterized by α, β, γ (see Section 2 for details).

Roughly speaking, the velocity field $\mathbf{V}$ is time-stationary, space-homogeneous and Gaussian. Its two-point correlation functions $\mathbf{R} = [R_{ij}]$, $R_{ij}(\mathbf{x}, t) = \langle V_i(\cdot, \cdot)V_j(\cdot + \mathbf{x}, \cdot + t)\rangle$, are given by the Fourier transform $R_{ij}(\mathbf{x}, t) = \int \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}}\,\hat{R}_{ij}(\mathbf{k}, t)\,\mathrm{d}\mathbf{k}$ with

$$\hat{R}_{ij}(\mathbf{k}, t) = \rho(|\mathbf{k}|^{2\beta}t)\varepsilon(\mathbf{k})(\delta_{ij} - k_ik_j|\mathbf{k}|^{-2})|\mathbf{k}|^{1-d} \tag{4}$$

where ρ is the time correlation (relaxation) function and ε the energy (shell) spectrum given by a power-low

$$\mathcal{E}(\mathbf{k}) = E_0|\mathbf{k}|^{1-2\alpha}, \quad E_0 > 0.$$

Here $\langle\cdot\rangle$ denotes the ensemble average. The factor $(\delta_{ij} - k_ik_j|\mathbf{k}|^{-2})$ ensures that the flow is divergence-free. If ρ is an *exponential* function, $\mathrm{e}^{-a_0|\mathbf{k}|^{2\beta}t}$, then the velocity field is an Ornstein-Uhlenbeck process which is Markovian.

Note that the spectrum is not integrable near $|\mathbf{k}| = \infty$ or $|\mathbf{k}| = 0$ for $\alpha \le 1$ or $\alpha \ge 1$, respectively. The infrared divergence (small $|\mathbf{k}|$) of the integral of velocity energy spectrum indicates non-homogeneous velocity and thus violates the space homogeneity assumption, whereas the ultraviolet divergence (large $|\mathbf{k}|$) of the integral would make the velocity a generalized, rather than ordinary, function (i.e., a distribution). To remove divergence in the spectrum, we introduce an ultraviolet cut-off

$$\mathcal{E}(\mathbf{k}) = 0, \quad |\mathbf{k}| > K, \qquad \text{for}\,\alpha \le 0, \tag{5}$$

and a infrared cut-off

$$\mathcal{E}(\mathbf{k}) = 0, \quad |\mathbf{k}| < \delta \ll 1, \qquad \text{for}\ \ \alpha \ge 1. \tag{6}$$

In the case of $\alpha < 1$, the energy containing scale is at the ultraviolet cut-off; in the case of $\alpha > 1$, the energy containing scale is at the infrared cut-off. It is convenient to write the cut-off energy spectrum as

$$\mathcal{E}(\mathbf{k}) = E_0|\mathbf{k}|^{1-2\alpha}I(|\mathbf{k}|), \tag{7}$$

where $I(|\mathbf{k}|)$ is the characteristic function of $[0, K]$, for $\alpha < 1$, of $[\delta, \infty)$, for $\alpha > 1$, and of $[\delta, K]$, for $\alpha = 1$ (see [20]). When we study the effect of an infrared cut-off, we will take an infrared cut-off $\delta = \varepsilon^{\gamma} > 0$.

If the infrared cut-off $\delta > 0$ is fixed, independent of ϵ, then the flow is mixing in time (i.e., correlation time is uniformly bounded, independent of wave number), and consequently, the scaling in (1) is diffusive, $q = 1$, and the limit is a Brownian motion [15]. In the case of the Kolmogorov–Obukov spectrum ($\alpha = 4/3$, $\beta = 1/3$, see [20,28]), $[\delta, K]$ represents the inertial range where K^{-1} is the dissipation length and δ^{-1} is the integral length. In general, the infrared cut-off is determined by the scale of external forcing and the size of physical domain. By letting δ to change with ϵ, as $\delta = \varepsilon^{\gamma}$, $\gamma > 0$, we vary the spatial scale of observation $1/\epsilon$ in relation to, e.g., the size of physical domain.

If the scaling limit exists, *statistically* independent of the initial point, and has stationary increments, then the transport process is said to be *homogenized*, and the up-scaling, or coarse-graining, procedure represented by (1) is called *homogenization*. The scaling is diffusive if $q = 1$, superdiffusive if $q < 1$, subdiffusive if $q > 1$. Sub- and super-diffusions are called anomalous diffusion.

The limit $\mathbf{Z}(t)$ may be Gaussian or non-Gaussian, Markovian or non-Markovian, even if the velocity field is Gaussian and Markovian. In general $\mathbf{Z}(t)$ has stationary increments as does $\boldsymbol{x}^{\varepsilon}(t)$ [39]. If $\mathbf{Z}(t)$ is self-similar and Gaussian then it can be characterized by a unique Hurst exponent H in its autocovariance function

$$\mathrm{Cov}(\mathbf{Z}(t_1), \mathbf{Z}(t_2)) = \tfrac{1}{2}\mathbf{C}\{|t_1|^{2H} + |t_2|^{2H} - |t_1 - t_2|^{2H}\}, \quad 0 < H \leq 1 \tag{8}$$

where $\mathbf{C}$ is the variance of $\mathbf{Z}(1)$. $H = 1$ corresponds to a *regular* (or *smooth*) motion: $H = 1/2$ a Brownian motion, $\mathbf{B}(t)$. Any other H corresponds to a fractional Brownian motion (FBM), $\mathbf{B}_H(t)$, which, after normalization, can be represented as

$$\mathbf{B}_H(t) = \int_{-\infty}^{0} (|t - t'|^{H-1/2} - |t'|^{H-1/2})\,\mathrm{d}\mathbf{B}(t') + \int_0^t |t - t'|^{H-1/2}\,\mathrm{d}\mathbf{B}(t'), \qquad 0 < H < 1, \tag{9}$$

as introduced in [36]. Eq. (9) defines the only mean-zero, continuous, Gaussian process that is self-similar (or self-affine), with the Hurst exponent H, has stationary increments, and satisfies $\mathbf{B}_H(0) = 0$ (see [41]). FBMs found in the present study are all persistent in the sense that $H > 1/2$. It is worth noting that, for $\alpha \geq 1$ with the critical cut-off ($\gamma = \gamma_c$), the limit process is not self-similar (cf. Regimes II$'$,III$'$ and IV$'$).

Non-Markovian limits are related to non-local homogenization [5,11,43,44]. Previously non-local homogenization has been shown to arise as a result of fast oscillation, rather than a scaling limit.

If $\mathbf{Z}(t)$ is non-Gaussian, then there may be a hierarchy of Hurst exponents corresponding to higher moments of the process. When the sequence of Hurst exponents diverges as the order of moment increases, the limit is *intermittent*. Intermittency effect may also manifest in multiple scaling exponents. We do not consider the problem of intermittency here.

In this paper, we do not address directly the question of existence and uniqueness of the scaling limit. Rather, we assume the existence and uniqueness of a non-trivial scaling limit, and seek to identify the scaling exponent and the second order Hurst exponent (the Hurst exponent, in the case of a Gaussian limit). In doing so, we point out relevant mathematical results that exist, or can be proved. We try to present a coherent physical picture of the whole phase diagram. To enhance our case, we often analyze the problems from several different perspectives.

The exponent q characterizes the time scale associated with transport observed on the space scale $1/\varepsilon$; the exponent H characterizes the time correlation property of successive increments on the observation scale (and therefore, the roughness of the limiting sample paths). Naturally, we ask if the dimensionally correct relation

$$H = 1/(2q) \tag{10}$$

holds? When (10) holds the limit process is invariant under the same scaling transformation (1). It turns out that relation (10) generally holds for $\alpha < 1$ but fails for $\alpha > 1$. If an additional infrared cut-off is made in the case of $\alpha < 1$ and if the cut-off is removed slower than some *critical* wave number, k_c, then (10) does not hold. In these situations, the inequality

$$H < 1/(2q) \tag{11}$$

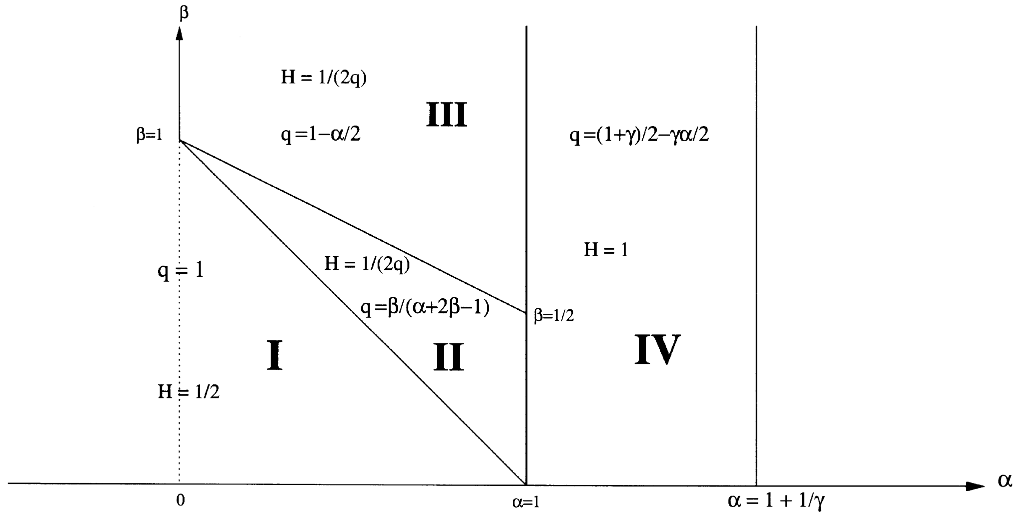

Fig. 1. Phase diagram with supercritical cut-offs: $\gamma > \max\{1/(\alpha + 2\beta - 1), 1\}$.

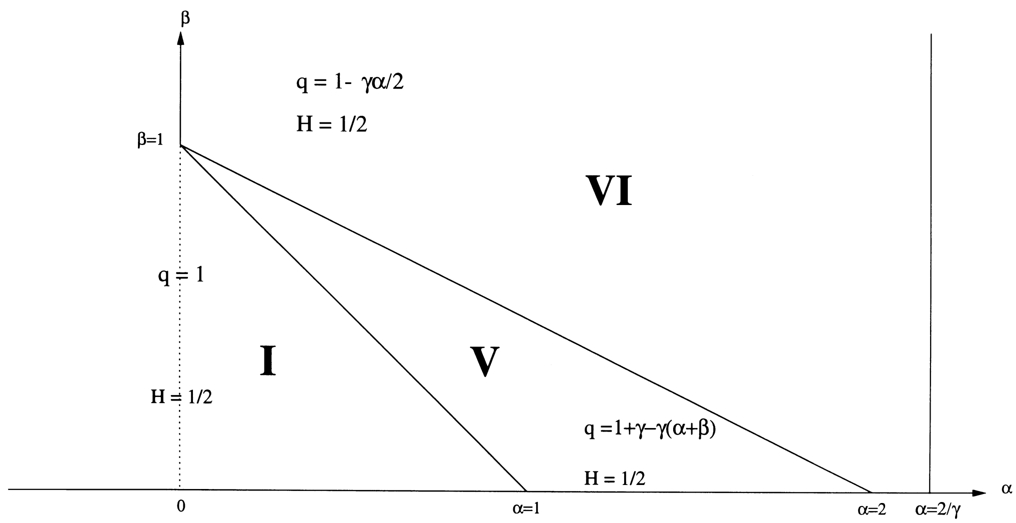

Fig. 2. Phase diagram with subcritical cut-offs: $\gamma > \max\{1/(\alpha + 2\beta - 1), 1\}$.

is in the place of (10). The inequality (11) is due to the fact that $2H$ characterizes the *covariances*, whereas $1/q$ characterizes the *variances*, of successive increments of turbulent motion on the observation scale.

In general the exponents q, H depend on the parameters α, β and the cut-off $\delta = \varepsilon^{\gamma}$ and can be expressed explicitly as functions of α, β, γ. Here it may be helpful to draw analogy to critical phenomena in statistical physics: we think of α, β, γ as order parameters, the scaling limit $\varepsilon \to 0$ as thermodynamic limit and the exponents q, H given by formulas of α, β, γ as phases. The phase diagram divides the space of order parameters, α, β, γ into regions associated with different formulas for q, H. Our results are summarized in Figs. 1–4. Since there are three parameters, the full phase diagram is three-dimensional. To simplify the presentation, we choose to portray the full diagram as several two-dimensional diagrams.

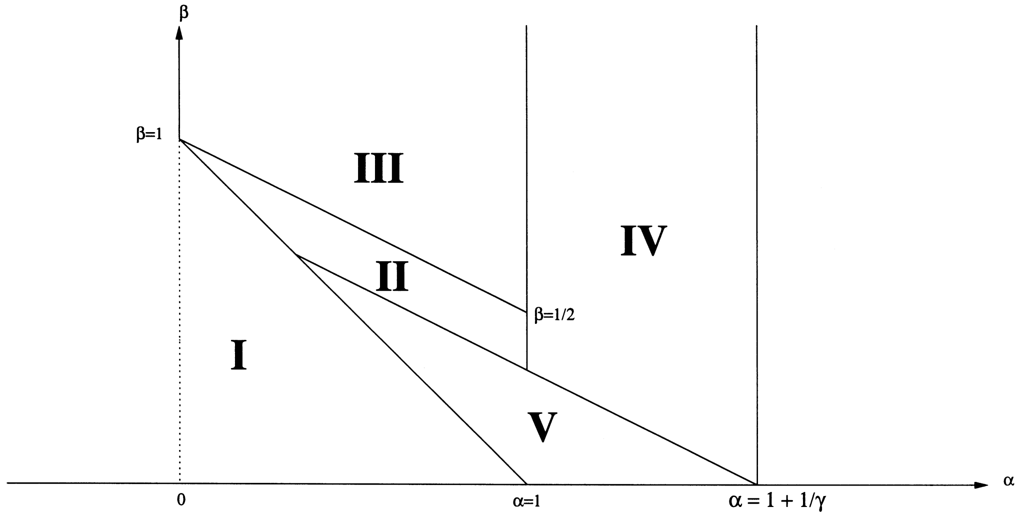

Fig. 3. Phase diagram at any cross section $\gamma = \text{constant} > 1$.

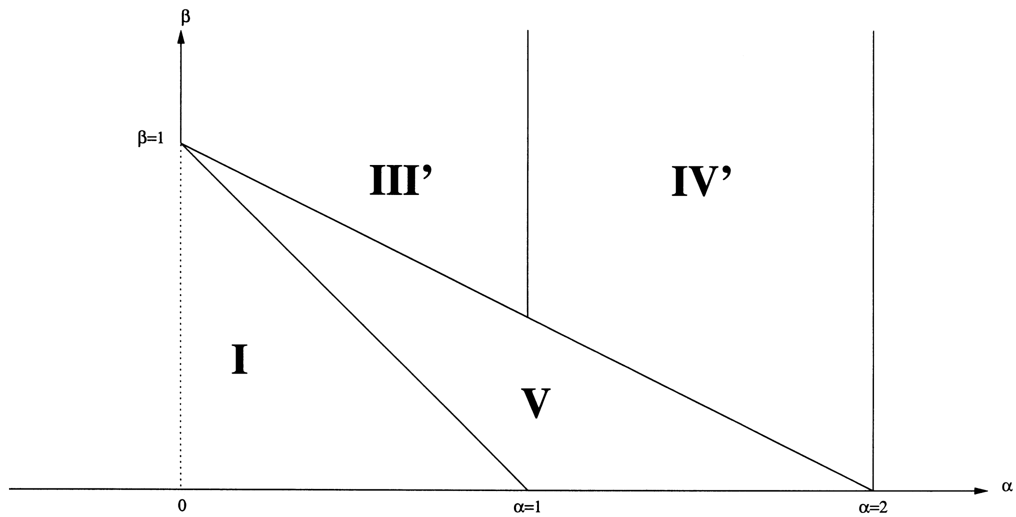

Fig. 4. Phase diagram at any cross-section $\gamma = 1$.

Note also that the phase diagrams are different from those in statistical mechanics in that our phases are continuum, not discrete: except for the diffusive regime, where $H = 1/2, q = 1$, H, q change from point to point, continuously or discontinuously. But their functional forms in relation to α, β, γ are discrete and divided by phase boundaries.

Phase diagram was first used by Avellaneda and Majda [1,2] to present scaling limits of turbulent transport in anisotropic, stratified flows of the form $\mathbf{V}(\mathbf{x}, t) = (v(x_2, t), 0)$, with $\mathbf{x} = (x_1, x_2)$. A different diagram for the same shear-layer flows was rigorously obtained by Zhang and Glimm [46] using a different approach. In the current paper we consider *isotropic* turbulent flows in two or three dimensions, although the results are applicable to the case of shear-layer flows when interpreted appropriately. Also, we do not attempt to derive the results rigorously here. Often we refer to existing theorems to indicate how in principle results may be proved, subject to technical modification, and to support the physical arguments invoked; they are not intended to be mathematical proofs. The proofs of many of the results are very technical and will be published elsewhere.

The effect of an infrared cut-off depends on whether the cut-off is *subcritical or supercritical*. For $\alpha < 1$, a supercritical cut-off, $\gamma > \gamma_c = \max\{1, 1/(\alpha + 2\beta - 1)\}$, does not affect the scaling law. For $\alpha \geq 1$, because the infrared cut-off corresponds to the energy-containing scale, the scaling limit is dominated by the infrared cut-off.

The supercritical diagram (Fig. 1) includes:

- Regime I: $\alpha + \beta < 1$ or $\alpha < 0$. The scaling is diffusive, $q = 1$, and the limit is a Brownian motion, $H = 1/2$.
- Regime II: $\alpha+\beta > 1, \alpha+2\beta < 2, \alpha < 1,\ \gamma > 1/(\alpha+2\beta-1)$. A FBM regime with the space-freezing property that the velocity dependence on space is negligible. The scaling is superdiffusive, $q = \beta/(\alpha + 2\beta - 1)$, and the limit is a fractional Brownian motion, with $H = 1/(2q)$.
- Regime III. $\alpha + 2\beta > 2, 0 < \alpha < 1, \gamma > 1$. A FBM regime with the time-freezing property that the velocity dependence on time is negligible. The scaling is superdiffusive, $q = 1-\alpha/2$, and the limit is a fractional Brownian motion, with $H = 1/(2q)$.
- Regime IV: $1 \leq \alpha < 2, \gamma > \max\{1, 1/(\alpha + 2\beta - 1)\}$. A regular (or smooth) motion regime with both the space-freezing and the time-freezing properties. The scaling is superdiffusive, $q = (1 + \gamma)/2 - \gamma\alpha/2$, and the limit is regular ($H = 1$).

The relation (10) is satisfied in all but Regime IV.

In the case of subcritical cut-offs, $\gamma < \gamma_c = \max\{1, (\alpha + 2\beta - 1)^{-1}\}$, the number of regimes shrinks as the significance of low wave numbers is reduced: part of Regime IV merges with Regime II, and part of Regime IV merges with Regime III. The scaling exponent now depends on the cut-off exponent γ explicitly. The limit is universally a Brownian motion across all regimes.

The subcritical diagram Fig. 2 includes:

- Regime I remains intact.
- Regime V: $\alpha+\beta > 1, \alpha+2\beta < 2, \gamma < 1/(\alpha+2\beta-1)$. Velocity decorrelation in time dominates the transport. The scaling is superdiffusive, $q = 1 + \gamma - \gamma(\alpha + \beta)$.
- Regime VI: $\alpha + 2\beta > 2, 0 < \alpha < 2/\gamma, \gamma < 1$. Velocity decorrelation in space dominates the transport. The scaling is superdiffusive, $q = 1 - \gamma\alpha/2$.

Finally, there are three regimes associated with critical cut-offs for which the limit process is not self-similar, and thus, the Hurst exponent is not well-defined (see Figs. 3 and 4).

- Regime II′: $\alpha + \beta > 1, \alpha + 2\beta < 2, 0 < \alpha < 1$ with $\gamma = (\alpha + 2\beta - 1)^{-1}$.
- Regime III′: $\alpha + 2\beta > 2, 0 < \alpha < 1$ with $\gamma = 1$.
- Regime IV′: $1 < \alpha < 1 + 1/\gamma$ with $\gamma = \max\{1, (\alpha + \beta - 1)^{-1}\}$.

Part of Regimes V and VI was first studied by Avellaneda and Majda [3] (see also [21,22]). The main difference in assumption and setup between this work and [3] is that they considered a partial diagram ($0 < \beta < 1/2, 0 < \alpha < 2$) with an infrared cut-off $\gamma = 1 \leq \gamma_c$ (see also [38]). Fig. 4 is a generalization of their work. The phase diagram of [23] was obtained entirely by certain scaling arguments, and is restricted to two dimensions.

In contrast to previous results [1–3,46], our main findings are: (i) the transport effect of the sampling drift and related critical wave number, which are introduced for the first time, (ii) fractional Brownian motion limit as a result of the critical wave numbers, (iii) the effect of infrared cut-offs, and (iv) the formulation of cut-off dependent eddy diffusivity and its associated variational principle without molecular diffusion. The variational principle gives a useful bound for the eddy diffusively.

The organization of the paper is as follows. In Section 2, we define the three-parameter family of Gaussian flows, whose transport properties are discussed in subsequent sections. In Section 3, we introduce the notions of sampling drift, critical wave numbers and eddy diffusively. We also formulate the variational principles that lead to general bounds for the cut-off dependent eddy diffusivity in terms of a fractional vector potential of the velocity field. Since the transition from ultraviolet to infrared cut-off in velocity occurs at $\alpha = 1$, we divide the discussion accordingly into two cases: $\alpha < 1$ and $\alpha > 1$. We consider the case $\alpha < 1$ in Section 4 and the case $\alpha > 1$ in Section 5. We conclude with various remarks in Section 6. In Appendix A we derive a variational principle for the cut-off dependent eddy diffusivity, without the presence of molecular diffusion.

2. Random velocity field

In this section, we describe some mathematical properties of the random velocity fields considered in this paper.

The most important property is stationarity in time and homogeneity in space (space–time stationarity for short), without which homogenization is unlikely to hold. It should be noted that, when formulated in a general, abstract framework as we will do momentarily, space–time stationarity encompasses space–time periodicity, quasi-periodicity and almost periodicity as well as random stationarity. This abstract formulation is also handy for formulating the variational principle for the eddy diffusivity (Section 3.2). Elsewhere, the paper can be understood without referring to the abstract formulation.

The variational principle in the absence of molecular diffusivity also uses explicitly the Markov property of the flow and the associated generator. A key turbulent diffusion theorem cited in the discussion of the diffusive regime (Section 4.1) was proved for certain Markovian velocity fields. For Markovian flows, the mixing property conveniently corresponds to the spectral gap of the generator. Elsewhere, the Markov property is not used explicitly and probably not needed.

Since we only use the spectral density explicitly in presenting the phase diagrams, it is safe to assume that the velocity fields are Gaussian. In particular, the Gaussian property is essential in the fractional-Brownian-motion regimes (II and III). Elsewhere, the Gaussian property is probably not important.

Let us begin with the abstract formulation of space–time stationarity, upon which we will define the Gaussian and Markov properties. Let $\boldsymbol{\Omega}$ be the space of steady, space-homogeneous velocity field and let P be a probability measure on $\boldsymbol{\Omega}$. Homogeneity in space can be canonically described by the invariance of the distribution P under the group of translations $\{\boldsymbol{\tau}_{\mathbf{x}}\}_{\mathbf{x}\in R^d}$ acting on $\boldsymbol{\Omega}$. We further assume that P is ergodic with respect to $\{\boldsymbol{\tau}_{\mathbf{x}}\}_{\mathbf{x}\in R^d}$ in the sense that the only invariant, measurable functions on $\boldsymbol{\Omega}$ under $\{\boldsymbol{\tau}_{\mathbf{x}}\}_{\mathbf{x}\in R^d}$ are constants. The measure P dictates the correlation of the velocity field in *space*, and in the case of Gaussian velocity fields, is determined by the energy spectrum.

Alternatively, we think of $\boldsymbol{\Omega}$ as the ensemble of elements ω, representing the randomness of the velocity field, which is distributed according to the measure P. A (prototypical) random velocity field is a vector-valued, random variable (i.e., a function on $\boldsymbol{\Omega}$), denoted by $\tilde{\mathbf{V}}(\omega)$. The realization or the sample of the (time independent) velocity field, $\mathbf{V}(\mathbf{x}, \omega)$, is the translate of $\tilde{\mathbf{V}}(\omega)$ on $\boldsymbol{\Omega}$, i.e., $\mathbf{V}(\mathbf{x}, \omega) = \tilde{\mathbf{V}}(\boldsymbol{\tau}_{\mathbf{x}}\omega)$. Since the measure P is invariant under the translations, the resulting velocity fields are space-homogeneous. We assume that $\tilde{\mathbf{V}}$ has zero mean

$$\langle \tilde{\mathbf{V}} \rangle = 0$$

and zero divergence

$$\nabla \cdot \tilde{\mathbf{V}} = 0, \quad \nabla = (\partial_1, \partial_2, \dots, \partial_d).$$

Partial derivative ∂_i is the infinitesimal generator of the subgroup of translation $\{\boldsymbol{\tau}_{x_i}\}_{x_i\in R}$. The Laplacian $\Delta := \nabla \cdot \nabla$ is defined as usual. As before, $\langle \cdot \rangle$ denotes the ensemble average with respect to the distribution P.

The time dependence of the velocity field is then introduced as a time-stationary stochastic process, $\omega(t)$, on the space $\boldsymbol{\Omega}$, which preserves the measure P. In other words, P is an invariant measure of the process $\omega(t)$. The realization of time dependent velocity field is then given by

$$\mathbf{V}(\mathbf{x}, t, \omega) = \tilde{\mathbf{V}}(\boldsymbol{\tau}_{\mathbf{x}}\omega(t)), \quad \omega(0) = \omega.$$

In this formulation, the temporal properties are conveniently separated from the spatial properties of the velocity field. Additional structures such as Gaussianity and Markovianity can be added on by imposing corresponding properties on P and $\omega(t)$. The space $\boldsymbol{\Omega}$ is usually infinite dimensional in suitable coordinates such as Fourier modes. This formulation is sufficiently general to describe periodic, quasi-periodic, almost periodic as well as random homogeneous velocity field (see, e.g., [10]).

We think of a Markovian velocity field as a sample path in $\boldsymbol{\Omega}$ of a Markov process $\omega(t)$. A Markovian, Gaussian velocity field corresponds to an *exponential* time correlation function ρ in (4) and admits the spectral representation

$$\mathbf{V}(\mathbf{x}, t) = \int_{R^d} \mathrm{e}^{i2\pi\mathbf{k}\cdot\mathbf{x}}\hat{\mathbf{V}}(\mathrm{d}\mathbf{k}, t),$$

where the stochastic measure $\hat{\mathbf{V}}(\mathrm{d}\mathbf{k}, t)$ is an Ornstein-Uhlenbeck process

$$\mathrm{d}_t\hat{\mathbf{V}}(\mathrm{d}\mathbf{k}, t) = -a_0|\mathbf{k}|^{2\beta}\hat{\mathbf{V}}(\mathrm{d}\mathbf{k}, t)\,\mathrm{d}t + |\mathbf{k}|^{\beta}\mathcal{E}^{1/2}(\mathbf{k})(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2})^{1/2}|\mathbf{k}|^{(1-d)/2}\mathbf{W}(\mathrm{d}\mathbf{k}, \mathrm{d}t) \tag{12}$$

and can be conveniently expressed in terms of Gaussian white noise $\mathbf{W}(\mathrm{d}\mathbf{k}, \mathrm{d}s)$

$$\hat{\mathbf{V}}(\mathrm{d}\mathbf{k}, t) = \int_{-\infty}^{t} \mathrm{e}^{-a_0|\mathbf{k}|^{2\beta}(t-s)}|\mathbf{k}|^{\beta}\mathcal{E}^{1/2}(\mathbf{k})(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2})^{1/2}|\mathbf{k}|^{(1-d)/2}\mathbf{W}(\mathrm{d}\mathbf{k}, \mathrm{d}s).$$

The Ornstein-Uhlenbeck process (12) has an invariant measure P that is a Gaussian distribution with zero mean and the variance matrix $\mathbf{R} = [\hat{R}_{ij}]$ given by (4). Then the exponential relaxation function corresponds to a generator $\mathcal{A}$ of the form

$$\mathcal{A} = \left(-\frac{1}{4\pi^2}\Delta\right)^{\beta}\mathcal{A}_0, \qquad \beta \geq 0, \tag{13}$$

where $\mathcal{A}_0$ is the generator of the process

$$\mathrm{d}_t\hat{\mathbf{V}}_0(\mathrm{d}\mathbf{k}, t) = -a_0\hat{\mathbf{V}}_0(\mathrm{d}\mathbf{k}, t)\,\mathrm{d}t + \mathcal{E}^{1/2}(\mathbf{k})(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2})^{1/2}|\mathbf{k}|^{(1-d)/2}\mathbf{W}(\mathrm{d}\mathbf{k}, \mathrm{d}t). \tag{14}$$

The operator $\mathcal{A}_0$ is symmetric with respect to the measure P and commutes with the translation $\tau_{\mathbf{x}}$, $\forall\mathbf{x} \in R^d$ because of homogeneity. As the process (14) is a time change of (12) and different wave numbers are independent, the measure P remains invariant with respect to (14). Also, because the time correlation function for (14) is exponential with an exponent a_0 uniformly bounded above zero, $\mathcal{A}_0$ has a spectral gap

$$-\langle\mathcal{A}_0 f f\rangle \geq a_0\langle f^2\rangle, \qquad a_0 > 0, \tag{15}$$

for all functions f, $\langle f\rangle = 0$, in the domain of $\mathcal{A}_0$.

The motion in such a temporally stationary, Markovian flow is also a temporally stationary, Markov process whose generator is

$$\mathcal{L} = \mathcal{A} + \tilde{\mathbf{V}}\cdot\nabla \tag{16}$$

when molecular diffusion is absent, and is

$$\mathcal{L} = \mathcal{A} + \kappa\Delta + \tilde{\mathbf{V}}\cdot\nabla \tag{17}$$

when molecular diffusion is present [15].

Now we make an observation which will be helpful in assessing the role of molecular diffusion. The generator (17) in conjunction with (13) and (15) suggests that the presence of molecular diffusion introduces a mechanism of generating a Lagrangian correlation in time comparable to $\beta = 1$ in the Eulerian correlation in time. For $\beta < 1$, the generator $\mathcal{A}$ dominates over $\kappa\Delta$ for low wave numbers, and if a fixed ultraviolet cut-off is also present, are also comparable to $\kappa\Delta$ for the other wave numbers. Thus, the effect of molecular diffusion is negligible for $\beta \le 1$ and $\alpha < 1$ in the limit of high Peclet number ($\kappa \to 0$).

In the sequel we shall use the notation of the fractional gradient of order β,

$$\nabla^{\beta} := (-\Delta)^{(\beta-1)/2}\nabla.$$

3. Transport properties of various wave numbers

To study motion in a flow with a power-law energy spectrum over a wide range of scales, it is convenient to decompose the energy spectrum into the *sampling drift* and the *fluctuating* velocity field, and to consider separately their distinctive transport properties. The relation between the sampling drift and the fluctuating velocity field is like that between a mean flow and the fluctuation.

3.1. Sampling drift and critical wave numbers

For each realization of random velocity field there is a non-zero sampling drift due to random fluctuation, depending on the scale of observation.

The volume-averaged flow on the observation scale consists of spatially non-fluctuating wave numbers on the observation scale, namely, essentially all $|\mathbf{k}| = O(\epsilon)$. The volume-averaged flow comprises three kinds of wave numbers: *supercritical, critical* and *subcritical* wave numbers depending on their variations in time on the observation scale. Critical and supercritical wave numbers compose the *sampling drift*.

The supercritical wave numbers are effectively uniform in time as well as in space in the sense that their correlation times are much larger than the timescale of observation, $|\mathbf{k}|^{-2\beta} \gg \varepsilon^{-2q}$, and satisfy

$$|\mathbf{k}| \ll \min\{\varepsilon^{q/\beta}, \varepsilon\}. \tag{18}$$

As such, they behave like a constant drift on the observation scale and transport particles *ballistically*. Among them, we pay special attention to those wave numbers that, on their own correlation timescales, transport particles over a distance larger than the observation scale, i.e.,

$$|\mathbf{k}|^{1-\alpha}|\mathbf{k}|^{-2\beta} \gg 1/\varepsilon, \tag{19}$$

where

$$\left(\int_{c_1|\mathbf{k}|\le|\mathbf{k}'|\le c_2|\mathbf{k}|} \mathcal{E}(\mathbf{k}')\,\mathrm{d}|\mathbf{k}'|\right)^{1/2} \sim |\mathbf{k}|^{1-\alpha} \qquad |\mathbf{k}| \ll 1, \tag{20}$$

is the amplitude associated with wave numbers of order $|\mathbf{k}| \ll 1$. Note that, for (19) to define a non-empty set of low wave numbers, we need $\alpha + 2\beta > 1$. For $\alpha + 2\beta \le 1$, the supercritical wave numbers do not contribute to the transport on the observation scale and are negligible asymptotically; for this reason they are refferred to as insignificant wave numbers.

Since we do not know the scaling exponent q ahead of time, we define the *critical* wave numbers to be the boundary of those *significant* supercritical wave numbers. Thus, the critical wave numbers are of the order $k_c = \varepsilon^{\gamma_c}$, with

$$\gamma_c = \max\{1, (\alpha+2\beta-1)^{-1}\} = \begin{cases} (\alpha+2\beta-1)^{-1}, & \text{for } 1<\alpha+2\beta<2, \\ 1, & \text{for } \alpha+2\beta \geq 2. \end{cases} \tag{21}$$

By (21), for $\alpha+2\beta>2$, the sampling drift is identical to the volume-averaged flow.

Insignificant supercritical wave numbers occur when the following conditions are satisfied: $\gamma_c > q/\beta > 1$. This leads immediately to $\alpha+2\beta<2$ and $\beta \leq 1$. The latter follows from $q \leq 1$ (see Section 4.1). As we will see later, this can only happen, in part of Regime I (with $q=1$) defined by $\alpha+\beta<1, \alpha+2\beta<2, \beta<1$.

The critical wave numbers have long-range correlation in time or in space and dominate the transport in the fractional-Brownian-motion regimes (Regimes II and III). The *subcritical* wave numbers are either temporally fluctuating, $|\mathbf{k}| \gg \varepsilon^{q/\beta}$, or spatially fluctuating $|\mathbf{k}| \gg \varepsilon$. Effectively, the subcritical wave numbers can be defined by $|\mathbf{k}| \gg k_c$, and by definition, include the insignificant supercritical wave numbers.

Denote by $\boldsymbol{c}_\varepsilon$ the sampling drift on scale $1/\varepsilon$. It has an amplitude of the order

$$\left(\int_{\delta\leq|\mathbf{k}|\leq k_c} \mathcal{E}(\mathbf{k})\,\mathrm{d}|\mathbf{k}|\right)^{1/2} \sim \begin{cases} |k_c^{1-\alpha}-\delta^{1-\alpha}|, & \text{for } \alpha\neq 1, \\ |\log k_c - \log\delta|, & \text{for } \alpha = 1. \end{cases} \tag{22}$$

Since the critical wave numbers dominate the sampling drift for $\alpha<1$, $\boldsymbol{c}_\varepsilon$ has a long-range correlation in space or time on the observation scale, so its transport effect is not ballistic. For $\alpha\geq 1$, $\boldsymbol{c}_\varepsilon$ is effectively frozen in time and its transport effect is ballistic.

Infrared cut-offs are classified accordingly: $\delta=\varepsilon^\gamma$ is *critical* if $\gamma=\gamma_c$, *supercritical* if $\gamma<\gamma_c$, and *subcritical* if $\gamma>\gamma_c$. We call γ_c the *critical exponent*.

From (18) and (19), we have the simple inequality for the scaling exponent

$$q \leq \begin{cases} \beta/(\alpha+2\beta-1), & \text{for } 1<\alpha+2\beta<2, \\ \beta, & \text{for } \alpha+2\beta\geq 2. \end{cases} \tag{23}$$

The equality in (23) is admissible because asymptotics is a continuum and cannot be fully resolved by power-laws. In this study, we restrict our attention to the power-law part of scaling behaviors.

For transport effect, besides the line $\alpha+2\beta=2$, the line $\alpha=1$ is also important for the following reasons. For $\alpha<1$, the sampling drift is dominated by the critical wave numbers, whereas, for $\alpha>1$, the sampling drift is dominated by wave numbers nearby the infrared cut-off. Moreover, in the case of $\alpha\geq 1$, the infrared cut-off corresponds to the energy containing scale, and therefore, dominates the transport as well as the flow. As a result, scaling laws of transport for $\alpha\geq 1$ are in general (infrared) cut-off dependent. The limit processes in the case of a *supercritical* cut-off, however, are always regular motions ($H=1$, Regime IV) as the effective constant drift dominates the transport.

Based on the supercritical wave numbers alone, the exit time τ (out of a ball of radius $1/\epsilon$) for $\alpha<1$ can be estimated by (cf. (20))

$$\tau \geq k_c^{\alpha-1}/\varepsilon = \begin{cases} \varepsilon^{-2\beta/(\alpha+2\beta-1)}, & \text{for } 1<\alpha+2\beta<2, \\ \varepsilon^{-2+\alpha}, & \text{for } \alpha+2\beta\geq 2. \end{cases} \tag{24}$$

It is easy to see that, the (asymptotic) equality in (24) is achieved when the combined effect of the supercritical and the critical wave numbers is considered since, for $\alpha<1$, the critical wave numbers are much stronger than the supercritical wave numbers in magnitude. For $\alpha\geq 1$, however, the transport is dominated by the wave numbers $|\mathbf{k}|\sim\delta$. So we have

$$\tau \geq \delta^{\alpha-1}/\varepsilon = \varepsilon^{-1-\gamma+\alpha\gamma} \quad (\text{with } \delta=\varepsilon^\gamma) \tag{25}$$

in the case of $\alpha\geq 1$.

As we will show by the variational method in Section 3.3 that the critical wave numbers dominate the transport in Regimes II and III. In the case of $\alpha < 1$, the exponent for $\alpha+2\beta < 2$ is less than or equal to 2(i.e., $2\beta/(\alpha+2\beta-1) \leq 2$) only if $\alpha+\beta \geq 1$; for $\alpha+2\beta \geq 2$, the exponent is less than or equal to 2(i.e., $2-\alpha < 2$) only if $\alpha \geq 0$. The former defines Regime II; the latter defines Regime III. In the case of $\alpha \geq 1$, any non-negative γ leads to $1+\gamma-\alpha\gamma \leq 2$ (the scaling is superballistic for $\gamma > 1$). In the remaining region (Regime I: $\alpha+\beta < 1$ or $\alpha < 0$), the supercritical wave numbers are negligible since the transport effect of the fluctuating wave numbers is at least *diffusive* as we will see later. Equating the exponent with $2q$, we have, from (24) and the remark following the scaling exponents for Regimes II, III (see Section 4), and from Eq. (25), the scaling exponent for Regime IV (see Section 5), both with *supercritical* cut-offs, $\gamma > \gamma_c$.

In the regimes where the critical wave numbers have a leading effect, the scaling limit is a fractional Brownian motion (Regimes II and III). Fractional Brownian motions arise as a result of long-range correlation of the critical wave numbers.

If the infrared cut-off is subcritical, i.e., $\delta \gg k_c$, wave numbers of the spectrum are either temporally or spatially fluctuating. Contrary to the fractional-Brownian-motion limit caused by the critical sampling drift, the limit is always a Brownian motion. But the scaling exponent may be superdiffusive ($q < 1$) due to low wave numbers in the vicinity of the cut-off.

3.2. Subcritical wave numbers: eddy diffusivity

To study the effect of subcritical, or fluctuating, wave numbers on transport, we think of turbulent motion as a superposition of a mean flow (i.e., $\boldsymbol{c}_\varepsilon$), and the fluctuating flow, following a spectral discretization.

We propose that the fluctuating wave numbers give rise to a fluctuating motion, on top of the mean flow, on the observation scale, and this fluctuating motion can be characterized by a notion of scale dependent eddy diffusivity introduced below. We then formulate two variational principles and use them to obtain general upper bounds for the (scale-dependent) eddy diffusivity.

Spectral discretization is motivated by a standard result of the ergodic theory for stationary processes that a stationary process is a limit of periodic processes (see [10]). We will use the periodic approximation in two different ways: In the first, we consider the periodic approximation in the *space* variables only and work with a subspace of $(\boldsymbol{\Omega}, P)$, the space $(\boldsymbol{\Omega}^n, P^{(n)})$ of time-independent, space-periodic velocity fields with period cell $[0, n]^d$ (see discussion below). In this approach, time randomness in the velocity field is represented as a Markov process on $(\boldsymbol{\Omega}^n, P^{(n)})$. In the second approach, we work with a sequence of space–time periodic fields with the (normalized) Lebesgue measure as the probability distribution on the space–time period cells as stated in the following lemma.

Lemma 1. *Let ω be a stationary process. Then there exists a sequence of periodic processes ω_n of period $l_n \to \infty$ in each variable, such that, the probability measure P_n obtained as the distribution of $\tau_{\mathbf{x}}\omega_n$ where $\mathbf{x}$ is random and distributed uniformly on the period cell $[0, l_n]^d$ converges weakly to the distribution of ω as $n \to \infty$.*

(See, for instance, [37] for a proof). We emphasize that spectral discretization is only a convenience for the formulation of the variational principles; it is neither essential nor necessary.

We now formulate the first approach more specifically. A spectrally discretized flow can be written as a sum $\boldsymbol{c}_\varepsilon + \mathbf{V}^{(\varepsilon,n)}$ where $\boldsymbol{c}_\varepsilon$ is the sampling drift (see Section 3.1 and Eq. (22)), and $\mathbf{V}^{(\varepsilon,n)}$ is the *spatial periodic* version of the fluctuating velocity field with a discrete spectrum $\mathbf{k} \in Z^{d/n}$, $\max\{k_c, \delta\} \ll |\mathbf{k}| \leq K$ and the amplitude $(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}/|\mathbf{k}|^2)\sqrt{\int_{|\mathbf{k}|\leq|\mathbf{k}'|\leq|\mathbf{k}|+1/n}\mathcal{E}(\mathbf{k}')\,\mathrm{d}|\mathbf{k}'|}$. The mesh size $1/n$ should tend to zero sufficiently fast, as $\epsilon \to 0$, to approximate the transport effect of the original fluctuating flow in view of the above lemma.

Equivalently, we replace the spectral measure $\hat{\mathbf{V}}(\mathrm{d}\mathbf{p}, t)$ by the discrete measure $\hat{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{k}, t)\delta_{\mathbf{k},\mathbf{p}}$, $\forall\mathbf{k} \in Z^d/n$, $\max\{k_c, \delta\} \ll |\mathbf{k}| \leq K$ with $\hat{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{k}, t)$ satisfying

$$\mathrm{d}_t\hat{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{k}, t) = -a_0|\mathbf{k}|^{2\beta}\hat{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{k}, t)\,\mathrm{d}t + |\mathbf{k}|^{\beta}\sqrt{\int_{|\mathbf{k}|\leq|\mathbf{k}'|\leq|\mathbf{k}|+1/n}\mathcal{E}(\mathbf{k}')\,\mathrm{d}|\mathbf{k}|}(\mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2})^{1/2}\,\mathrm{d}_t\mathbf{W}(\mathbf{k}, t), \tag{26}$$

where $\mathbf{W}(\mathbf{k}, t)$, $\forall\mathbf{k} \in Z^d/n$, $\max\{k_c, \delta\} \ll |\mathbf{k}| \leq K$ are independent standard Brownian motions. As discussed in Section 3.1, the sampling drift $\boldsymbol{c}_\varepsilon$ is steady for $\alpha + 2\beta > 2$ or $\alpha \geq 1$; it is unsteady for $\alpha + 2\beta \leq 2, \alpha < 1$.

The time-stationary, space-periodic field $\mathbf{V}^{(\varepsilon,n)}(\mathbf{x}, t, \omega_n)$, $\omega \in \boldsymbol{\Omega}^{(n)}$ is a Markovian flow and can be represented as a translate, $\mathbf{V}^{(\varepsilon,n)}(\mathbf{x}, t, \omega_n) = \tilde{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{x}, \omega_n(t))$, of steady, space-periodic field $\tilde{\mathbf{V}}^{(\varepsilon,n)}(\mathbf{x}, \omega_n)$, where $\omega_n(t)$, $w_n(0) = 0$ is a Markov process on $\boldsymbol{\Omega}^{(n)}$. As usual, we write ω_n explicitly only to emphasize its role.

For fixed ε, n, the displacement, $\mathbf{x}(t)$, in the periodic flow, $\boldsymbol{c}_\varepsilon + \mathbf{V}^{(\varepsilon,n)}$, is the sum of a mean motion, $\int_0^t \boldsymbol{c}_\varepsilon(s)\,\mathrm{d}s$, and the fluctuation, $\mathbf{x}(t) - \int_0^t \boldsymbol{c}_\varepsilon(s)\,\mathrm{d}s$. After a proper rescaling $t \to \lambda^2 t, \mathbf{x} \to \lambda\mathbf{x}, \lambda \to \infty$, the fluctuation converges to a Brownian motion by a turbulent diffusion theorem for mixing flows [15]. Let $\mathcal{A}^{(\varepsilon,n)}$ be the generator for $\boldsymbol{c}^\varepsilon(t) + \mathbf{V}^{(\varepsilon,n)}(\mathbf{x}, t)$. The diffusion coefficients, $D_{ij}^{(\varepsilon,n)}$, of the limiting Brownian motion are determined from the random, *space-periodic* solution $\chi_j^{(\varepsilon,n)}$ (i.e., $\chi_j^{(\varepsilon,n)}$ can be viewed as a function defined on $\boldsymbol{\Omega}^{(n)}$) of the abstract cell problem (cf. (16), see also [15])

$$\mathcal{L}^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)} := \mathcal{A}^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)} + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi_j^{(\varepsilon,n)} = -\tilde{V}_j^{(\varepsilon,n)}, \quad \text{in } \boldsymbol{\Omega}^{(n)}, \quad \forall i, j \tag{27}$$

$$\begin{aligned} D_{ij}^{(\varepsilon,n)} &:= \tfrac{1}{2}(\langle\tilde{V}_j^{(\varepsilon,n)}\chi_i^{(\varepsilon,n)}\rangle_n + \langle\tilde{V}_i^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\rangle_n) = -\tfrac{1}{2}(\langle\mathcal{L}^{(\varepsilon,n)}\chi_i^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\rangle_n + \langle\mathcal{L}^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\chi_i^{(\varepsilon,n)}\rangle_n) \\ &= -\tfrac{1}{2}(\langle\mathcal{A}^{(\varepsilon,n)}\chi_i^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\rangle_n + \langle\mathcal{A}^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\chi_i^{(\varepsilon,n)}\rangle_n) = \langle\nabla^\beta\chi_i^{(\varepsilon,n)}\cdot\mathcal{A}_0^{(\varepsilon,n)}\nabla^\beta\chi_j^{(\varepsilon,n)}\rangle_n, \quad \forall i, j. \end{aligned} \tag{28}$$

with the periodic boundary condition, where $\langle\cdot\rangle_n$ is the average with respect to $P^{(n)}$. Here we have used the following identity

$$\begin{aligned} &\langle[(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi_i^{(\varepsilon,n)}]\chi_j^{(\varepsilon,n)}\rangle_n + \langle[(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi_j^{(\varepsilon,n)}]\chi_i^{(\varepsilon,n)}\rangle_n \\ &\quad = \nabla\cdot\langle(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\chi_i^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\rangle_n = 0, \end{aligned} \tag{29}$$

which follows from the incompressibility of $\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}$ and the space-homogeneity of $\langle(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\chi_i^{(\varepsilon,n)}\chi_j^{(\varepsilon,n)}\rangle_n$.

The problem (27) is well-posed and has a unique solution, up to a constant (which does not affect (28)). In Appendix A, we derive the minimum principle

$$D^{(\varepsilon,n)}(\mathbf{e}) := \mathbf{D}^{(\varepsilon,n)}\mathbf{e}\cdot\mathbf{e} = \inf_f\{-\langle\mathcal{A}^{(\varepsilon,n)}ff\rangle_n - \langle\mathcal{A}^{(\varepsilon,n)}f'f'\rangle_n\} \tag{30}$$

with the space-periodic functions f', f related by

$$\mathcal{A}^{(\varepsilon,n)}f' + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla f + \tilde{\mathbf{V}}^{(\varepsilon,n)}\cdot\mathbf{e} = 0, \quad \text{in } \boldsymbol{\Omega}^{(n)}. \tag{31}$$

It should be noted that the explicit form of the generator is not used for the variational formulation.

In the limit $n \to \infty$, the abstract cell problem (27)–(28) becomes

$$\mathcal{L}\chi_j^{(\varepsilon)} + \tilde{\mathbf{V}}_j^{(\varepsilon)} = 0, \qquad \text{in } \boldsymbol{\Omega}. \tag{32}$$

We also have

$$D_{ij}^{(\varepsilon)} = \lim_{n\to\infty} D_{ij}^{(\varepsilon,n)} = \tfrac{1}{2}(\langle\tilde{V}_j^\varepsilon\chi_i^{(\varepsilon)}\rangle + \langle\tilde{V}_i^{(\varepsilon)}\chi_j^{(\varepsilon)}\rangle) = \langle\nabla^\beta\chi_i^{(\varepsilon)}\cdot\mathcal{A}_0\nabla^\beta\chi_j^{(\varepsilon)}\rangle, \quad \forall i, j, \tag{33}$$

following from (28). It is clear from (33) that the matrix $\mathbf{D}^{(\varepsilon)} = [D_{ij}^{(\varepsilon)}]$ is symmetric and positive-definite. We think of $\mathbf{D}^{(\varepsilon)}$ as a measure of turbulent dispersion caused by eddies composed of subcritical wave numbers in interaction with the sampling drift. We call it the *eddy diffusivity*. If the increments of the fluctuation of particle motion have divergent step sizes as $\varepsilon \to 0$, then the eddy diffusivity is cut-off dependent. Eq. (33) indicates the right solution space for (32) in the limit of $\varepsilon \to 0$: $L^2_\beta(\boldsymbol{\Omega})$, the space of functions with homogeneous, square integrable fractional gradient of order β.

From another perspective, the variance of the fluctuation $\mathbf{x}(t) - \int_0^t \boldsymbol{c}_\varepsilon(s)\,\mathrm{d}s$ after the rescaling $t \to \lambda^2 t, \mathbf{x} \to \lambda\mathbf{x}$, $\lambda \to \infty$ can be expressed as the time integral

$$2\int_0^t D_{ij}^{\varepsilon,\lambda}(s)\,\mathrm{d}s$$

of the Lagrangian velocity autocorrelation

$$D_{ij}^{\varepsilon,\lambda}(s) = \frac{1}{2}\int_0^{\lambda^2 s} (\langle V_i^{(\varepsilon)}(\mathbf{x}(s), s) V_j^{(\varepsilon)}(\boldsymbol{x}(s'), s')\rangle + \langle V_j^{(\varepsilon)}(\boldsymbol{x}(s), s) V_i^{(\varepsilon)}(\mathbf{x}(s'), s')\rangle)\,\mathrm{d}s'. \tag{34}$$

Because the Lagrangian velocity $\mathbf{V}^{(\varepsilon)}(\mathbf{x}(t), t)$ is a stationary Markov process [15,39], (34) can be rewritten as

$$\begin{aligned} D_{ij}^{\varepsilon,\lambda}(s) &= \frac{1}{2}\int_0^{\lambda^2 s} \{\langle V_i^{(\varepsilon)}(0,0) V_j^{(\varepsilon)}(\boldsymbol{x}(s'), s')\rangle + \langle V_j^{(\varepsilon)}(0,0) V_i^{(\varepsilon)}(\mathbf{x}(s'), s')\rangle\}\,\mathrm{d}s' \\ &= \frac{1}{2}\int_0^{\lambda^2 s} \{\langle \tilde{V}_i^{(\varepsilon)} \exp(\mathcal{L}s') \tilde{V}_j^{(\varepsilon)}\rangle + \langle \tilde{V}_j^{(\varepsilon)} \exp(\mathcal{L}s') \tilde{V}_i^{(\varepsilon)}\rangle\}\,\mathrm{d}s'. \end{aligned}$$

In the limit $\lambda \to \infty$, $D_{ij}^{\varepsilon,\lambda}(s)$ tends to the following expressions

$$-\tfrac{1}{2}(\langle \tilde{V}_i^{(\varepsilon)} \mathcal{L}^{-1} \tilde{V}_j^{(\varepsilon)}\rangle + \langle \tilde{V}_j^{(\varepsilon)} \mathcal{L}^{-1} \tilde{V}_i^{(\varepsilon)}\rangle) = \tfrac{1}{2}(\langle \tilde{V}_i^{(\varepsilon)} \chi_j^{(\varepsilon)}\rangle + \langle \tilde{V}_j^{(\varepsilon)} \chi_i^{(\varepsilon)}\rangle) = D_{ij}^{(\varepsilon)}, \tag{35}$$

where $\chi_j^{(\varepsilon)}$ is the solution of (32).

When molecular diffusion is present, we denote the eddy diffusivity by $\mathbf{D}_\kappa^{(\varepsilon,n)}$. As before, $\mathbf{D}_\kappa^{(\varepsilon,n)}$ can be characterized variationally by adding the terms, $\kappa, \kappa\langle\nabla f \cdot \nabla f\rangle_n, \kappa\langle\nabla f' \cdot \nabla f'\rangle_n$ to (30) and a Laplacian term, $\kappa\Delta f'$, to (31).

We turn to the second approach of space–time periodic approximation. Let $\mathbf{V}^{(\varepsilon,n,\lambda)}(\mathbf{x}, t)$ be the approximating sequence of space–time periodic fields, as stated in Lemma 1, with increasing space period n and time period λ, for the velocity field $\mathbf{V}^\varepsilon$ with a subcritical cut-off. We work with the space–time period cell problem in which time randomness in the Lagrangian dynamics is absent. To formulate a variational principle in this case, we need to reinstate the molecular diffusion here.

For fixed λ, n, the effective diffusivity, $\mathbf{D}_\kappa^{(\varepsilon,n,\lambda)}$, in the flow $\mathbf{V}^{(\varepsilon,n,\lambda)}(\mathbf{x}, t)$ exists and can be given by

$$D_\kappa^{(\varepsilon,n,\lambda)}(\mathbf{e}) = \inf_f \frac{1}{\lambda}\int_0^\lambda \frac{1}{n^d}\int_{[0,n]^d} \kappa(1 + \nabla f \cdot \nabla f + \nabla f' \cdot \boldsymbol{\nabla} f')\,\mathrm{d}\mathbf{x}\,\mathrm{d}t, \tag{36}$$

where f, f' are both temporally and spatially periodic with the period λ, n, respectively, and are related to each other by the following equation

$$\kappa\Delta f' = -\frac{\partial f}{\partial t} - (\boldsymbol{c}_\varepsilon + \mathbf{V}^{(\varepsilon,n,\lambda)}) \cdot \boldsymbol{\nabla} f - \mathbf{V}^{(\varepsilon,n,\lambda)} \cdot \mathbf{e}$$

(see [17,19]). The eddy diffusivity $\mathbf{D}_\kappa^{(\varepsilon)}$ in the presence of molecular diffusion is the large scale limit of $D_\kappa^{(\varepsilon,n,\lambda)}$, i.e.,

$$\mathbf{D}_\kappa^{(\varepsilon)} = \lim_{n,\lambda\to\infty} \mathbf{D}_\kappa^{(\varepsilon,n,\lambda)}. \tag{37}$$

The variational principle (36)–(37) is more useful than (30)–(31) when the temporal randomness of the velocity is negligible as in Regime III. Another advantage for working with the space–time periodic setting is that a dual variational principle can be formulated for the inverse of $D_\kappa^{(\varepsilon,n,\lambda)}$ and can be used to obtain the lower bound for $D_\kappa^{(\varepsilon,n,\lambda)}$ (see [12,17,18]).

3.3. *Variational bounds: fractional vector potential*

3.3.1. *Case 1: supercritical cut-off*

When the sampling drift is present, i.e., $\gamma > \gamma_c$, we show by the variational principles the following upper bounds on the growth rate of the eddy diffusivity

$$D_{ij}^{(\varepsilon)} \le C, \quad \text{for } \alpha+\beta<1 \text{ or } \alpha<0, \qquad D_{ij}^{(\varepsilon)} \ll \begin{cases} \varepsilon^{2\gamma_c(1-\alpha-\beta)}, & \text{for } \alpha+\beta>1, \\ \log(1/\varepsilon^{\gamma_c}), & \text{for } \alpha+\beta=1, \\ \varepsilon^{-\alpha}, & \text{for } \alpha>0, \\ \sqrt{\log(1/\varepsilon^{\gamma_c})}, & \text{for } \alpha=0. \end{cases} \tag{38}$$

$\forall i, j$, for some constant $C>0$. Note that $\varepsilon^{-\alpha}$ is a better bound than $\varepsilon^{2\gamma_c(1-\alpha-\beta)}$ for $\alpha+2\beta>2$.

Take the trivial trial function $f=0$ in (30) and eliminate the first term in the functional. We calculate the second term in (30) by studying the equation

$$\mathcal{A}^{(\varepsilon,n)} f' + \mathbf{V}^{(\varepsilon,n)}\cdot\mathbf{e} = 0, \tag{39}$$

(cf. (31)). Consider the *fractional* vector potential (the fractional stream function) in three dimensions (in two dimensions) $\tilde{\mathbf{H}}_\beta^{(\varepsilon,n)}$ of order β defined by

$$\tilde{\mathbf{H}}_\beta^{(\varepsilon,n)} = (-\Delta)^{-\beta/2}\tilde{\mathbf{V}}^{(\varepsilon,n)} \tag{40}$$

via the Fourier transform. This means that $\tilde{\mathbf{H}}_\beta^{(\varepsilon)} = \lim_{n\to\infty}\tilde{\mathbf{H}}_\beta^{(\varepsilon,n)}$ has the energy spectrum

$$E_0|\mathbf{k}|^{1-2(\alpha+\beta)} \tag{41}$$

with a subcritical infrared cut-off. The usual vector potential and the stream function correspond to $\beta=1$. What is significant is that, for $\alpha+\beta<1$, (41) is integrable near $\mathbf{k}=0$ uniformly as the infrared cut-off is removed, and thus defines a homogeneous, square integrable $\mathbf{H}_\beta$ that is cut-off independent. For $\alpha+\beta\ge 1$, (41) is not square integrable uniformly as the infrared cut-off is removed, and the second moment of $\mathbf{H}_\beta^{(\varepsilon)}$ grows like

$$\langle|\mathbf{H}_\beta^{(\varepsilon)}|^2\rangle \ll \varepsilon^{2\gamma_c(1-\alpha-\beta)}, \quad \text{for } \alpha+\beta>1 \tag{42}$$

and $\langle|\mathbf{H}_\beta^{(\varepsilon)}|^2\rangle \ll \log(1/\varepsilon^{\gamma_c})$, for $\alpha+\beta=1$ as $\varepsilon\to 0$.

In terms of $\mathbf{H}_\beta^{(\varepsilon,n)}$, (39) becomes

$$\mathcal{A}^{(\varepsilon,n)} f' + (-\Delta)^{\beta/2}\mathbf{H}_\beta^{(\varepsilon,n)}\cdot\mathbf{e} = 0. \tag{43}$$

A straightforward energy estimate for (43) gives

$$-\langle \mathcal{A}^{(\varepsilon,n)} f' f' \rangle_n = \langle (-\boldsymbol{\Delta})^{\beta/2} \mathbf{H}_\beta^{(\varepsilon,n)} \cdot \mathbf{e} f' \rangle_n \tag{44}$$

$$\leq \sqrt{\langle |\mathbf{H}_\beta^{(\varepsilon,n)} \cdot \mathbf{e}|^2 \rangle_n} \sqrt{\langle |(-\Delta)^{\beta/2} f'|^2 \rangle_n} \tag{45}$$

$$\leq \sqrt{\langle |\mathbf{H}_\beta^{(\varepsilon,n)} \cdot \mathbf{e}|^2 \rangle_n} \sqrt{\frac{1}{a_0} \langle -\mathcal{A}^{(\varepsilon,n)} f' f' \rangle_n}. \tag{46}$$

Therefore,

$$-\langle \mathcal{A}^{(\varepsilon,n)} f' f' \rangle_n \leq \frac{1}{a_0} \langle |\boldsymbol{H}_\beta^{(\varepsilon,n)} \cdot \mathbf{e}|^2 \rangle_n \tag{47}$$

which, in the limit $n \to \infty$, is much less than $\varepsilon^{2\gamma_c(1-\alpha-\beta)}$ for small ε.

It is worth noting that the right side of (47) is, up to a factor independent of ε, what one gets in replacing the Lagrangian autocorrelation in (34) by the Eulerian autocorrelation $R_{ij}^{(\varepsilon)}(0, s - s') := \langle V_i^{(\varepsilon)}(0, s) V_j^{(\varepsilon)}(0, s') \rangle$, i.e.,

$$\int_0^\infty R_{ij}^{(\varepsilon)}(0, s)\, \mathrm{d}s \tag{48}$$

which is called the (Eulerian) Taylor–Kubo formula, used extensively in the literature to approximate the eddy diffusivity since Taylor's work [42] (see also [35]). The physical significance of the bound (47) is that the eddy diffusivity $D_{ij}^{(\varepsilon)}$ of the fluctuation is bounded, as the infrared cut-off is removed, by constant times the Eulerian Taylor–Kubo formula (48); the eddy diffusivity may be much smaller than (48) due to the spatial decorrelation of velocity.

A different upper bound for the eddy diffusivity can be obtained by using the second variational principle (36). First we note that the eddy diffusivity in the presence of molecular diffusion would be enhanced if we freeze the time variable of the velocity field $\mathbf{V}^{(\varepsilon)}$. This can be easily seen as follows. Let $\bar{\mathbf{D}}_\kappa^{(\varepsilon,n)}$ be the eddy diffusivity for the frozen velocity field $\mathbf{V}^{(\varepsilon,n)}(\mathbf{x}, 0)$

$$\bar{D}_\kappa^{(\varepsilon,n)} \mathbf{e} := \bar{\mathbf{D}}_\kappa^{(\varepsilon,n)} \mathbf{e} \cdot \mathbf{e} = \inf_f \frac{1}{n^d} \int_{[0,n]^d} \kappa (1 + \nabla f \cdot \nabla f + \nabla f' \cdot \nabla f')\, \mathrm{d}\mathbf{x} \tag{49}$$

where f, f' are spatially periodic with period cell $[0, n]^d$ and are related by

$$\kappa \Delta f' = -(\boldsymbol{c}_\varepsilon + \mathbf{V}^{(\varepsilon,n)}(\mathbf{x}, 0)) \cdot \nabla f - \mathbf{V}^{(\varepsilon,n)}(\boldsymbol{x}, 0) \cdot \mathbf{e}.$$

Since time independent trial functions f are admissible in (36), (49) is larger than $D_\kappa^{(\varepsilon,n,\lambda)}$, $\forall \lambda > 0$, given by (36).

Using the trivial trial function $f = 0$ in (49) and the same energy estimate as above we have the upper bound

$$\bar{\mathbf{D}}_\kappa^{(\varepsilon)} := \lim_{n \to \infty} \bar{\mathbf{D}}_\kappa^{(\varepsilon,n)} \leq \langle |\mathbf{H}_1^{(\varepsilon)}|^2 \rangle.$$

A better bound, however, can be obtained for steady, isotropic flows by a duality argument in conjunction with the variational method (see [12]):

$$\bar{\mathbf{D}}_\kappa^{(\varepsilon)} \leq \sqrt{\langle |\mathbf{H}_1^{(\varepsilon)}|^2 \rangle} \begin{cases} \leq C, & \text{for } \alpha < 0, \\ \ll \varepsilon^{-\alpha}, & \text{for } \alpha > 0, \\ \ll \sqrt{\log(1/\varepsilon^{\gamma_c})}, & \text{for } \alpha = 0, \end{cases} \tag{50}$$

which agrees with results by other approaches (such as RNG calculation of [7] and Green's function method of [27]).

When the sampling drift is present, the estimates (38) can be used to compare the transport effects of the sampling drift and the subcritical wave numbers. For $\alpha+\beta > 1$, we have from (38) the bound for the timescale of the fluctuation of particle motion.

$$\varepsilon^{-2}/D^{(\varepsilon)}(\mathbf{e}) \gg \varepsilon^{-2}\varepsilon^{2\gamma_c(\alpha+\beta-1)} = \varepsilon^{-2\beta/(\alpha+2\beta-1)},$$

which, in Regime II, is the timescale of observation as determined from the sampling drift alone (cf. (24) and the remark following). Therefore, the transport in Regime II is dominated by the sampling drift.

For $\alpha + 2\beta \geq 2$, (50) implies the bound for the timescale associated with the fluctuation of particle motion

$$\varepsilon^{-2}/D^{(\varepsilon)}(\mathbf{e}) \gg \varepsilon^{-2}\varepsilon^{\alpha} = \varepsilon^{-2(1-\alpha/2)},$$

which, in Regime III, is the timescale of observation as determined from the sampling drift alone (cf. (24) and the remark following). Thus, again, the transport in Regime III is dominated by the sampling drift.

3.3.2. Case 2: subcritical cut-off

When the sampling drift is absent, i.e., $\gamma < \gamma_c$, or negligible, instead of (38), we have

$$D_{ij}^{(\varepsilon)} \leq C, \quad \text{for } \alpha+\beta < 1 \text{ or } \alpha < 0, \qquad D_{ij}^{(\varepsilon)} \leq \begin{cases} C\varepsilon^{2\gamma(1-\alpha-\beta)}, & \text{for } \alpha+\beta > 1, \\ C\log(1/\varepsilon^{\gamma}), & \text{for } \alpha+\beta = 1, \\ C\varepsilon^{-\gamma\alpha}, & \text{for } \alpha > 0, \\ C\sqrt{\log(1/\varepsilon^{\gamma})}, & \text{for } \alpha = 0, \end{cases}, \tag{51}$$

$\forall i, j$, for some constant $C > 0$. Note again that $\varepsilon^{-\gamma\alpha}$ is a better bound than $\varepsilon^{2\gamma(1-\alpha-\beta)}$, for $\alpha + 2\beta > 2$.

The estimates (51) are derived by the same energy estimate as before. In this case, $\boldsymbol{c}_\varepsilon = 0$ and the velocity field consists entirely of the subcritical wave numbers. As a result, $\ll$ in (38) becomes $\leq$ in (51).

When the sampling drift is absent, the estimates (51) yield a lower bound for the scaling exponent. For $\alpha + \beta > 1$, $\gamma < \gamma_c$ the fluctuation of particle motion is $1/\varepsilon$ and is much less than

$$\sqrt{\varepsilon^{2\gamma_c(1-\alpha-\beta)}\varepsilon^{-2q}} = \varepsilon^{\gamma_c(1-\alpha-\beta)-q}.$$

Thus, we have

$$q \geq 1 + \gamma - \gamma(\alpha+\beta), \quad \text{for } \alpha+\beta > 1. \tag{52}$$

The bound (52) is sharp when temporal fluctuations of the velocity fields are the dominant mechanism for transport as in Regime V.

For $\alpha + 2\beta \geq 2$, $\gamma < \gamma_c$, (51) implies

$$\varepsilon^{-1} \leq \sqrt{\varepsilon^{-\gamma\alpha}\varepsilon^{-2q}} = \varepsilon^{-q-\gamma\alpha/2}.$$

Thus, we have

$$q \geq 1 - \gamma\alpha/2, \quad \text{for } \alpha+2\beta \geq 2, \tag{53}$$

which turns out to be sharp in Regime VI.

In the case of subcritical infrared cut-offs, it is often useful to know if the wave numbers $|\mathbf{k}| \sim \delta$ dominate the transport or not. For this purpose, we modify the previous variational method to estimate the transport effect of the wave numbers much larger than the subcritical infrared cut-off. We replace $\boldsymbol{c}_\varepsilon$ by the velocity field $\mathbf{U}_\varepsilon$ consisting entirely of wave numbers $|\mathbf{k}| \sim \delta = \varepsilon^{\gamma}$ and $\mathbf{V}^{(\varepsilon)}$ by the velocity field consisting of the wave numbers $|\mathbf{k}| \gg \varepsilon^{\gamma}$.

After other corresponding modifications are made, the variational method and the subsequent energy estimate after substitution of the trivial trial function work the same way. We get the upper bound for the contribution, denoted by $\dot{\mathbf{D}}^{(\varepsilon)} = [\dot{D}_{ij}^{(\varepsilon)}]$, of wave number $|\mathbf{k}| \gg \varepsilon^{\gamma}$ to the total eddy diffusivity $D_{ij}^{(\varepsilon)}$:

$$\dot{D}_{ij}^{(\varepsilon)} \leq C, \quad \text{for } \alpha+\beta<1 \text{ or } \alpha<0, \qquad \dot{D}_{ij}^{(\varepsilon)} \ll \begin{cases} \varepsilon^{2\gamma(1-\alpha-\beta)}, & \text{for } \alpha+\beta>1, \\ \log(1/\varepsilon^{\gamma}), & \text{for } \alpha+\beta=1, \\ \varepsilon^{-\gamma\alpha}, & \text{for } \alpha>0, \\ \sqrt{\log(1/\varepsilon^{\gamma})}, & \text{for } \alpha=0. \end{cases} \tag{54}$$

$\forall_{i\,j}$, for some constant $C>0$. For specific applications of bounds (54) see discussions for Regimes V and VI.

4. Phase diagram for $\alpha < 1$

4.1. Regime I: diffusive limits

First of all, as discussed in Section 3.1, the sampling drift is negligible in this regime: In (24), if $\alpha+\beta<1$ and $\alpha+2\beta<2$, then $2\beta/(\alpha+2\beta-2)>2$; if $\alpha<0$ and $\alpha+2\beta\geq 2$, then $2-\alpha>2$. In either case, the transport would be dominated by the fluctuating wave numbers.

When $\alpha+\beta<1$, (38) implies

$$0 \leq \liminf_{\varepsilon\to 0} D^{(\varepsilon)}(\mathbf{e}) \leq \limsup_{\varepsilon\to 0} D^{(\varepsilon)}(\mathbf{e}) < \infty. \tag{55}$$

As we will see below that $D^{(\varepsilon)}(\mathbf{e})$ should not vanish in the limit even with $\lambda=0$, so the scaling is diffusive $q=1$ and the scaling limit should be a Brownian motion ($H=1/2$). The limit $D^*(\mathbf{e}) = \lim_{\varepsilon\to 0} D^{(\varepsilon)}(\mathbf{e})$, if exists, is the (scale independent) eddy diffusivity. Similarly, for $\alpha<0$, the variational bound (50) implies the diffusive scaling limit.

The eddy diffusivity probably does not vanish in the absence of molecular diffusion for the following reason. From the turbulent diffusion theorem for *mixing* flows, proved in [15], we know that, for $\beta=0, \alpha<1$, the scaling is diffusive ($q=1$) and the limit is a Brownian motion. As β increases, the velocity correlation in time increases and so should the rate of transport. But the upper bound (55) for the eddy diffusivity tells us that it cannot enhance transport to the extent of changing the scaling limit as long as $\alpha+\beta<1$ (this scenerio has been rigorously justified in the region $\alpha<0, \beta\leq 1$ in a different turbulent diffusion theorem for *non-mixing* flows, proved in [15]).

For $\alpha<0$, the (ordinary) vector potentials for the flows are time-stationary, space-homogeneous and have finite moments. Then the diffusion limit theorem of [14] holds for such flows if molecular diffusion is present (i.e., $q=1, H=1/2$ if $\kappa>0$). The effective diffusivity can be determined from a pair of variational principles [19], one of which is (36). This is manifest in the existence of homogeneous (ordinary) vector potentials when $\alpha<0$. As shown in [19], for *steady flows*, the existence of space-homogeneous (ordinary) vector potentials is the *sharp* condition for a diffusive scaling limit with molecular diffusion. As time dependence of velocity becomes important with $\beta\leq 1$, the phase boundary defined by $\alpha+\beta=1, \beta\leq 1$ points to the fact that the existence of space-homogeneous, *fractional* vector potentials becomes the criterion for the diffusive scaling limit.

For $\alpha+\beta>1, \alpha\leq 0$ (thus, $\beta>1$ and $\alpha+2\beta>2$), both the sampling drift and high wave numbers are negligible. The effect of molecular diffusion may not be negligible for homogenization but the scaling law should remain the same in the limit of vanishing molecular diffusion (see Section 6.1).

4.2. Regime II: space-freezing property

As we have seen from the analysis of the sampling drift and the applications of variational bounds, for

$$\gamma > \gamma_c, \quad \alpha+\beta > 1, \quad \alpha+2\beta < 2, \quad \alpha < 1,$$

the sampling drift dominates the transport, and therefore,

$$q = \frac{\beta}{\alpha+2\beta-1} \tag{56}$$

by (24) (and the discussion afterward). Moreover, since the sampling drift is asymptotically uniform in space, the displacement can be approximated asymptotically by

$$\mathbf{x}^{\varepsilon}(0) + \varepsilon \int_0^{t/\varepsilon^{2q}} \mathbf{V}(\boldsymbol{x}^{\varepsilon}(0)/\varepsilon s)\,\mathrm{d}s. \tag{57}$$

Eq. (57) is called the space-freezing approximation, in which the space dependence of the Lagrangian velocity is suppressed. Eq. (57) defines a Gaussian process with stationary increments. It is easy to check that (57) converges to a fractional Brownian motion $\mathbf{B}_H(t)$ by computing its covariance tensor

$$\langle \mathbf{B}_H(t) \otimes \mathbf{B}_H(t) \rangle = \mathbf{C} t^{2H}$$

with the Hurst exponent

$$H = \frac{1}{2} + \frac{\alpha+\beta-1}{2\beta} = 1/(2q) > \frac{1}{2}$$

and the coefficient

$$\mathbf{C} = E_0 \int_{R^d} \frac{\mathrm{e}^{-a_0|\mathbf{k}|^{2\beta}} - 1 + a_0|\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} (\mathbf{I} - \mathbf{k}\otimes\mathbf{k}|\mathbf{k}|^{-2})|\mathbf{k}|^{1-d}\mathrm{d}\mathbf{k}.$$

This scaling limit was first obtained in [16] by a different, rigorous approach.

4.3. Regime V: subcritical cut-off

If the cut-off is supercritical, $\delta \ll k_c$, the sampling drift is effectively intact, so the frozen path approximation (57) holds along with the FBM limit with q given by (56).

If the cut-off is subcritical, $\delta \gg k_c$, the sampling drift is absent and the transport is determined by the fluctuating velocity field. We further decompose the subcritical wave numbers into those $|\mathbf{k}| \sim \delta = \varepsilon^{\gamma}$ and those much larger. We have made the estimate (54) for the contribution of the latter to the eddy diffusivity.

By a simple spectral calculation, the velocity field $\mathbf{U}_{\varepsilon}(\mathbf{x}, t)$ consisting of the wave numbers $|\mathbf{k}| \sim \delta$ can be approximated by

$$\delta^{1-\alpha}\mathbf{U}(\delta\mathbf{x}, \delta^{2\beta}t), \tag{58}$$

where $\mathbf{U}$ has the energy spectrum (7) for $|\mathbf{k}| \in [1, C]$ with a sufficiently large constant C. Substituting (58) into the equation of motion we have

$$\mathrm{d}\mathbf{x}^{\varepsilon} = \varepsilon^{\gamma(1-\alpha)-2q+1}\mathbf{U}(\mathbf{x}^{\varepsilon}(t)/\varepsilon^{1-\gamma}, t/\varepsilon^{2q-2\beta\gamma})\,\mathrm{d}t. \tag{59}$$

The effect of the wave numbers $|\mathbf{k}| \gg \delta$ is like adding a turbulent diffusivity to Eq. (59), i.e.,

$$\mathrm{d}\mathbf{x}^{\varepsilon} = \varepsilon^{\gamma(1-\alpha)-2q+1}\mathbf{U}(\mathbf{x}^{\varepsilon}(t)/\varepsilon^{1-\gamma}, t/\varepsilon^{2q-2\beta\gamma})\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t), \tag{60}$$

where $\mathbf{B}(t)$ is the standard Browian motion and $\dot{\mathbf{D}}^{(\varepsilon)}$ is the portion of the eddy diffusivity coming from the wave numbers $|\mathbf{k}| \gg \delta$ (cf. the discussion preceding (54)).

Since $\mathbf{U}$ is a mixing flow, we expect the limit of (60) to be a Brownian motion. We also expect the time variable in (59) to dominate, so we equate $\varepsilon^{\gamma(1-\alpha)-2q+1} = \varepsilon^{2q-2\beta\gamma}$ and arrive at the expression

$$q = 1 + \gamma - \gamma(\alpha + \beta) \tag{61}$$

in view of a generalized 'diffusive' scaling of (59). We check that the space variable in (59) is indeed relatively slow in the sense

$$(1-\gamma)/(q-\beta\gamma) < 1, \tag{62}$$

for $\alpha + 2\beta < 2$. With (61), Eq. (60) becomes

$$\mathrm{d}\mathbf{x}^{\varepsilon} = \eta_{\varepsilon}^{-1}\mathbf{U}(\mathbf{x}/\eta_{\varepsilon}^{(1-\gamma)/(q-\beta\gamma)}, t/\eta_{\varepsilon}^{2})\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t), \quad \text{with } \eta_{\varepsilon} = \varepsilon^{1-\beta\gamma}. \tag{63}$$

The bound (54) and (61) imply that $\varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}} \ll \varepsilon^{1-q+\gamma-\gamma(\alpha+\beta)} = 1$ as ε tends to zero. Subcriticality, $\gamma < 1/(\alpha + 2\beta - 1)$, implies $\eta_{\varepsilon} \to 0$. Eq. (63) satisfies the conditions of the diffusion limit theorem of [29,30] for mixing flows with a generalized 'diffusive' scaling (63)–(62). The limit is a Brownian motion with the diffusion coefficients given by the Eulerian Taylor–Kubo formula

$$\int_0^{\infty} \langle \mathbf{U}(0,t) \otimes \mathbf{U}(0,0)\rangle \mathrm{d}t. \tag{64}$$

The condition $\alpha + \beta > 1$ implies $q < 1$.

The scaling law with $H = 1/2, q$, given by (61), and the eddy diffusivity given by the Eulerian Taylor–Kubo formula (64) holds in the other part of Regime $V(\alpha \geq 1)$ as well (see Section 5).

4.4. Regime II′: critical cut-off

When the infrared cut-off is critical, i.e., $\gamma = \gamma_c = (\alpha + 2\beta - 1)^{-1}$ the critical wave numbers are still present. Therefore, the scaling exponent is given by (56).

Rescaling the velocity field $\mathbf{U}_{\varepsilon}$ consisting entirely of wave numbers $|\mathbf{k}| \sim \varepsilon^{\gamma_c}$ but $|\mathbf{k}| < \varepsilon^{\gamma_c}$ by (58), we have, instead of (60), the equation

$$\mathrm{d}\mathbf{x}^{\varepsilon} = \mathbf{U}(\mathbf{x}^{\varepsilon}(t)/\varepsilon^{1-\gamma_c}, t)\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\mathbf{D}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t),$$

where $\mathbf{U}$ has the energy spectrum (28) supported in $[1,\infty)$ and $\mathbf{D}^{(\varepsilon)}$ is the eddy diffusivity.

Since $\gamma_c > 1$ and $q < 1$, we have in the limit $\varepsilon \to 0$

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(0,t)\,\mathrm{d}t. \tag{65}$$

Because the energy spectrum of $\mathbf{U}$ does not have small wave numbers, the process $\mathbf{Z}$ is not self-similar. Thus, the Hurst exponent is not well-defined. However, the long time asymptotics of (65) is a Brownian motion, due to the mixing property of $\mathbf{U}$, so we may associate the asymptotic Hurst exponent 1/2 to the process defined by (65).

4.5. Regime III: time-freezing property

In this regime, the sampling drift is a time independent, mean zero random variable whose second moment is of order

$$\int_0^{k_c} |\mathbf{k}|^{1-2\alpha}\,\mathrm{d}|\mathbf{k}| \sim \epsilon^{2-2\alpha}.$$

The mixing time for the sampling drift is not less than $1/\epsilon^{2\beta}$. On this timescale the sampling drift transports particles over the distances not less than $\epsilon^{1-\alpha}\epsilon^{-2\beta} = \epsilon^{1-\alpha-2\beta}$ which is not less than the spatial observation scale $1/\epsilon$ if $\alpha + 2\beta \geq 2$. Thus, the sampling drift appears steady on the observation scale. The time to exit a ball of radius $1/\varepsilon$, based on the sampling drift alone is $\epsilon^{-1}\epsilon^{\alpha-1} = \epsilon^{\alpha-2}$.

From (50) it follows that the time to exit a ball of radius $1/\varepsilon$, with its center moving by the sampling drift, is $\ll \epsilon^{\alpha-2}$, as a result of the spatially fluctuating, subcritical wave numbers ($|\mathbf{k}| \gg \varepsilon$). Thus, by (24) and the remark following, the effect of the subcritical wave numbers is dominated by that of the sampling drift. Therefore, we have

$$q = 1 - \alpha/2. \tag{66}$$

The scaling exponent (66) is superdiffusive for $\alpha > 0$. Note that

$$q - \beta/(\alpha + 2\beta - 1) = (\alpha/2 + \beta - 1)(1 - \alpha)/(\alpha + 2\beta - 1) < 0$$

for $1 < \alpha + 2\beta < 2$, $\alpha < 1$, and thus, for the same α, the rate of transport in Regime II is smaller (since faster decorrelation in time tends to slow down the transport).

Since the transport is dominated by the effectively steady sampling drift, we may consider the velocity field $\mathbf{U}_\varepsilon$ consisting entirely of the wave numbers $|\mathbf{k}| \sim k_c = \varepsilon$ and freeze the time variable in the resulting velocity field as $\varepsilon \to 0$. Rescaling as in (58), we have the equation

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \varepsilon^{2-\alpha-2q}\mathbf{U}(\mathbf{x}^\varepsilon(t), 0)\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\kappa}\,\mathrm{d}\boldsymbol{w}(t) \tag{67}$$

in the presence of molecular diffusion.

Eq. (67) has a limit if and only if (66) holds, which also make the diffusion term vanish in the limit for $\alpha > 0$. Formally, the limit process $\mathbf{Z}$ satisfies

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(\mathbf{Z}(t), 0)\,\mathrm{d}t, \tag{68}$$

where $\mathbf{U}$ has the energy spectrum (7) supported in $|\mathbf{k}| \in (0, \infty)$. The supercritical cut-off is now removed by the rescaling (58). This indicates the limit $\mathbf{Z}$ is self-similar.

It should be noted that the velocity field $\mathbf{U}$ is a generalized function because of the ultraviolet divergence in the energy spectrum of $\mathbf{U}$. Thus, Eq. (68) is not well-defined in the ordinary sense. Study of transport in generalized velocity fields is interesting by itself, but we do not pursue it here. Our only purpose is to use the energy spectrum of the velocity field as an indicator of the self-similarity of the limit process and to show how the space and the time dependence of the velocity field enter the equation. The same remark applies to the same situation in the sequel as well as (65) and will not be repeated.

We now identify the Hurst exponent of $\mathbf{Z}$. From (8), we have the asymptotics for the covariance of the successive increments on the time scale $t \sim \varepsilon^{-2q}$

$$\langle(\mathbf{x}(2t) - \mathbf{x}(t)) \cdot (\mathbf{x}(t) - \mathbf{x}(0))\rangle \sim t^{2H} \sim \varepsilon^{-4qH}. \tag{69}$$

On the other hand, by (24) and the remark following, the covariance of the successive increments on the time scale $t \sim \varepsilon^{-2q}$ is of order $(k_c^{1-\alpha}\varepsilon^{-2q})^2$. Equating it with (69), we have $H = 1/(2q)$. We hypothesize that the limit be a fractional Brownian motion.

Molecular diffusion probably has no significant effect on the scaling law, though the presence of molecular diffusion is needed in the variational principle (36). As remarked after (17) in Section 2, the molecular diffusion is negligible for $\beta \le 1$. For $\beta \ge 1$, as larger β gives rise to longer correlation times, the scaling law should have equal or smaller scaling exponent q. However, since the time-freezing property has already set in for $\beta < 1$ and resulted in a scaling exponent independent of β, the absence of molecular diffusion would not change the scaling exponent for $\beta \ge 1$.

4.6. Regime VI: subcritical cut-off

The results of the previous section hold for any supercritical cut-off ($\gamma > \gamma_c = 1$) as the sampling drift is essentially intact and dominates the transport.

For a subcritical cutoff we separate the wave numbers $|\mathbf{k}| \sim \delta$ from $|\mathbf{k}| \gg \delta$ and rescale the equation as in Section 4.3 to obtain (60). As before, we expect the limit to be a Brownian motion. In this case, however, we expect the space variable in the velocity field to dominate the transport. So we equate $\varepsilon^{\gamma(1-\alpha)-2q+1} = \varepsilon^{\gamma-1}$ and obtain the scaling exponent

$$q = 1 - \gamma\alpha/2. \tag{70}$$

Rewriting (60) with $\eta_\varepsilon = \varepsilon^{1-\gamma}$ we have

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \eta_\varepsilon^{-1}\mathbf{U}(\mathbf{x}^\varepsilon(t)/\eta_\varepsilon, t/\eta_\varepsilon^{2(q-\beta\gamma)/(1-\gamma)})\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t) + \varepsilon^{1-q}\sqrt{2\kappa}\,\mathrm{d}\boldsymbol{w}(t). \tag{71}$$

The bound (54) and (70) imply that $\varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}} \ll \varepsilon^{1-q-\gamma\alpha/2} = 1$. Since $q < 1$ the factor in front of the molecular diffusion also tends to zero as $\varepsilon \to 0$. The time variable is relatively slow in the sense

$$\eta_\varepsilon^{2(q-\beta\gamma)/(1-\gamma)} \gg \eta_\varepsilon^2$$

as $\alpha + 2\beta > 2$.

The velocity field $\mathbf{U}$ has a fixed infrared cut-off, and consequently, gives rise to a space-homogeneous vector potential, so, as suggested by the diffusion limits theorem of [14], the limit should be a Brownian motion.

The diffusion limit theorem of [14], however, does not apply directly because it was proved with a non-vanishing molecular diffusion. Generalizing the theorem of [14] to the situation with a vanishing molecular diffusion such as Eq. (71) remains a challenging problem in turbulent transport (see also [32,33]).

4.7. Regime III′: critical cut-off

When the cut-off is critical, i.e., $\gamma = 1$, the critical wave numbers are still present and dominates the transport. Therefore, the scaling exponent is given by (66).

The difference is that Eq. (67) now has the limit satisfying Eq. (68) with the energy spectrum of $\mathbf{U}$ supported in $|\mathbf{k}| \in [1, \infty)$ rather than $(0, \infty)$. As a consequence, the limit is not self-similar.

4.8. Phase boundary

The transport for the phase boundary $\alpha + 2\beta = 2$, $0 < \alpha < 1$ contrasts interestingly to regime on either side of the boundary.

When the cut-off is supercritical, the sampling drift is present and dominates the transport. Thus, we expect $q = \beta$ from (56). Indeed, with that and $\gamma_c = 1$, we have as before the asymptotic equation

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \mathbf{U}(\mathbf{x}^\varepsilon(t), t)\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\mathbf{D}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t),$$

where $\mathbf{U}$ has the energy spectrum (28) supported in $|\mathbf{k}| \in (0, \infty)$. In the limit, the diffusion term vanishes (cf. (38)) and the limit $\mathbf{Z}$ satisfies the equation

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(\mathbf{Z}(t), t)\,\mathrm{d}t \tag{72}$$

whose solution is expected to be a fractional Brownian motion.

When the cut-off is subcritical, we expect $q = 1 - \gamma + \gamma\beta$ from (61). With that and $\eta_\varepsilon = \varepsilon^{1-\gamma}$, we have the asymptotic equation

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \eta_\varepsilon^{-1}\mathbf{U}(\mathbf{x}/\eta_\varepsilon, t/\eta_\varepsilon^2)\,\mathrm{d}t + \varepsilon^{1-q}\sqrt{2\dot{\mathbf{D}}^{(\varepsilon)}}\,\mathrm{d}\mathbf{B}(t), \tag{73}$$

where the energy spectrum of $\mathbf{U}$ is supported in $|\mathbf{k}| \in [1, \infty)$. The diffusion term dies out in the limit as before (cf. (54)) and the limit $\mathbf{Z}$ is a Brownian motion due to the spectral gap in $\mathbf{U}$ by a turbulent diffusion theorem of [15].

Contrary to Regimes II, II′, V, III, III′ and VI, both the time and the space dependence of the velocity field in (72) and (73) affect the transport.

5. Phase diagram for $\alpha > 1$

If the infrared cut-off threshold δ is fixed as ε tends to zero then the velocity is mixing in time, and by the turbulent diffusion theory of [15], the scaling limit is a Brownian motion ($H = 1/2$).

Anomalous scaling limits arise when δ is coupled to the spatial observation scale: $\delta = \varepsilon^\gamma$, $\gamma > 0$. The exponent γ characterizes the relation between the spatial observation scale $1/\varepsilon$ and the energy containing scale $1/\delta$. In this case, superdiffusive scaling results from divergent mean kinetic energy as the infrared cut-off is removed; in particular, when, the cut-off is supercritical, $\gamma > \max\{(\alpha + 2\beta - 1)^{-1}, 1\}$, (Regime IV), the energy containing scale is larger than the spatial observation scale, and it results in a super-ballistic scaling $q < 1/2$ and a regular limit ($H = 1$).

Since the transport is dominated by wave numbers $|\mathbf{k}| \sim \delta$, it is natural to rescale the velocity $\mathbf{V}$ as

$$\mathbf{V}(\mathbf{x}, t) = \delta^{1-\alpha}\mathbf{U}(\delta\mathbf{x}, \delta^{2\beta}t), \tag{74}$$

where the velocity field $\mathbf{U}$ has the energy spectrum (7) supported in $|\mathbf{k}| \in [1, \infty)$. Contrary to $\mathbf{U}$ occurring in the case of $\alpha \le 1$, the velocity field $\mathbf{U}$ in the case of $\alpha > 1$ is an ordinary function, since there is no ultraviolet or infrared divergence, and temporally mixing due to the spectral gap in $\mathbf{U}$.

In terms of $\mathbf{U}$, the equation of motion becomes

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \delta^{1-\alpha}\varepsilon^{1-2q}\mathbf{U}(\delta\mathbf{x}^\varepsilon(t)/\varepsilon, \delta^{2\beta}t/\varepsilon^{2q})\,\mathrm{d}t = \varepsilon^{\gamma(1-\alpha)-2q+1}\mathbf{U}(\mathbf{x}^\varepsilon(t)/\varepsilon^{1-\gamma}, t/\varepsilon^{2q-2\beta\gamma})\,\mathrm{d}t. \tag{75}$$

Now that the infrared cut-off of $\mathbf{U}$ is 1, the limit is expected to be a Brownian motion as long as either space or time variable is fast. Depending on the parameters, two types of diffusion limit theorems in the literature are pertinent to the limit; one is based on velocity decorrelation in time [24,29,30,34] and the other based on velocity decorrelation in space [14].

Eliminating the infrared divergence by rescaling is also the approach of Avellaneda and Majda [3], in which the case of a critical cut-off $\gamma = 1$ was considered in the region $\beta < 1/2$, $0 < \alpha < 2$. Here, we adopt the same idea of rescaling and generalize their results by using new limit theorems which were not available to them.

5.1. $\gamma \geq 1$: *Regimes V, IV and IV′*

For $\gamma \geq 1$, Eq. (75) does not have fast space variables. To have a non-trivial limit, we must have $2q-1+\gamma(\alpha-1) \geq 0$: For $\gamma \geq 1$, space variable is not fast in (75). To have a non-trivial limit, we must have $2q-1+\gamma(\alpha-1) > 0$ or $2q-1+\gamma(\alpha-1) = 0$. The former case gives rise to Regimes V whereas the latter gives rise to Regimes IV or IV′.

5.1.1. *Regime V*

When

$$2q-1+\gamma(\alpha-1) > 0, \tag{76}$$

U in (75) has a large multiplier, so a non-trivial scaling limit requires rapid time relaxation, i.e.,

$$q > \gamma\beta. \tag{77}$$

By choosing a generalized 'diffusive' scaling for Eq. (75), i.e., $(\delta^{\alpha-1}\varepsilon^{2q-1})^{-1} = \delta^{\beta}/\varepsilon^{q}$ or

$$q = 1+\gamma-\gamma(\alpha+\beta), \tag{78}$$

(75) becomes

$$\mathrm{d}\mathbf{x}^{\varepsilon}(t) = \eta_{\varepsilon}^{-1}\mathbf{U}(\eta_{\varepsilon}^{(\gamma-1)/(q-\beta\gamma)}\mathbf{x}^{\varepsilon}(t), t/\eta_{\varepsilon}^{2})\,\mathrm{d}t \tag{79}$$

with $\eta_{\varepsilon} = \varepsilon^{q-\beta\gamma}$. Eq. (79) has the form of the classical diffusion limit theorem [6,24,34]. (Moreover, the velocity field **U** is smooth and satisfies the mixing condition of Rosenblatt [40] even for $\beta > 0$ since **U** has no small **k** components) Thus, the process $\mathbf{x}^{\varepsilon}(t)$ converges to a Brownian motion ($H = 1/2$) with diffusion coefficients given by the Eulerian Taylor–Kubo formula (64).

However, there is one constraint to be considered: (78) must be consistent with (77), i.e., η_{ε} must tend to zero with ε. This means $\gamma < 1/(\alpha+2\beta-1)$, a subcritical cut-off.

5.1.2. *Regime IV: smooth motion*

If the cut-off is supercritical, as discussed in Section 3.1, the transport in this regime is dominated by the sampling drift that is, in turn, dominated by the wave numbers near by the infrared cut-off. Time as well as space dependence of the velocity field are irrelevant. Because both space and time variables are slow in the velocity field, non-trivial scaling limit holds only if $\delta^{\alpha-1}\varepsilon^{2q-1} = 1$. Thus we have

$$q = (1+\gamma)/2 - \gamma\alpha/2. \tag{80}$$

Consistency, $0 < q < \gamma\beta$, then implies that $\gamma > 1/(\alpha+2\beta-1)$ and

$$\alpha < 1 + 1/\gamma.$$

The limit process $\mathbf{Z}(t)$ is advected by a constant drift

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(0,0)\,\mathrm{d}t \tag{81}$$

and is regular, or smooth ($H = 1$).

We note that the limit (81) is independent of the initial condition $\mathbf{x}^{\varepsilon}(0)$, is self-similar and has a well-defined Hurst exponent.

5.1.3. Regime IV′: critical cut-off

When $2q - 1 + \gamma(\alpha - 1) = 0$, or, equivalently,

$$q = (1+\gamma)/2 - \gamma\alpha/2, \tag{82}$$

and

$$q = \gamma\beta, \tag{83}$$

a non-trivial limit results. Combining (82) and (83), we have $\alpha + 2\beta = 1 + 1/\gamma$ or

$$\gamma = 1/(\alpha + 2\beta - 1) \tag{84}$$

which defines a critical cut-off for $\alpha + 2\beta < 2$. The limit $\mathbf{Z}(t)$ satisfying

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(0, t)\,\mathrm{d}t \tag{85}$$

is a smooth, Gaussian process. The Hurst exponent is not strictly well-defined for $\mathbf{Z}(t)$ due to lack of self-similarity. On large timescales, however, (85) has a Brownian motion limit as $\mathbf{U}$ is temporally mixing, and thus, an asymptotic Hurst exponent $H = 1/2$. This case is similar to Regime V.

In particular, when $\gamma = 1 = 1/(\alpha + 2\beta - 1)$ and $q = \beta$, Eq. (75) is independent of ε, i.e., $\mathbf{x}^\varepsilon(t) = \mathbf{Z}(t)$ with

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(\mathbf{Z}(t), t)\,\mathrm{d}t \tag{86}$$

The Hurst exponent is not strictly well-defined for $\mathbf{Z}(t)$ due to lack of self-similarity. But, as the velocity field is temporally mixing, the long-time limit of $\mathbf{Z}(t)$ is a Brownian motion, by the turbulent diffusion theorm of [15]. So $H = 1/2$ is the asymptotic Hurst exponent.

Another critical cut-off is $\gamma = 1$ for $\alpha + 2\beta > 2$. The equation of motion (75) becomes

$$\mathrm{d}\mathbf{x}^\varepsilon(t) = \varepsilon^{2-\alpha-2q}\mathbf{U}(\mathbf{x}^\varepsilon(t), t/\varepsilon^{2q-2\beta})\,\mathrm{d}t$$

which has a non-trivial limit when $q = \beta = 1 - \alpha/2$. The limiting process $\mathbf{Z}(t)$ satisfies

$$\mathrm{d}\mathbf{Z}(t) = \mathbf{U}(\mathbf{Z}(t), 0)\,\mathrm{d}t \tag{87}$$

which is regular for finite times, but is not self-similar. Thus the Hurst exponent is not well-defined. It is not clear whether an asymptotic Hurst exponent is well-defined either, since we do not know if, without molecular diffusion, motion in three-dimensional, steady flows like (87) can be homogenized or not. Bounded and unbounded streamlines may co-exist in steady flows, and if so, the resulting limit would depend on initial conditions (see discussion in Section 6.1).

5.2. $\gamma < 1$: Regimes V and VI

With $\gamma < 1$, fast space variables now enter the picture. There are two regimes depending on whether time dominates over space decorrelation or not.

5.2.1. Regime V: dominant time relaxation

This regime occurs when $\delta^\beta/\varepsilon^q \gg \delta/\varepsilon$, or equivalently,

$$q > 1 - \gamma + \gamma\beta. \tag{88}$$

Then, by choosing the 'diffusive' scaling, $(\delta^{\alpha-1}\varepsilon^{2q-1})^{-1} = \delta^{\beta}/\varepsilon^{q}$, i.e.,

$$q = 1 + \gamma - \gamma(\alpha + \beta), \tag{89}$$

Eq. (75) can be rewritten as (79), except that the space variable is also fast, albeit not fast enough to have an impact in the diffusive scaling. In this case, the generalized limit theorems proved in [29,30] are applicable and they extend the validity of the Taylor–Kubo formula to our situation. (Like the classical diffusion limit theorem, the generalized limit theorems also require the mixing condition and regularity on the velocity **U**, both of which are satisfied here.) The limit is a Brownian motion ($H = 1/2$) with diffusion coefficients given by the Eulerian Taylor–Kubo formula (64).

Condition (88) requires that

$$\alpha + 2\beta < 2. \tag{90}$$

5.2.2. *Regime VI: dominant space decorrelation*

For

$$q < 1 - \gamma + \gamma\beta \tag{91}$$

velocity dependence on space now dominates over the dependence on time in the diffusive scaling of Eq. (75). By choosing the scaling $(\delta^{\alpha-1}\varepsilon^{2q-1})^{-1} = \delta/\varepsilon$, or equivalently,

$$q = 1 - \gamma\alpha/2, \tag{92}$$

Eq. (75) is rewritten as

$$\mathrm{d}\mathbf{x}^{\varepsilon}(t) = \eta_{\varepsilon}^{-1}\mathbf{U}(\mathbf{x}^{\varepsilon}(t)/\eta_{\varepsilon},\ t/\eta_{\varepsilon}^{2(q-\beta\gamma)/(1-\gamma)})\,\mathrm{d}t, \quad \text{with}\ \ \eta_{\varepsilon} = \varepsilon^{1-\gamma}. \tag{93}$$

The limit of (93) should be a Brownian motion as **U** gives rise to a space-homogeneous vector potential (cf. the discussion in Section 4.5).

Consistency ((91) and $q > 0$) requires that $\alpha + 2\beta > 2$, $\gamma < 2/\alpha$.

5.2.3. *Phase boundary*

On the phase boundary $\alpha + 2\beta = 2$, Eq. (93) becomes

$$\mathrm{d}\mathbf{x}^{\varepsilon}(t) = \eta_{\varepsilon}^{-1}\mathbf{U}(\mathbf{x}^{\varepsilon}(t)/\eta_{\varepsilon},\ t/\eta_{\varepsilon}^{2})\,\mathrm{d}t \tag{94}$$

with a temporally mixing flow **U**. Space and time correlations play comparable roles in (94). From the turbulent diffusion theorem for mixing flows [15], it follows that the solution $\mathbf{x}^{\varepsilon}(t)$ has a Brownian motion limit ($H = 1/2$).

6. Conclusions

The supercritical and subcritical diagrams (Figs. 1 and 2) are divided by the line, $\alpha + \beta = 1$, and the line, $\alpha + 2\beta = 2$, and/or the vertical lines $\alpha = 0, 1, 1 + 1/\gamma, 2/\gamma$. First of all, $\alpha + \beta < 1$ or $\alpha < 0$ defines a cut-off independent diffusive regime, in which the sampling drift is negligible. Outside of the diffusive regime, the line, $\alpha + \beta = 1$, is the cross-over between short-ranged and long-ranged velocity correlations; the latter manifests in the fact that the sampling drift dominates the transport and subcritical wave numbers are negligible. The line, $\alpha + 2\beta = 2$, is the cross-over between velocity dependence on space and on time; in the region above the line, velocity dependence on time is negligible, whereas in the region below the line velocity dependence on space is negligible.

Figs. 3 and 4 are cross-sections of the full three-dimensional phase diagram at $\gamma = \text{constant} > 1$ and $\gamma = 1$, respectively. In Fig. 3, the cut-off is supercritical for $\alpha + 2\beta > 1 + 1/\gamma$ and subcritical for $\alpha + 2\beta < 1 + 1/\gamma$. In Fig. 4, the cut-off is subcritical for $\alpha + 2\beta < 2$ and critical for $\alpha + 2\beta \geq 2$. Cut-offs with $\gamma < 1$ are subcritical, and thus, covered in Fig. 2.

The limit is one of the three kinds: Brownian motion ($H = 1/2$), persistent fractional Brownian motion ($1/2 < H < 1$) or regular, or smooth, motion ($H = 1$). The relation $H = 1/(2q)$ holds for $\alpha < 1$ with supercritical infrared cut-off but neither for subcritical cut-offs nor for $\alpha > 1$ (in these situations, $H < 1/(2q)$, instead). For the critical cut-off $\gamma = \gamma_c$, the Hurst exponent is not well-defined. However, an asymptotic Hurst exponent may be defined and it is equal to 1/2. The diffusive regime ($q = 1, H = 1/2$) is most robust in that the scaling law is independent of any infrared cut-offs. The fractional Brownian motion limit of Regime II and III as well as the regular motion limit of Regime IV are not affected by supercritical cut-offs. All other regimes are cut-off dependent explicitly.

In the case of subcritical infrared cut-offs, with the rescaling of the velocity field, the diagram can be understood by means of three types of diffusion limit theorems in the literature: (i) one for which the spatial dependence of velocity is negligible and the effective diffusivity is explicitly given by the Eulerian Taylor–Kubo formula [8,24,25,29,30], (ii) another for which the temporal dependence of velocity is negligible, but molecular diffusion is assumed to be present and the effective diffusivity is implicitly given by a pair of variational principles [13,14,19] or a Stieltjes integral formula [4], and (iii) the other for which space and time dependence of velocity play comparable roles and is referred to as turbulent diffusion theorems (two such theorems are proved in [15]). As in (ii), the turbulent eddy diffusivity can be written as a variational principle similar to (30). After rescaling, Type (i) limit theorems apply to Regime V; Type (ii) limit theorems apply to Regime VI and part of Regime I; Type (iii) limit theorems apply to part of Regime I or on the phase boundary $\alpha + 2\beta = 2$, $0 < \alpha < 2$. All these types of limit theorems are insensitive to the dimension.

Similarly, there should be three types of fractional-Brownian-motion limit theorems: one completely determined by time dependence (Regime II), another determined by the space dependence (Regime III) and the other determined by both the time and space dependence of velocity (the phase boundary $\alpha + 2\beta = 2$, $0 < \alpha < 1$).

In the case of supercritical infrared cut-offs new phenomena emerge: dominant sampling drift, fractional Brownian motion limits, critical infrared cut-off and related cut-off dependent effects. Although these phenomena are introduced and analyzed for motion in three-dimensional, isotropic flows, they also arise in two-dimensional flows or anisotropic flows such as random shear-layer flows. An important difference lies in the role in molecular diffusion which is much more prominent for anisotropic or two-dimensional flows (see discussion in the next section). New variational principles for the cut-off dependent eddy diffusivity are formulated and used to obtain general bounds for the eddy diffusivity.

Contrary to subcritical and supercritical cut-offs, regimes (II′, III′, and IV′) with critical cut-offs produce limits that are not self-similar and do not possess a well-defined Hurst exponent.

Scaling limts of turbulent transport in flows with a non-zero mean drift have different phase diagrams (see [2,26,45]) and will be reported in a forthcoming paper.

6.1. Role of molecular diffusion

Molecular diffusion has at least two roles: (i) to eliminate possible dependence of scaling limit on the initial point, as particles may be trapped by closed or bounded streamlines, so that the process may be homogenized; (ii) to reduce dynamic velocity correlation in time, and thus, change the scaling law to one with larger scaling exponent

q. The first role of molecular diffusion is prominent for transport in steady flows; without it, localized streamlines would prevent homogenization from happening (see [18]). In this connection, molecular diffusion also helps to blend the effects of streamlines of different scaling behaviors. In the present work, we assume homogenization and focus on the second role of molecular diffusion.

Molecular diffusion is negligible in Regimes II, III, IV, where the sampling drift dominates the transport, as well as Regime V and part of Regime I (i.e., $\alpha+\beta<1$), where velocity decorrelation in time is significant. But its effect is not so clear in Regimes VI and the other part of Regime I (i.e., $\alpha+\beta\geq 1, \alpha<0$), where the spatially fluctuating wave numbers dominate.

In this regard, when high wave numbers are negligible, as for $\alpha+\beta>1$, one can go further by comparing the term representing molecular diffusion, $\kappa\Delta$, and the term representing the flow, $\mathcal{A}$, in (17), and see that, for $\beta\leq 1$, the effect of molecular diffusion should not affect the scaling law. As the velocity dependence on space dominates over that on time, the scaling law is independent of β. In view of the discussion in Section 4.5 on effect of molecular diffusion, we expect the scaling law for $\beta>1$ can be extrapolated from that for $\beta\leq 1$ to conclude the scaling law of Regime VI is independent of molecular diffusion as well as β.

As α on the phase boundary ($\alpha=0$) is bigger than that in the region ($\alpha+\beta\geq 1, \alpha<0$), for given β, the scaling exponent q in the region, with a vanishing molecular diffusion, should not be less than that of the phase boundary, which is 1. Therefore, $q=1$ in this region and the diffusivity stugs bounded with a vanishing molecular diffusion.

Acknowledgements

I thank an anonymous referee for careful reading and useful remarks which led to improvement of the paper. The research is supported in part by National Science Foundation Grant DMS-9600119 and DMS-9971322.

Appendix A. Variational principle for eddy diffusivity

To derive the variational principle (30) we consider a pair of period cell problems for an arbitrary constant unit vector $\mathbf{e}$

$$\mathcal{A}^{(\varepsilon,n)}\chi_+^{(\varepsilon,n)}+(\boldsymbol{c}_\varepsilon+\tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi_+^{(\varepsilon,n)}+\tilde{\mathbf{V}}^{(\varepsilon,n)}\cdot\mathbf{e}=0,\quad \text{in } \boldsymbol{\Omega}^{(n)} \tag{A.1}$$

$$\mathcal{A}^{(\varepsilon,n)}\chi_-^{(\varepsilon,n)}-(\boldsymbol{c}_\varepsilon+\tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi_-^{(\varepsilon,n)}-\tilde{\mathbf{V}}^{(\varepsilon,n)}\cdot\mathbf{e}=0,\quad \text{in } \boldsymbol{\Omega}^{(n)} \tag{A.2}$$

where both $\chi_+^{(\varepsilon,n)}$ and $\chi_-^{(\varepsilon,n)}$ satisfy the periodice boundary condition. Note that (A.2) is simply the adjoint of (A.1) as $\tilde{\mathbf{V}}^{(\varepsilon,n)}$ is divergence free.

Adding and subtracting (A.1) and (A.2) we obtain

$$\mathcal{A}^{(\varepsilon,n)}\chi^{(\varepsilon,n)}+(\boldsymbol{c}_\varepsilon+\tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi^{(\varepsilon,n)'}=0 \tag{A.3}$$

$$\mathcal{A}^{(\varepsilon,n)}\chi^{(\varepsilon,n)'}+(\boldsymbol{c}_\varepsilon+\tilde{\mathbf{V}}^{(\varepsilon,n)})\cdot\nabla\chi^{(\varepsilon,n)}-\tilde{\mathbf{V}}^{(\varepsilon,n)}\cdot\mathbf{e}=0 \tag{A.4}$$

where

$$\chi^{(\varepsilon,n)}=\tfrac{1}{2}(\chi_+^{(\varepsilon,n)}+\chi_-^{(\varepsilon,n)}),\qquad \chi^{(\varepsilon,n)'}=\tfrac{1}{2}(\chi_+^{(\varepsilon,n)}-\chi_-^{(\varepsilon,n)}). \tag{A.5}$$

First we established some useful identities for

$$D^{(\varepsilon,n)}(\mathbf{e})=\mathbf{D}^{(\varepsilon,n)}\mathbf{e}\cdot\mathbf{e}=-\langle\mathcal{A}^{(\varepsilon,n)}\chi_+^{(\varepsilon,n)'}\chi_+^{(\varepsilon,n)}\rangle_n.$$

Proposition 1.

$$D^{(\varepsilon,n)}(\mathbf{e}) = \langle \chi_+^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n = -\langle \chi_-^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n = -\langle \mathcal{A}^{(\varepsilon,n)} \chi_-^{(\varepsilon,n)} \chi_-^{(\varepsilon,n)}\rangle_n. \tag{A.6}$$

The first identify in (A.6) follows from integration by parts after multiplication of Eq. (A.1) by $\chi_+^{(\varepsilon,n)}$. To verify the second, we make use Eqs. (A.1), (A.2) and the divergence free property of $\tilde{\mathbf{V}}^{(\varepsilon,n)}$ in the following calculation

$$\begin{aligned} D^{(\varepsilon,n)}(\mathbf{e}) &= \langle(-\mathcal{A}^{(\varepsilon,n)} - (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla)\chi_+^{(\varepsilon,n)} \chi_+^{(\varepsilon,n)}\rangle_n \\ &= \langle[-\mathcal{A}^{(\varepsilon,n)}\chi_+^{(\varepsilon,n)} - (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla)\chi_+^{(\varepsilon,n)} - \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}]\chi_+^{(\varepsilon,n)}\rangle_n + \langle \chi_+^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n \\ &\underset{(A.1)}{=} \langle[-\mathcal{A}^{(\varepsilon,n)}\chi_+^{(\varepsilon,n)} - (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla)\chi_+^{(\varepsilon,n)} - \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}]\chi_-^{(\varepsilon,n)}\rangle_n + \langle \chi_+^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n \\ &= \langle[-\mathcal{A}^{(\varepsilon,n)}\chi_-^{(\varepsilon,n)} + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla)\chi_-^{(\varepsilon,n)}]\chi_+^{(\varepsilon,n)}\rangle_n - \langle \chi_-^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n + \langle \chi_+^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n \\ &= \langle[-\mathcal{A}^{(\varepsilon,n)}\chi_-^{(\varepsilon,n)} + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla)\chi_-^{(\varepsilon,n)} + \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}]\chi_+^{(\varepsilon,n)}\rangle_n - \langle \chi_-^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n \\ &\underset{(A.7)}{=} \langle \chi_-^{(\varepsilon,n)} \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \mathbf{e}\rangle_n. \end{aligned}$$

Here we have used the identity

$$\langle[(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla\chi_+^{(\varepsilon,n)}]\chi_+^{(\varepsilon,n)}\rangle_n = \tfrac{1}{2}\nabla \cdot \langle(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})(\chi_+^{(\varepsilon,n)})^2\rangle_n = 0$$

as a result of the incompressibility of $\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}$ and the space-homogeneity of $\langle(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)})(\chi_+^{(\varepsilon,n)})^2\rangle_n$.

Thus, in view of (A.5), the following result in clear.

Proposition 2.

$$D^{(\varepsilon,n)}(\mathbf{e}) = \langle \chi^{(\varepsilon,n)'} \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e}\rangle_n = -\langle \mathcal{A}^{(\varepsilon,n)} \chi^{(\varepsilon,n)'} \chi^{(\varepsilon,n)'}\rangle_n = \langle \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \nabla\chi^{(\varepsilon,n)} \chi^{(\varepsilon,n)'}\rangle_n.$$

Next, we derive the variational principle (30).

Let g be the minimizer of the convex functional in (30) and g' be the periodic solution of the equation

$$\mathcal{A}^{(\varepsilon,n)} g' + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla g + \tilde{\mathbf{V}}^{(\varepsilon,n)} \cdot \mathbf{e} = 0. \tag{A.7}$$

Taking the first variation of the functional in (30) at g we have

$$-\langle \mathcal{A}^{(\varepsilon,n)} g\delta g\rangle_n - \langle \mathcal{A}^{(\varepsilon,n)} g'\delta g'\rangle_n = 0 \tag{A.8}$$

where the variation $\delta g'$ is related to the variation δg by

$$\mathcal{A}^{(\varepsilon,n)}\delta g' + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla\delta g = 0 \tag{A.9}$$

following (A.7). Substituting (A.9) into (A.8) and integrating by parts we get

$$\langle \mathcal{A}^{(\varepsilon,n)} g\delta g\rangle_n + \langle(\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla g'\delta g\rangle_n = 0$$

for all admissible variations δg. Thus

$$\mathcal{A}^{(\varepsilon,n)} g + (\boldsymbol{c}_\varepsilon + \tilde{\mathbf{V}}^{(\varepsilon,n)}) \cdot \nabla g' = 0 \tag{A.10}$$

Since Eqs. (A.3) and (A.4) (also (A.7), (A.10)) are well posed, we conclude that $g = \chi^{(\varepsilon,n)}$, $g' = \chi^{(\varepsilon,n)'}$ up to constants.

By reversing the above argument, it is easy to see that $g = \chi^{(\varepsilon,n)}$, $g' = \chi^{(\varepsilon,n)'}$ with $\chi^{(\varepsilon,n)}$, $\chi^{(\varepsilon,n)'}$ given by (A.5) are the minimizer of (30).

References

[1] M. Avellaneda, A.J. Majda, Mathematical model with exact renormalization for turbulent transport, Comm. Math. Phys. 131 (1990) 381–429.
[2] M. Avellaneda, A.J. Majda, Mathematical model with exact renormalization for turbulent transport, II, Comm. Math. Phys. 146 (1992) 139–204.
[3] M. Avellaneda, A.J. Majda, Renormalization theory for eddy diffusivity in turbulent transport, Phys. Rev. Lett. 68(20) (1992) 3028–3031.
[4] M. Avellaneda, M. Vergassola, Stieltjes integral representation of effective diffusivities in time-dependent flows, Phys. Rev. E 52(3B) (1995) 3249–3251.
[5] A. Bensoussan, J.L. Lions, G.C. Papanicolaou, Asymptotic Analysis for Periodic Structures, North-Holland, Amsterdam, 1978.
[6] A.N. Borodin, Theory Prob. Appl. (Engl. Transl.) 22 (1978) 482–497.
[7] J.-P. Bouchaud, A. Georges, Anomalous diffusion in disordered media: statistical mechanics, models and physical applications, Phys. Rep. 195(4&5) (1990) 127–293.
[8] R.A. Carmona, J.P. Fouque, in: E. Bolthausen, M. Dozzi, F. Russo (Eds.), Diffusion approximation for the advection diffusion of a passive scalar by a space–time Gaussian velocity field, Seminar on Stochastic Analysis, Random Fields and Applications, Birkhauser, Basel, 1994 pp. 37–50.
[9] R.A. Carmona, L. Xu, Homogenization for time dependent 2d incompressible Gaussian flows, Ann. Appl. Prob. 7 (1997) 265–279.
[10] J.L. Doob, Stochastic Processes, Wiley, New York (1953).
[11] W.E. Homogenization of linear and nonlinear transport equations, Comm. Pure Appl. Math. 45 (3) (1992) 301–326.
[12] A. Fannjiang, Anomalous diffusion in random flows, in: K.M. Golden, G.R. Grimmett, R.D. James, G.W. Milton, P.N. Sen (Eds.), IMA Proceedings, vol. 99, Mathematics of Multiscale Materials, 1998, pp. 81–99.
[13] A. Fannjiang, T. Komorowski, A martingale approach to homogenization of unbounded random flows, Ann. Prob. 25(4) (1997) 1872–1894.
[14] A. Fannjiang, T. Komorowski, An invariance principle for diffusion in turbulence, Ann. Prob. 27 (2) (1999) 751–781.
[15] A. Fannjiang, T. Komorowski, Turbulent diffusion in Markovian flows, Ann. Appl. Prob. 9 (3) (1999) 591–610.
[16] A. Fannjiang, T. Komorowski, Fractional-Brownian-motion limit for a model of turbulent transport, to appear.
[17] A. Fannjiang, G.C. Papanicolaou, Convection enhanced diffusion for periodic flows, SIAM J. Appl. Math. 54(2) (1994) 333–408.
[18] A. Fannjiang, G.C. Papanicolaou, Convection enhanced diffusion in random flows, J. Stat. Phys. 88(5–6) (1997) 1033–1076.
[19] A. Fannjiang, G.C. Papanicolaou, Diffusion in turbulence, Prob. Theo. Rel. Fields 105 (1996) 279–334.
[20] U. Frisch, Turbulence: the legacy of A.N. Kolmogorov, Cambridge University Press, Cambridge, 1995.
[21] J. Glimm, D.H. Sharp, A random field model for anomalous diffusion in heterogeneous porous media, J. Stat. Phys. 62 (1991) 415–424.
[22] J. Glimm, D.H. Sharp, Stochastic partial differential equations; Selected applications in continuum physics, Los Alamos National Laboratory, preprint LAUR-96-4917.
[23] J. Kalda, On transport of passive scalar in 2d turbulent flow, in: J. Douglas Jr., U. Hornung (Eds.), Flow in porous media: proceedings of the Oberwolfach Conference, 21–27 June 1992, Birkhauser, Basel, 1993.
[24] H. Kesten, G. Papanicolaou, A limit theorem in turbulent diffusion, Comm. Math. Phys. 65 (1979) 97–128.
[25] R.Z. Khasminski, A limit theorem for solutions of differential equations with a random right hand side, Theory Prob. Appl. 11 (1966) 390–406.
[26] D.L. Koch, J.F. Brady, Anomalous diffusion in heterogeneous porous media, Phys. of Fluids 31(5) (1988) 965–973.
[27] D.L. Koch, J.F. Brady, Anomalous diffusion due to long-range velocity fluctuations in the absence of a mean flow, Phys. Fluids A 1 (1989) 47.
[28] A.N. Kolmogorov, A refinement of previous hypotheses concerning the local structure of turbulence in a viscous incompressible fluid at high Reynolds number, J. Fluid Mech. 13 (1962) 82–85.
[29] T. Komorowski, Diffusion approximation for the advection of particles in a strongly turbulent random environment, Ann. Prob. 24 (1996) 346–376.
[30] T. Komorowski, Application of the parametric method to diffusions in a turbulent Gaussian environment, Stochastic Processes and their Applications 74 (1998) 165–193.
[31] L. Koralov, Effective diffusivity of stationary vector fields with short time correlations, Random Oper. Stochastic Eqs. 5(4) (1997) 303–324.
[32] R.H. Kraichnan, Diffusion by a random velocity field, Phys. Fluids 13(1) (1970) 22–31.
[33] R.H. Kraichnan, Eddy viscosity and diffusivity: exact formulas and approximation, Complex Systems 1 (1987) 805–820.
[34] R. Kubo, Stochastic Liouville Equation, J. Math. Phys. 4 (1963) 174–183.
[35] J.L. Lumley, The mathematical nature of the problem of relating Lagrangian and Eulerian statistical functions in turbulence, Méchanique de la turbulence. Coll. Intern. du CNRS à Marseille. Ed. CNRS, Paris, 1962.
[36] B.B. Mandelbrot, J.W. Van Ness, Fractional Brownian motions, fractonal noises and applications, SIAM Rev. 10 (1968) 422–437.
[37] K.R. Parthasarathy, On the category of ergodic measures, Ill. J. Math. 5 (1961) 648–655.
[38] L. Piterbarg, Short-correlation approximation in models of turbulent diffusion, in: S.A. Molchanov, W.A. Woyczynski (Eds.), Stochastic Models in Geosystems, IMA, vol. 85, Springer, Berlin, 1997, pp. 313–352.
[39] S.C. Port, C.J. Stone, Random measures and their application to motion in an incompressible fluid, J. Appl. Prob. 13 (1976) 498–506.
[40] M. Rosenblatt, Markov Processes, Structure and Asymptotic Behavior, Springer, Berlin, 1971.

[41] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance, Chapman & Hall, New York, 1994.
[42] G.I. Taylor, Diffusions by continuous movement, Proc. London Math. Soc. Ser. 2(20) (1923) 196–211.
[43] L. Tartar, Nonlinear effects induced by homogenization, in: F. Colombini et al. (Eds.), Partial differential equations and the calculus of variations: essays in honor of Ennio De Giorgi, Birkhauser, Boston, 1989.
[44] L. Tartar, Memory effects and homogenization, Arch. Rational Mech. Anal. 111(2) (1990) 121–133 .
[45] Q. Zhang, The asymptotic scaling behavior of mixing induced by a random velocity field, Adv. Appl. Math. 16(1) (1995) 23–58.
[46] Q. Zhang, J. Glimm, Inertial range scaling of laminar shear flow as a model of turbulent transport, Comm. Math. Phys. 146(2) (1992) 217–229.