1

Fractional Brownian Motion in a Limit of Turbulent Transport

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May 30, 2001

Abstract We show that the motion of a particle advected by a random Gaussian velocity field with long-range correlations converges to a fractional Brownian motion in the long time.

Keywords Turbulent diffusion, mixing, fractional Brownian motion.

AMS subject classification Primary 60F05, 76F05, 76R50; Secondary 58F25.

Abbreviated title fractional Brownian motion limit.

*The research of Fannjiang is supported in part by National Science Foundation Grant No. DMS-9600119. This work was finished while Komorowki was visiting Department of Statistics, University of California, Berkeley.

1 Introduction

The motion of a particle advected by a random velocity field is governed by

$$\frac{d\boldsymbol{x}(t)}{dt} = \mathbf{V}(t, \boldsymbol{x}(t)) \tag{1}$$

where $\mathbf{V}(t, \boldsymbol{x}) = (V_1(t, \boldsymbol{x}), \cdots, V_d(t, \boldsymbol{x}))$ is a random, mean-zero, time-stationary, space-homogeneous incompressible velocity field in dimension $d \geq 2$ in dimension $d \geq 2$...

In certain situations, it is believed that the *convergence* of the Taylor-Kubo formula ([14], [8]) given by

$$\int_0^\infty \{\mathbf{E}[V_i(t, \mathbf{0})V_j(0, \mathbf{0})] + \mathbf{E}[V_j(t, \mathbf{0})V_i(0, \mathbf{0})]\} \, dt \tag{2}$$

is a criterion for convergence of turbulent motion to Brownian motion in the long time limit. Indeed, it has been shown that the solution of

$$\frac{d\boldsymbol{x}^\varepsilon(t)}{dt} = \frac{1}{\varepsilon}\mathbf{V}(\frac{t}{\varepsilon^2}, \boldsymbol{x}^\varepsilon(t)), \quad \boldsymbol{x}^\varepsilon(0) = 0 \tag{3}$$

converges in law, as $\varepsilon \to 0$, to the Brownian motion with diffusion coefficients given by the Taylor-Kubo formula when the velocity field is sufficiently *mixing in time* (see [7], [6], [9], [2]). Moreover, the solution of (3) converges to the same Brownian motion for a family of *non-mixing* Gaussian, Markovian flows with power-law spectra as long as the Taylor-Kubo formula converges (see [3]). In this paper, for the same family of power-law spectra, we show that, when the Taylor-Kubo formula *diverges*, the solution of the following equation

$$\frac{d\boldsymbol{x}_\varepsilon(t)}{dt} = \varepsilon^{1-2\delta}\mathbf{V}(\frac{t}{\varepsilon^{2\delta}}, \boldsymbol{x}_\varepsilon(t)), \quad \boldsymbol{x}^\varepsilon(0) = 0, \tag{4}$$

with some $\delta \neq 1$ depending on the velocity spectrum, converges, as $\varepsilon \to 0$, to a fractional Brownian motion (FBM), as introduced in [10] (see also [13]).

We define the family of velocity fields with power-law spectra as follows. Let $(\Omega, \mathcal{V}, P)$ be a probability space of which each element is a velocity field $\mathbf{V}(t, \boldsymbol{x})$, $(t, \boldsymbol{x}) \in R \times R^d$ satisfying the following properties.

H 1) $\mathbf{V}(t, \boldsymbol{x})$ is time stationary, space-homogeneous and centered, i.e., $\mathbf{E}\{\mathbf{V}\} = \mathbf{0}$, and Gaussian. Here $\mathbf{E}$ stands for the expectation with respect to the probablity measure P.

H 2) The two-point correlation tensor $\mathbf{R} = [R_{ij}]$ is given by

$$R_{ij}(t, \boldsymbol{x}) = \mathbf{E}\left[V_i(t, \boldsymbol{x})V_j(0, \mathbf{0})\right] = \int_{R^d} \cos\left(\mathbf{k} \cdot \boldsymbol{x}\right) e^{-|\mathbf{k}|^{2\beta}t} \hat{\mathbf{R}}_{ij}(\mathbf{k}) d\mathbf{k} \tag{5}$$

with the spatial spectral density

$$\hat{\mathbf{R}}(\mathbf{k}) = \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}}\left(\mathbf{I} - \frac{\mathbf{k} \otimes \underline{\mathbf{k}}}{|\mathbf{k}|^2}\right), \tag{6}$$

where $a : [0, +\infty) \to R_+$ is a compactly supported, continuous, nonnegative function. The factor $\mathbf{I} - \mathbf{k} \otimes \mathbf{k}/|\mathbf{k}|^2$ in (6) is a result of incompressibility.

H 3) $\alpha < 1$, $\beta \geq 0$ and $\alpha + \beta > 1$.

It can be readily checked that the correlation function (5) is temporally integrable and, hence, the Taylor-Kubo formula is convergent if and only if $\alpha + \beta < 1$.

The function $\exp(-|\mathbf{k}|^{2\beta} t)$ in (5) is called the *time correlation function* of the flow $\mathbf{V}$. For $\beta > 0$, the velocity field lacks the spectral gap and, thus, is not mixing in time. As the time correlation function is exponential, the Gaussian velocity field is an Ornstein-Uhlenbeck process. Because the function a has a compact support we may assume, without loss of generality, that $\mathbf{V}$ is jointly continuous in both $(t, \boldsymbol{x})$ and is C^∞ in $\boldsymbol{x}$ almost surely. For $\alpha < 1$, the spectral density $\hat{\mathbf{R}}(\mathbf{k})$ is integrable in $\mathbf{k}$ and, thus, (5)-(6) defines a random velocity field with a finite second moment. The exponent α is directly related to the decay exponent of $\mathbf{R}$. Namely $|\mathbf{R}|(0, \boldsymbol{x}) \sim |\boldsymbol{x}|^{\alpha-1}$ for $|\boldsymbol{x}| \gg 1$. As α increases to one, the decay exponent of $\mathbf{R}$ decreases to zero.

Our main result is summarized in the following theorem.

Theorem 1 *Under the assumptions H 1)- H 3), the solution of eq. (4) with the scaling exponent*

$$\delta := \frac{\beta}{\alpha + 2\beta - 1} \tag{7}$$

converges in law, as ε tends to zero, to a fractional Brownian motion $\mathbf{B}_H(t)$ that is to a Gaussian process with stationary increments whoe covariance is given by

$$\mathbf{E}\left[\mathbf{B}_H(t) \otimes \mathbf{B}_H(t)\right] = \mathbf{D} t^{2H}, \tag{8}$$

with the coefficients $\mathbf{D}$

$$\mathbf{D} = \int\limits_{R^d} \frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha + 4\beta - 1}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}\right) \frac{a(0)}{|\mathbf{k}|^{d-1}} d\mathbf{k} \tag{9}$$

and the Hurst exponent H

$$1/2 < H = 1/2 + \frac{\alpha + \beta - 1}{2\beta} < 1. \tag{10}$$

Remark. Molecular diffusion can be added to the equation of motion so that instead of (1) we may consider an Itô stochastic differential equation

$$d\boldsymbol{x}(t) = \mathbf{V}(t, \boldsymbol{x}(t))dt + \sqrt{2\kappa} d\mathbf{B}(t)$$

with $\mathbf{B}(t)$, $t \geq 0$ the standard Brownian motion, independent of $\mathbf{V}$ and $\kappa \geq 0$. This however would not influence our results.

2 Multiple stochastic integrals

By the Spectral Theorem (see, e.g., [1]) we assume without loss of any generality that there exist two independent, identically distributed, real vector valued, Gaussian spectral measures $\hat{\mathbf{V}}_l(t, \cdot)$, $l = 0, 1$ such that

$$\mathbf{V}(t, \boldsymbol{x}) = \int \hat{\mathbf{V}}_0(t, \boldsymbol{x}, d\mathbf{k}), \tag{11}$$

where

$$\hat{\mathbf{V}}_0(t, \boldsymbol{x}, d\mathbf{k}) := c_0(\mathbf{k} \cdot \boldsymbol{x}) \hat{\mathbf{V}}_0(t, d\mathbf{k}) + c_1(\mathbf{k} \cdot \boldsymbol{x}) \hat{\mathbf{V}}_1(t, d\mathbf{k})$$

with $c_0(\phi) \equiv \cos(\phi), c_1(\phi) \equiv \sin(\phi)$. Define also

$$\hat{\mathbf{V}}_1(t, \boldsymbol{x}, d\mathbf{k}) := -c_1(\mathbf{k} \cdot \boldsymbol{x}) \hat{\mathbf{V}}_0(t, d\mathbf{k}) + c_0(\mathbf{k} \cdot \underline{\mathrm{x}}) \hat{\mathbf{V}}_1(t, d\mathbf{k}).$$

We have the relations

$$\partial\hat{\mathbf{V}}_0(t,\boldsymbol{x},d\mathbf{k})/\partial x_j \;=\; k_j\hat{\mathbf{V}}_1(t,\boldsymbol{x},d\mathbf{k}), \tag{12}$$

$$\partial\hat{\mathbf{V}}_1(t,\boldsymbol{x},d\mathbf{k})/\partial x_j \;=\; -k_j\hat{\mathbf{V}}_0(t,\boldsymbol{x},d\mathbf{k}). \tag{13}$$

Clearly $\int\hat{\mathbf{V}}_1(t,\boldsymbol{x},d\mathbf{k})$ is a random field distributed identically to and independently of $\mathbf{V}$. We define the *multiple stochastic integral*

$$\int\cdots\int\psi(\mathbf{k}_1,\cdots,\mathbf{k}_N)\widehat{\mathbf{V}}_{l_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}_1)\otimes\cdots\o times\widehat{\mathbf{V}}_{l_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}_N) \tag{14}$$

for any $l_1,\cdots,l_N\in\{0,1\}$ and a suitable family of functions ψ by using the Fubini theorem (see (15) below). For $\psi_1,\cdots,\psi_N\in\mathcal{S}(R^d)$, the Schwartz space, and $l_1,\cdots,l_N\in\{0,1\}$ we set

$$\begin{aligned}&\int\cdots\int\psi_1(\mathbf{k}_1)\cdots\psi_N(\mathbf{k}_N)\widehat{\mathbf{V}}_{l_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}_N) \qquad (15)\\ :=\;&\int\psi_1(\mathbf{k}_1)\widehat{\mathbf{V}}_{l_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\int\psi_N(\mathbf{k}_N)\widehat{\mathbf{V}}_{l_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}_N).\end{aligned}$$

We then extend the definition of multiple integration to the closure $\mathcal{H}$ of the Schwartz space $\mathcal{S}((R^d)^N,R)$ under the norm

$$\|\psi\|^2:=\int\cdots\int\psi(\mathbf{k}_1,\cdots,\mathbf{k}_N)\psi(\mathbf{k}'_1,\cdots,\mathbf{k}'_N) \tag{16}$$

$$\mathbf{E}\left[\widehat{\mathbf{V}}_{l_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}_N)\cdot\widehat{\mathbf{V}}_{l_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}'_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}'_N)\right].$$

The expectation is to be calculated by the formal rule

$$\mathbf{E}\left[\widehat{V}_{l,i}(t,\boldsymbol{x},d\mathbf{k})\widehat{V}_{l',i'}(t',\boldsymbol{x}',d\mathbf{k}')\right]=e^{-|\mathbf{k}|^{2\beta}|t-t'|}\delta_{l,l'}c_0(\mathbf{k}\cdot(\boldsymbol{x}-\boldsymbol{x}'))\widehat{R}_{i,i'}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')d\mathbf{k}d\mathbf{k}'.$$

This approach to spectral integration follows [12].

When $\mathbf{i}=(i_1,\cdots,i_d),i_1,\cdots,i_d\in\{1,2,\cdots,d\}$ is fixed and $\mathbf{l}=(l_1,\cdots,l_N),l_1,\cdots,l_N\in\{0,1\}$ we shall denote the corresponding component of the stochastic integral by $\Psi_{\mathbf{l},\mathbf{i}}$.

Note that $\Psi_{\mathbf{l},\mathbf{i}}\in H^N(\mathbf{V})$ - the Hilbert space obtained as a completion of the space of N-th degree polynomials in variables $\int\psi(\mathbf{k})\widehat{\mathbf{V}}(t,\boldsymbol{x},\mathbf{k})$ with respect to the standard L^2 norm.

Proposition 1 *For any $(t_1,\boldsymbol{x}_1),\cdots,(t_N,\boldsymbol{x}_N)\in R\times R^d$ and $p>0$, $\Psi_{\mathbf{l},\mathbf{i}}$ belongs to $L^p(\Omega)$ and*

$$(\mathbf{E}|\Psi_{\mathbf{l},\mathbf{i}}|^p)^{1/p}\le C\left(\mathbf{E}|\Psi_{\mathbf{l},\mathbf{i}}|^2\right)^{1/2} \tag{17}$$

with the constant C depending only on p,N and the dimension d. Moreover, $\Psi_{\mathbf{l},\mathbf{i}}$ is differentiable in the mean square sense with

$$\nabla_{\boldsymbol{x}_j}\Psi_{\mathbf{l},\mathbf{i}}(t_1,\cdots,t_N,\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N)=(-1)^{l_j}\int\cdots\int\mathbf{k}_j\psi(\mathbf{k}_1,\cdots,\mathbf{k}_N) \tag{18}$$

$$\widehat{V}_{l_1,i_1}(t_1,\boldsymbol{x}_1,d\mathbf{k}_1)\cdots\widehat{V}_{1-l_j,i_j}(t_j,\boldsymbol{x}_j,d\mathbf{k}_j)\cdots\widehat{V}_{l_N,i_N}(t_N,\boldsymbol{x}_N,d\mathbf{k}_N).$$

The proof of Proposition 1 is standard and follows directly from the well known hypercontractivity property for Gaussian measures (see, e.g., [5], Theorem 5.1. and its corollaries), so we do not repeat it here.

The field $\mathbf{V}$ is Markovian i.e.

$$\mathbf{E}\left[\int \psi(\mathbf{k})\widehat{\mathbf{V}}_l(t,\boldsymbol{x},d\mathbf{k}) \mid \mathcal{V}_{-\infty,s}\right] = \int e^{-|\mathbf{k}|^{2\beta}(t-s)}\psi(\mathbf{k})\widehat{\mathbf{V}}_l(s,\boldsymbol{x},d\mathbf{k}), \quad l=0,1, \tag{19}$$

for all $\psi \in \mathcal{S}(R^d, R)$, where $\mathcal{V}_{a,b}$ denotes the σ-algebra generated by random variables $\mathbf{V}(t,\boldsymbol{x})$, for $t \in [a,b]$ and $\boldsymbol{x} \in R^d$.

To calculate a mathematical expectation of multiple product of Gaussian random variables, it is convenient to use a graphical representation, borrowed from quantum field theory. We refer to, e.g., Glimm and Jaffe [4] and Janson [5]. A *Feynman diagram* $\mathcal{F}$ (of order $n \geq 0$ and rank $r \geq 0$) is a graph consisting of a set $B(\mathcal{F})$ of n vertices and a set $E(\mathcal{F})$ of r edges without common endpoints. So there are r pairs of vertices, each joined by an edge, and $n-2r$ unpaired vertices, called *free vertices*. $B(\mathcal{F})$ is a set of positive integers. An edge whose endpoints are $m, n \in B$ is represented by $\widehat{mn}$ (unless otherwise specified, we always assume $m<n$); and an edge includes its endpoints. A diagram $\mathcal{F}$ is said to be *based on* $B(\mathcal{F})$. Denote the set of free vertices by $A(\mathcal{F})$, so $A(\mathcal{F}) = \mathcal{F} \setminus E(\mathcal{F})$. The diagram is *complete* if $A(\mathcal{F})$ is empty and *incomplete*, otherwise. Denote by $\mathcal{G}(B)$ the set of all diagrams based on B, by $\mathcal{G}_c(B)$ the set of all complete diagrams based on B and by $\mathcal{G}_i(B)$ the set of all incomplete diagrams based on B. A diagram $\mathcal{F}' \in \mathcal{G}_c(B)$ is called a *completion* of $\mathcal{F} \in \mathcal{G}_i(B)$ if $E(\mathcal{F}) \subseteq E(\mathcal{F}')$.

Let $B = \{1,2,3,...,n\}$. Denote by $\mathcal{F}_{|k}$ the sub-diagram of $\mathcal{F}$, based on $\{1,\cdots,k\}$. Define $A_k(\mathcal{F}) = A(\mathcal{F}_{|k})$. A special class of diagrams, denoted by $\mathcal{G}_s(B)$, plays an important role in the subsequent analysis: a diagram $\mathcal{F}$ of order n belongs to $\mathcal{G}_s(B)$ if $A_k(\mathcal{F})$ is not empty for all $k=1,...,n$. We shall adopt the following multiindex notation. For any $P \in Z^+$, multiindex $\mathbf{n} = (n_1,\cdots,n_P)$, $|\mathbf{n}|$ stands for $\sum n_p$. If $P' \leq P$ we denote $\mathbf{n}_{|P'} := (n_1,\cdots,n_{P'})$. In addition if k is any number we set $\mathbf{n}\cdot k := (n_1,\cdots,n_P,k)$. We work out the conditional expectation for multiple spectral integrals using the Markov property (19).

Proposition 2 *For any function $\psi \in \mathcal{H}$ and $l_1,\cdots,l_N \in \{0,1\}$, $i_1,\cdots,i_N \in \{1,\cdots,d\}$,*

$$\mathbf{E}\left[\int\cdots\int \psi(\mathbf{k}_1,\cdots,\mathbf{k}_N)\widehat{V}_{l_1,i_1}(t,\boldsymbol{x}_1,d\mathbf{k}_1)\cdots\widehat{V}_{l_N,i_N}(t,\boldsymbol{x}_N,d\mathbf{k}_N) \mid \mathcal{V}_{-\infty,s}\right] = \tag{20}$$

$$\sum_{\mathcal{F}\in\mathcal{G}(\{1,\ldots,N\})}\int\cdots\int \exp\left\{-\sum_{m\in A(\mathcal{F})}|\mathbf{k}_m|^{2\beta}(t-s)\right\}\psi(\mathbf{k}_1,\cdots,\mathbf{k}_N)\widehat{V}_{s,\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N}(d\mathbf{k}_1,\cdots,d\mathbf{k}_N;\mathcal{F})$$

with

$$\widehat{V}_{s,\boldsymbol{x}_1,\cdots,\boldsymbol{x}_N}(d\mathbf{k}_1,\cdots,d\mathbf{k}_N;\mathcal{F}) := \prod_{m\in A(\mathcal{F})}\widehat{V}_{l_m,i_m}(s,\boldsymbol{x}_m,d\mathbf{k}_m) \tag{21}$$

$$\times\prod_{\widehat{mn}\in E(\mathcal{F})}\left[1-e^{-\left(|\mathbf{k}_m|^{2\beta}+|\mathbf{k}_n|^{2\beta}\right)(t-s)}\right]\mathbf{E}[\widehat{V}_{l_m,i_m}(s,\boldsymbol{x}_m,d\mathbf{k}_m)\widehat{V}_{l_n,i_n}(s,\boldsymbol{x}_n,d\mathbf{k}_n)].$$

Proof. Without loss of generality we consider $\psi(\mathbf{k}_1,\cdots,\mathbf{k}_N) = \mathbf{1}_{A_1}(\mathbf{k}_1)\cdots\mathbf{1}_{A_N}(\mathbf{k}_N)$ for some Borel sets $A_1,\cdots,A_N$. Note that $\widehat{\mathbf{V}}_l(t,A_i) = \widehat{\mathbf{V}}_l^0(t,A_i) + \widehat{\mathbf{V}}_l^1(t,A_i)$ where $\widehat{\mathbf{V}}_l^0(t,\cdot)$ is the orthogonal projection of $\widehat{\mathbf{V}}_l(t,\cdot)$ on $L^2_{-\infty,t}$ and $\widehat{\mathbf{V}}_l^1(t,\cdot)$ its complement. Here $L^2_{a,b}$ denotes L^2 closure of the linear span over $\mathbf{V}(s,\boldsymbol{x})$, $a \leq s \leq b$, $\boldsymbol{x} \in R^d$. The conditional expectation in (20) equals

$$\sum_{\mathcal{F}\in\mathcal{G}(\{1,\ldots,N\})}\prod_{\widehat{mn}\in E(\mathcal{F})}\mathbf{E}\left[\widehat{V}^1_{i_m,l_m}(t,A_m)\widehat{V}^1_{i_n,l_n}(t,A_n)\right]\prod_{m\in A(\mathcal{F})}\widehat{V}^0_{i_m,l_m}(t,A_m).$$

The statement follows upon the application of the relations

$$\widehat{\mathbf{V}}_l^0(t,A) = \int\limits_A e^{-|\mathbf{k}|^{2\beta}(t-s)}\widehat{\mathbf{V}}_l(s,d\mathbf{k})$$

and

$$\mathbf{E}\left[\widehat{\mathbf{V}}_l^1(t,A)\otimes\widehat{\mathbf{V}}_{l'}^1(t,B)\right] =$$

$$\int\limits_A\int\limits_B \delta_{l,l'}\left\{\mathbf{E}\left[\widehat{\mathbf{V}}_l(t,d\mathbf{k})\otimes\widehat{\mathbf{V}}_{l'}(t,d\mathbf{k}')\right] - \mathbf{E}\left[\widehat{\mathbf{V}}_l^0(t,d\mathbf{k})\otimes\widehat{\mathbf{V}}_{l'}^0(t,d\mathbf{k}')\right]\right\}.$$

□

3 Proof of tightness.

Let us start with the following result, which establishes, among other things, that the family of continuous trajectory processes $\boldsymbol{x}_\varepsilon(t)$, $t\ge 0$ is tight.

Lemma 1 *For the family of trajectories given by (4) we have*

$$\lim_{\varepsilon\downarrow 0}\mathbf{E}\left[(\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_\varepsilon(u))\otimes(\boldsymbol{x}_\varepsilon(t)-\boldsymbol{x}_\varepsilon(u))\right] = \mathbf{D}(t-u)^{2H},\ \ \textit{if}\ t>u$$

where H, $\mathbf{D}$ are given by (9), (10) respectively.

Proof. First let us observe that since $\boldsymbol{x}_\varepsilon(t)$ has stationary increments it is enough to prove the lemma for $u=0$. By the stationarity of $\mathbf{V}(s,\varepsilon\boldsymbol{x}(s))$ (see [11]), we can write that

$$\lim_{\varepsilon\downarrow 0}\mathbf{E}\left[\boldsymbol{x}_\varepsilon(t)\otimes\boldsymbol{x}_\varepsilon(t)\right] = \lim_{\varepsilon\downarrow 0}\varepsilon^2\int\limits_0^{t/\varepsilon^{2\delta}} ds\int\limits_0^s \mathbf{E}\left[\mathbf{V}(s',\varepsilon\boldsymbol{x}(s'))\otimes\mathbf{V}(0,\mathbf{0})\right]ds'. \tag{22}$$

Thus (22) equals

$$2\sum_{n=1}^N \mathcal{I}_n + \mathcal{R}_N \tag{23}$$

where

$$\mathcal{I}_n = \varepsilon^{n+1}\int\limits_0^{t/\varepsilon^{2\delta}} ds\int\limits_0^s ds_1\cdots\int\limits_0^{s_{n-1}}\mathbf{E}\left[\mathbf{W}_{n-1}(s_1,\cdots,s_n,\mathbf{0})\otimes\mathbf{V}(0,\mathbf{0})\right]ds_n$$

and

$$\mathbf{W}_0(s_1,\boldsymbol{x}) = \mathbf{V}(s_1,\boldsymbol{x})$$

$$\mathbf{W}_n(s_1,\cdots,s_{n+1},\boldsymbol{x}) = \mathbf{V}(s_{n+1},\boldsymbol{x})\cdot\nabla\mathbf{W}_{n-1}(s_1,\cdots,s_n,\boldsymbol{x}) \qquad \text{for } n=1,2,\cdots$$

with the remainder term

$$\mathcal{R}_N = 2\varepsilon^{N+2}\int\limits_0^{t/\varepsilon^{2\delta}} ds\int\limits_0^s ds_1\cdots\int\limits_0^{s_N}\mathbf{E}\left[\mathbf{W}_N(s_1,\cdots,s_{N+1},\varepsilon\boldsymbol{x}(s_{N+1}))\otimes\mathbf{V}(0,\mathbf{0})\right]ds_{N+1}. \tag{24}$$

Estimates of $\mathcal{I}_n$. Elementary calculations show that

$$\lim_{\varepsilon\downarrow 0}\mathcal{I}_1 = \mathbf{D}t^{2H}.$$

Since $\mathbf{V}$ is Gaussian we deduce that

$$\mathbf{E}\mathcal{I}_n = \mathbf{0}$$

when $n \geq 2$ is even. We now show that

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\mathcal{I}_n = \mathbf{0} \tag{25}$$

for $n \geq 3$ odd. The i, j-th entry of the matrix $\mathcal{I}_n$ is given by

$$\mathcal{I}^n_{i,j} = 2\varepsilon^{n+1} \int_0^{t/\varepsilon^{2\delta}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E}\left[\mathbf{E}_0 W_{n-1,i}(s_1, \cdots, s_n, \mathbf{0}) V_j(0, \mathbf{0})\right] ds_n. \tag{26}$$

We would like to express the conditional expectation appearing in (26) in terms of spectral measures associated with the velocity field. To do so we introduce first the so-called *proper* functions of order $n, \sigma : \{1, \cdots, n\} \to \{0, 1\}$ that appear in the statement of the next lemma. The proper function of order 1 is unique and is given by $\sigma(1) = 0$. Any proper function, σ', of order $n+1$ is generated from a proper function σ of order n as follows. For some $p \leq n$,

$$\begin{aligned}
\sigma'(n+1) &:= 0 \\
\sigma'(k) &:= \sigma(k) \quad \text{for} \quad k \leq n \text{ and } k \neq p \\
\sigma'(p) &:= 1 - \sigma(p).
\end{aligned} \tag{27}$$

In other words, each proper function σ of order n generates n different proper functions of order $n+1$. Thus, the total number of proper functions of order n is $(n-1)!$. In the sequel, we sometimes write σ_k instead of $\sigma(k)$.

Lemma 2 *Let $n \geq 1$ and $s_1 \geq s_2 \geq \cdots \geq s_n \geq s_{n+1}$, $i \in \{1, \cdots, d\}$, $\boldsymbol{x} \in R^d$. We have then that*

$$\begin{aligned}
\mathbf{E}_{s_{n+1}} W_{n-1,i}(s_1, \cdots, s_n, \boldsymbol{x}) := \mathbf{E}\left[W_{n-1,i}(s_1, \cdots, s_n, \boldsymbol{x}) \mid \mathcal{V}_{-\infty, s_{n+1}}\right] = \\
\sum \int \cdots \int \varphi^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \exp\{-\sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_n - s_{n+1})\} \times \\
P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{i_m, \sigma_m}(s_{n+1}, \boldsymbol{x}, d\mathbf{k}_m),
\end{aligned} \tag{28}$$

where $\varphi^{(n)}_{\mathbf{i},\sigma}$ are some functions with $\sup |\varphi^{(n)}_{\mathbf{i},\sigma}| \leq 1$

$$P_{n-1}(\mathcal{F}) = \prod_{j=1}^{n-1} \left(\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m| \right) \exp\{-\sum_{m \in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_j - s_{j+1})\} \tag{29}$$

$$Q(\mathcal{F}) = \prod_{\widehat{mm'} \in E(\mathcal{F})} \mathbf{E}\left[\widehat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \widehat{V}_{i_{m'}, \sigma_{m'}}(0, d\mathbf{k}_{m'})\right]. \tag{30}$$

The summation is over all multiindices $\mathbf{i}$ of length n, whose first component equals i, all $\mathcal{F} \in \mathcal{G}_s$ and all proper functions σ of order n.

Before proving Lemma 2, we apply it to show (25). Notice that according to (28)

$$\int_0^{t/\varepsilon^{2\delta}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E}_0 W_{n-1,i}(s_1,\cdots,s_n,\mathbf{0}) ds_n =$$

$$\sum \int_0^{t/\varepsilon^{2\delta}} ds \int \cdots \int \tilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_n) \exp\{-\sum_{m\in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s\} P'_{n-1}(\mathcal{F}) Q(\mathcal{F}) \prod_{m\in A_n(\mathcal{F})} \widehat{V}_{i_m,\sigma_m}(0,d\mathbf{k}_m)$$

for $i = 1,\cdots,d$. Here, adopting the convention $s_{n+1} := 0$, we set

$$\widetilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_n) :=$$

$$\frac{\int_0^s ds_1 \cdots \int_0^{s_{n-1}} \varphi^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_n) \times \prod_{j=1}^n \exp\left\{-\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_j - s_{j+1})\right\} ds_1\cdots ds_n}{\int_0^s ds_1 \cdots \int_0^s \prod_{j=1}^n \exp\left\{-\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s_j\right\} ds_1\cdots ds_n}$$

and

$$P'_{n-1}(\mathcal{F}) = \prod_{j=1}^{n-1} \left\{ \left(\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m| \right) \times \frac{1-\exp\{-\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta} s\}}{\sum_{m\in A_j(\mathcal{F})} |\mathbf{k}_m|^{2\beta}} \right\}.$$

It is elementary to check that, due to $|\varphi^{(n)}_{\mathbf{i},\sigma}| \le 1$,

$$|\widetilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_n)| \le 1. \tag{31}$$

Using Lemma 2 we infer that the left hand side of (26) equals

$$2\varepsilon^{n+1} \sum \int_0^{t/\varepsilon^{2\delta}} ds \int \cdots \int \frac{\tilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_n)}{\sum_{m\in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta}} P'_{n-1}(\mathcal{F}) Q(\mathcal{F}) \mathbf{E}\left[\prod_{m\in A_n(\mathcal{F})\cup\{n+1\}} \widehat{V}_{i_m,\sigma_m}(0,d\mathbf{k}_m)\right]. \tag{32}$$

Here the summation extends over all multiindices $\mathbf{i} = (i_1,\cdots,i_{n+1})$ such that $i_1 = i$, $i_{n+1} = j$, all Feynman diagrams $\mathcal{F} \in \mathcal{G}_s$ and all proper functions σ of order n. Using an elementary inequality stating that

$$\frac{1-e^{-x/\varepsilon^{2\delta}}}{x} \le \frac{C}{\varepsilon^{2\delta} + x}$$

for a certain constant C independent of ε, x we conclude that the absolute value of (32) is less than or equal to

$$2t\varepsilon^{n+1-2\delta} \sum \int_0^K \cdots \int_0^K \frac{p_{n-1,\varepsilon}(\mathcal{F})}{\varepsilon^{2\delta} + \sum_{m\in A_n(\mathcal{F})} k_m^{2\beta}} \prod_{\widehat{mm'}} \frac{\delta(k_m - k_{m'}) dk_m dk_{m'}}{k_m^{2\alpha-1}} \tag{33}$$

with

$$p_{n-1,\varepsilon}(\mathcal{F}) := \prod_{j=1}^{n-1} \frac{\sum_{m\in A_j(\mathcal{F})} k_m}{\varepsilon^{2\delta} + \sum_{m\in A_j(\mathcal{F})} k_m^{2\beta}}.$$

Suppose first that $2\beta > 1$. There exists then a constant C, depending only on n, β and K such that, for any $m_j \in A_j(\mathcal{F})$,

$$\frac{\sum\limits_{m\in A_j(\mathcal{F})} k_m}{\varepsilon^{2\delta} + \sum\limits_{m\in A_j(\mathcal{F})} k_m^{2\beta}} \le C\frac{\varepsilon^{\delta/\beta} + k_{m_j}}{\varepsilon^{2\delta} + k_{m_j}^{2\beta}}. \tag{34}$$

So,

$$\frac{p_{n-1,\varepsilon}(\mathcal{F})}{\varepsilon^{2\delta} + \sum\limits_{m\in A_n(\mathcal{F})} k_m^{2\beta}} \le C\prod_{j=1}^{n-1} \frac{\varepsilon^{\delta/\beta} + k_{m_j}}{\varepsilon^{2\delta} + k_{m_j}^{2\beta}} \times \frac{1}{\varepsilon^{2\delta} + k_{m_n}^{2\beta}} \tag{35}$$

for any choice of $m_j \in A_j(\mathcal{F})$. Let $m_j := j$ if j is *not* the right endpoint of an edge of the diagram $\mathcal{F}$. Otherwise, let m_j be the closest free vertex to the left of j.

Consequently, the expression on the right hand side of (35) is less than or equal to

$$C\prod_{\widehat{mm'}\in E(\mathcal{F})} \frac{\varepsilon^{\delta/\beta} + k_m}{\varepsilon^{2\delta} + k_m^{2\beta}} \times \prod_{m\in A_n(\mathcal{F})\cup\{n+1\}} \frac{(\varepsilon^{\delta/\beta} + k_m)^{q_m+1-\delta_{m,m_n}}}{(\varepsilon^{2\delta} + k_m^{2\beta})^{q_m+1}} \tag{36}$$

where $E(\mathcal{F})$ denotes the set of the edges of the diagram $\mathcal{F}$ and $q_m \ge 0$ is the number of *left* endpoints between a free vertex $m \in A_n(\mathcal{F})$ and the next free vertex $m' \in A_n(\mathcal{F})$, also by a convention $q_{n+1} := -1$. Note that

$$c_n + 2\sum_{m\in A_n(\mathcal{F})\cup\{n+1\}} q_m = n-2, \tag{37}$$

with c_n denoting the cardinality of the set $A_n(\mathcal{F})$. Applying (36) to (33) we deduce that

$$|\mathcal{I}_n| \le Ct\varepsilon^{n+1-2\delta}\sum\left(\int_0^K \frac{(\varepsilon^{\delta/\beta} + k)dk}{(\varepsilon^{2\delta} + k^{2\beta})k^{2\alpha-1}}\right)^{(n-c_n)/2} \times \tag{38}$$

$$\prod_{\widehat{mm'}\in\mathcal{F}'}\int_0^K \frac{(k+\varepsilon^{\delta/\beta})^{2+q_m+q_{m'}-r_{m,m'}}}{(k^{2\beta}+\varepsilon^{2\delta})^{2+q_m+q_{m'}}} \times \frac{dk}{k^{2\alpha-1}}.$$

Here the summation extends over all Feynman diagrams $\mathcal{F}$ from $\mathcal{G}_s(\{1,\cdots,n\})$ and all complete diagrams $\mathcal{F}'$ made of the vertices of $A_n(\mathcal{F}) \cup \{n+1\}$ and $r_{m,m'} := \delta_{m,m_n} + \delta_{m',m_n}$. Using the definition of δ, see (7), it is elementary to observe that

$$\int_0^K \frac{(\varepsilon^{\delta/\beta} + k)dk}{(\varepsilon^{2\delta} + k^{2\beta})k^{2\alpha-1}} \le C(1+\varepsilon^{\gamma})$$

and

$$\int_0^K \frac{(k+\varepsilon^{\delta/\beta})^{2+q_m+q_{m'}-r_{m,m'}}}{(k^{2\beta}+\varepsilon^{2\delta})^{2+q_m+q_{m'}}} \times \frac{dk}{k^{2\alpha-1}} \le C(1+\varepsilon^{\gamma(\widehat{mm'})})$$

with

$$\gamma := \frac{3-2\alpha-2\beta}{\alpha+2\beta-1}, \tag{39}$$

$$\gamma(\widehat{mm'}) := \frac{4-2\alpha-4\beta+(q_m+q_{m'})(1-2\beta)-r_{m,m'}}{\alpha+2\underline{t}-1}. \tag{40}$$

Hence we obtain

$$|\mathcal{I}_n| \le C\varepsilon^{n+1-2\delta}(1+\varepsilon^{(n-c_n)\gamma/2})\varepsilon^{\kappa} \tag{41}$$

with

$$\kappa := \frac{2r(2-\alpha-2\beta)+\sum'[(1-2\beta)(q_m+q_{m'})-r_{m,m'}]}{\alpha+2\beta-1}.$$

Here the summation $\sum'$ extends over the edges $\widehat{mm'}$ of the diagram $\mathcal{F}'$ for which $\gamma(\widehat{mm'})<0$ and

$$r \le (c_n+1)/2 \tag{42}$$

denotes the number of such edges. In case there are no such edges we set $\kappa := 0$. The estimate of the right hand side of (41) consists of considering all possible situations depending on signs of the expressions $3-2\alpha-2\beta$, $2-\alpha-2\beta$ appearing in (39) and (40). As shown in Figure 2, there are three cases corresponding to three regions (A, B and C) in the (α,β) plane. In each region we can deduce that

$$|\mathcal{I}_n| \le C\varepsilon^{(n-1)\mu}. \tag{43}$$

Indeed, in Region A we have

$$\kappa \ge \frac{4(1-\alpha-\beta)+(1-2\beta)(n-c_n)}{2(\alpha+2\beta-1)}$$

and $\mu := (2(\alpha+\beta)-1)/2(\alpha+2\beta-1)$; in Region B we have

$$\kappa \ge \frac{-2\alpha+c_n(3-2\alpha-2\beta)+(1-2\beta)n}{2(\alpha+2\beta-1)}$$

and $\mu := (2(\alpha+\beta)-1)/2(\alpha+2\beta-1)$; in Region C we have κ as in the previous case and $\mu := 1/(\alpha+2\beta-1)$;

When, on the other hand $2\beta<1$, we conclude that $p_{n-1,\varepsilon}(\mathcal{F}) \le C$ for a certain constant $C>0$. From (33) we obtain that

$$|\mathcal{I}_n| \le Ct\varepsilon^{n+1-2\delta}\sum\int_0^K\cdots\int_0^K \frac{1}{\varepsilon^{2\text{ẹ}lta}+\sum\limits_{m\in A_n(\mathcal{F})}k_m^{2\beta}}\prod_{\widehat{mm'}}\frac{\delta(k_m-k_{m'})dk_m dk_{m'}}{k_m^{2\alpha-1}} \le \tag{44}$$

$$Ct\varepsilon^{n+1-2\delta}\varepsilon^{2(1-\alpha-\beta)\delta/\beta}\int_0^{K/\varepsilon^{-\delta/\beta}}\frac{dk}{(k^{2\beta}+1)k^{2\alpha-1}} \le Ct\varepsilon^{n-1}.$$

In conclusion, we deduce that all terms $\mathcal{I}_n$ vanish as $\varepsilon\downarrow 0$ when $n\ge 2$.

Estimates of $\mathcal{R}_N$. Note that according to (24)

$$\mathcal{R}_N = 2\varepsilon^{N+2}\int_0^{t/\varepsilon^{2\delta}} ds\int_0^s ds_1\cdots\int_0^{s_N}\mathbf{E}\left[\mathbf{E}_{s_{N+1}}\mathbf{W}_N(s_1,\cdots,s_{N+1},\varepsilon\boldsymbol{x}(s_{N+1}))\otimes\mathbf{V}(0,\mathbf{0})\right]ds_{N+1}.$$

By the Cauchy-Schwartz inequality we get that

$$|\mathcal{R}_N|^2 \le 4t^2\varepsilon^{4(1-\delta)+2N}\mathbf{E}|\mathbf{V}(0,\mathbf{0})|^2\times \tag{45}$$

$$\max_{0\le s\le t/\varepsilon^{2\delta}}\mathbf{E}\left|\int\cdots\int_{s\ge s_1\ge\cdots\ge s_{N+1}\ge 0}\mathbf{E}_{s_{N+1}}\mathbf{W}_N(s_1,\cdots,s_N,s_{N+1},\varepsilon\boldsymbol{x}(s_{N+1}))ds_1\cdots ds_{N+1}\right|^2.$$

The stationarity of the Lagrangian velocity field implies that the maximum in (45) is equal to

$$\max_{0\le s\le t/\varepsilon^{2\delta}} \mathbf{E}\left|\int_0^s ds' \int\cdots\int_{s'\ge s_1\ge\cdots\ge s_N\ge 0} \mathbf{E}_0\mathbf{W}_N(s_1,\cdots,s_N,0,\mathbf{0})ds_1\cdots ds_N\right|^2 \le \tag{46}$$

$$C\max_{0\le s\le t/\varepsilon^{2\delta}} \mathbf{E}\left|\int_0^s ds' \int\cdots\int_{s'\ge s_1\ge\cdots\ge s_N\ge 0} \mathbf{E}_0\nabla\mathbf{W}_{N-1}(s_1,\cdots,s_{N-1},s_N,\mathbf{0})ds_1\cdots ds_N\right|^2 \times \mathbf{E}\left|\mathbf{V}(0,\mathbf{0})\right|^2.$$

In the last line we used that $\mathbf{W}_N\in H^N(\mathbf{V})$ and the resulting hypercontractive property of L^p-norms with respect to a Gaussian measure on the space $H^N(\mathbf{V})$ (cf. Proposition 1). Using subsequently Lemma 2 to represent the conditional expectations on the right hand side of (46) we deduce that the left hand side of the above formula is less than or equal to

$$C\frac{t^2}{\varepsilon^{4\delta}}\times\mathbf{E}\left|\sum\int\cdots\int\psi_{\mathbf{i},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_N)P_N(\mathcal{F})Q(\mathcal{F})\prod_{m\in A_N(\mathcal{F})}\widehat{V}_{i_m,\sigma_m}(0,d\mathbf{k}_m)\right|^2 \tag{47}$$

with some $|\psi_{\mathbf{i},\sigma}|\le 1$. The summation above extends over all Feynman diagrams $\mathcal{F}\in\mathcal{G}_s$, the relevant proper functions σ and multiindices $\mathbf{i}$.

Thus, we have

$$\mathcal{R}_N^2\le Ct^4\varepsilon^{2N+4(1-2\delta)}\sum\int_0^K\cdots\int_0^K p_{N,\varepsilon}^2(\mathcal{F})\prod_{\widehat{mm'}}\frac{\delta(k_m-k_{m'})dk_mdk_{m'}}{k_m^{2\alpha-1}}. \tag{48}$$

Here the summation extends over all possible diagrams $\mathcal{F}\in\mathcal{G}_s(\{1,\cdots,N\})$, $\mathcal{F}'\in\mathcal{G}_c(A_N(\mathcal{F})\cup N+A_N(\mathcal{F}))$. The product is taken over all edges of $\mathcal{F}'$ with $A_N(\mathcal{F})$ denoting the set of free edges of $\mathcal{F}$. Let c_N be the cardinality of $A_N(\mathcal{F})$. Arguing as in (38), when $2\beta>1$, we obtain that for $q_m\ge 0$, $m\in A_N(\mathcal{F})\cup N+A_N(\mathcal{F})$ as in (36) satisfying

$$2c_N+2\sum q_m=2N$$

we have

$$|\mathcal{R}_N|^2\le Ct^4\varepsilon^{2N+4(1-2\delta)}\sum\left(\int_0^K\frac{(\varepsilon^{\delta/\beta}+k)dk}{(\varepsilon^{2\delta}+k^{2\beta})k^{2\alpha-1}}\right)^{N-c_N}\times \tag{49}$$

$$\prod_{\widehat{mm'}}\int_0^K\left(\frac{k+\varepsilon^{\delta/\beta}}{k^{2\beta}+\varepsilon^{2\delta}}\right)^{2+q_m+q_{m'}}\times\frac{dk}{k^{2\alpha-1}}.$$

The ranges of the sum and the product in (49) remain the same as in (48). Repeating the argument made after (42) we deduce that there exists $\mu_1>0$ such that

$$|\mathcal{R}_N|^2\le Ct^4\varepsilon^{N\mu_1-8\delta}. \tag{50}$$

The same inequality, with $\mu_1=1$, holds also when $2\beta\le 1$. This can be deduced repeating the corresponding argument used to obtain (44). We infer therefore that $\mathcal{R}_N$ vanishes as $\varepsilon\downarrow 0$ for a sufficiently large N. In conclusion, we proved that the left hand side of (22) tends to $\mathbf{D}t^{2H}$ as $\varepsilon\downarrow 0$, provided that $\alpha+\beta>1$. □

Remark. The foregoing argument can be used to infer, via an application of the hypercontractivity property of the L^p norms over Gaussian measures on $H^N(\mathbf{V})$, that for an arbitrary $p \geq 1$ and $T > 0$ there exists a constant $C > 0$ such that for any $T \geq t \geq s \geq 0$, $\varepsilon > 0$

$$\mathbf{E}|\boldsymbol{x}_\varepsilon(t) - \boldsymbol{x}_\varepsilon(s)|^p \leq C(t-s)^{2Hp}. \tag{51}$$

Proof of Lemma 2. We prove the lemma by induction. The case $n = 1$ is obvious by choosing $\varphi_i^{(0)} \equiv 1$. Suppose that the result holds for n. For the sake of convenience we assume with no loss of any generality that $s_{n+2} = 0$, then

$$\mathbf{E}_0 W_{n+1,i}(s_1, \cdots, s_{n+1}, \boldsymbol{x}) = \mathbf{E}_0 \left\{ \mathbf{V}(s_{n+1}, \boldsymbol{x}) \cdot \nabla \mathbf{E}_{s_{n+1}} W_{n,i}(s_1, \cdots, s_n, \boldsymbol{x}) \right\}. \tag{52}$$

By virtue of the inductive assumption we can represent $\mathbf{E}_{s_{n+1}} W_{n,i}$ using (28) and as a result (52) becomes

$$\sum \mathbf{E}_0 \left[\int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \exp\{- \sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_n - s_{n+1})\} P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \right. \tag{53}$$

$$\left. \widehat{\mathbf{V}}_0(s_{n+1}, \boldsymbol{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left\{ \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{i_m,\sigma_m}(s_{n+1}, \boldsymbol{x}, d\mathbf{k}_m) \right\} \right].$$

To calculate (53) we decompose each $\widehat{V}_{\sigma,i}(s, \boldsymbol{x}, d\mathbf{k})$ as

$$\widehat{V}_{\sigma,i}(s, \boldsymbol{x}, d\mathbf{k}) = \widehat{V}_{\sigma,i}^0(s, \boldsymbol{x}, d\mathbf{k}) + \widehat{V}_{\sigma,i}^1(s, \boldsymbol{x}, d\mathbf{k}) \tag{54}$$

and use (13)-(12) where

$$\widehat{V}_{\sigma,i}^0(s, \boldsymbol{x}, d\mathbf{k}) = e^{-|\mathbf{k}|^{2\beta}(s-t)} \widehat{V}_{\sigma,i}(t, \boldsymbol{x}, d\mathbf{k}) \tag{55}$$

is the orthogonal projection of $\widehat{V}_{\sigma,i}$ on $\mathcal{V}_{-\infty,t}$. Expression (53) becomes

$$\sum \mathbf{E}_0 \left[\int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \exp\{- \sum_{m \in A_n(\mathcal{F})} |\mathbf{k}_m|^{2\beta}(s_n - s_{n+1})\} P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \mathcal{K}(\mathcal{F}) \right] \tag{56}$$

with

$$\mathcal{K}(\mathcal{F}) := \sum_{\substack{\varrho = \{\varrho_j\} \\ j \in A_n(\mathcal{F}) \cup \{n+1\}}} \widehat{\mathbf{V}}_0^{\varrho_{n+1}}(s, \boldsymbol{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left\{ \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^{\varrho_m}(s, \boldsymbol{x}, d\mathbf{k}_m) \right\}.$$

The term corresponding to $\varrho_j \equiv 1$ vanishes, as is clear from the following calculation,

$$\mathbf{E} \left\{ \int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \right. \tag{57}$$

$$\left. \widehat{\mathbf{V}}_0^1(s, \boldsymbol{x}, d\mathbf{k}_{n+1}) \cdot \nabla \left(\prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^1(s, \boldsymbol{x}, d\mathbf{k}_m) \right) \right\} =$$

$$\nabla \cdot \mathbf{E} \left\{ \int \cdots \int \varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) P_{n-1}(\mathcal{F}) Q(\mathcal{F}) \widehat{\mathbf{V}}_0^1(s, \boldsymbol{x}, d\mathbf{k}_{n+1}) \prod_{m \in A_n(\mathcal{F})} \widehat{V}_{\sigma_m, i_m}^1(s, \boldsymbol{x}, d\mathbf{k}_m) \right\} = 0$$

by homogeneity of the velocity field. By (13)-(12)

$$\widehat{\mathbf{V}}_0(s,\boldsymbol{x},d\mathbf{k}_{n+1})\cdot\nabla\left\{\prod_{m\in A_n(\mathcal{F})}\widehat{V}_{\sigma_m,i_m}(s,\boldsymbol{x},d\mathbf{k}_m)\right\} \tag{58}$$
$$=\sum_{m'\in A_n(\mathcal{F})}\mathbf{k}_{m'}\cdot\widehat{\mathbf{V}}_0(s,\boldsymbol{x},d\mathbf{k}_{n+1})\prod_{m\in A_n(\mathcal{F})}\widehat{V}_{\sigma_m^{m'},i_m}(s,\boldsymbol{x},d\mathbf{k}_m)$$

where

$$\sigma_m^{m'}:=\begin{cases}1-\sigma_{m'} & \text{if} \quad m'=m\\ \sigma_m & \text{otherwise.}\end{cases} \tag{59}$$

By (55), (54), (58) and the definition (29), (56) further reduces to

$$\sum\int\cdots\int\sum_{i_{n+1}=1}^{d}\sum_{m'\in A_n(\mathcal{F})}\sum_{\mathcal{F}'}\varphi_{\mathbf{i},\sigma}^{(n)}\frac{k_{m',i_{n+1}}}{\sum\limits_{m\in A_n(\mathcal{F})}|\mathbf{k}_m|} \tag{60}$$
$$\exp\{-\sum_{m\in A(\mathcal{F}')}|\mathbf{k}_m|^{2\beta}s_{n+1}\}P_n(\mathcal{F})Q(\mathcal{F})\prod_{m\in A(\mathcal{F}')}\widehat{V}_{\sigma_m^{m'},i_m}(t,\boldsymbol{x},d\mathbf{k}_m)$$
$$\left.\prod_{\widehat{pq}\in E(\mathcal{F}')}\left[1-e^{-(|\mathbf{k}_p|^{2\beta}+|\mathbf{k}_q|^{2\beta})(s-t)}\right]\mathbf{E}\left[\widehat{V}_{\tilde{\sigma}_{p,m'},i_p}(0,\mathbf{0},d\mathbf{k}_p)\widehat{V}_{\tilde{\sigma}_{q,m'},i_q}(0,\mathbf{0},d\mathbf{k}_q)\right]\right\}$$

with $\tilde{\sigma}_{1,m'}=0$ and $\tilde{\sigma}_{j+1,m'}=\sigma_j^{m'}$ and all incomplete Feynman diagrams $\mathcal{F}'$ based on the set $A_n(\mathcal{F})\cup\{n+1\}$. Lemma 2 follows with

$$\varphi_{\mathbf{i},\sigma'_{m'}}^{(n+1)}(\mathbf{k}_1,\cdots,\mathbf{k}_{n+1}):=\varphi_{\mathbf{i},\sigma}^{(n)}(\mathbf{k}_1,\cdots,\mathbf{k}_n)\frac{k_{m',i_{n+1}}}{\sum\limits_{m'\in A_n(\mathcal{F})}|\mathbf{k}_{m'}|}\prod_{\widehat{pq}\in E(\mathcal{F}')}\left[1-e^{-(|\mathbf{k}_p|^{2\beta}+|\mathbf{k}_q|^{2\beta})s_{n+1}}\right]$$

□

4 Proof of weak convergence

It is a straightforward matter to verify that the Gaussian processes

$$\boldsymbol{y}_\varepsilon(t):=\varepsilon\int_0^{t/\varepsilon^{2\delta}}\mathbf{V}(s,\mathbf{0})ds \quad t\geq 0. \tag{61}$$

converge weakly to the fractional Brownian Motion $\mathbf{B}_H(t)$, $t\geq 0$ given by (8). In addition we have

$$\limsup_{\varepsilon\downarrow 0}\mathbf{E}|\boldsymbol{y}_\varepsilon(t)|^p<+\infty$$

for any $p\geq 1$, $t\geq 0$.

We now prove that

$$\lim_{\varepsilon\downarrow 0}\mathbf{E}\left\{[x_{\varepsilon,i_1}(t_1)-x_{\varepsilon,i_1}(t_2)]^{p_1}\cdots[x_{\varepsilon,i_M}(t_M)-x_{\varepsilon,i_M}(t_{M+1})]^{p_M}\right\}= \tag{62}$$
$$\mathbf{E}\left\{[B_{H,i_1}(t_1)-B_{H,i_1}(t_2)]^{p_1}\cdots[B_{H,i_M}(t_M)-B_{H,i_M}(t_{M+1})]^{p_M}\right\}.$$

(62) implies that the limiting law of the family of processes $\boldsymbol{x}_\varepsilon(t)$, $t\ge 0$, whose tightness, as $\varepsilon\downarrow 0$, has been established in the previous section, is that of the fractional Brownian motion $B_H(t)$, $t\ge 0$. Equation (62) is a consequence of

$$\lim_{\varepsilon\downarrow 0}|\mathbf{E}\{[x_{\varepsilon,i_1}(t_1)-x_{\varepsilon,i_1}(t_2)]^{p_1}\cdots[x_{\varepsilon,i_M}(t_M)-x_{\varepsilon,i_M}(t_{M+1})]^{p_M}- \tag{63}$$

$$[y_{\varepsilon,i_1}(t_1)-y_{\varepsilon,i_1}(t_2)]^{p_1}\cdots[y_{\varepsilon,i_M}(t_M)-y_{\varepsilon,i_M}(t_{M+1})]^{p_M}\}|=0$$

with $\boldsymbol{y}_\varepsilon(t)=(y_{\varepsilon,1}(t),\cdots,y_{\varepsilon,d}(t))$. Eq. (63) follows from the next lemma.

Lemma 3 *For any positive integers M, $p_1,\cdots,p_M$, multiindices $\mathbf{i}_j\in\{1,\cdots,d\}^{p_j}$ for $j=1,\cdots,M$ and $t_1\ge\cdots\ge t_M\ge t_{M+1}=0$ we have*

$$\lim_{\varepsilon\downarrow 0}\left|\mathbf{E}\left[Z^{(p_1)}_{\varepsilon,\mathbf{i}_1}(t_2,t_1)\cdots Z^{(p_M)}_{\varepsilon,\mathbf{i}_M}(t_{M+1},t_M)-W^{(p_1)}_{\varepsilon,\mathbf{i}_1}(t_2,t_1)\cdots W^{(p_M)}_{\varepsilon,\mathbf{i}_M}(t_{M+1},t_M)\right]\right|=0. \tag{64}$$

Here for any integer $N\ge 1$, multiindex $\mathbf{i}=(i_1,\cdots,i_N)\in\{1\cdots,d\}^N$ and $t\ge s$ we define

$$Z^{(N)}_{\varepsilon,\mathbf{i}}(s,t):=\varepsilon^N\int\cdots\int_{\triangle_N(s,t)}\prod_{p=1}^N V_{i_p}(s_p,\varepsilon\boldsymbol{x}(s_p))ds_1\cdots ds_N$$

and

$$W^{(N)}_{\varepsilon,\mathbf{i}}(s,t):=\varepsilon^N\int\cdots\int_{\triangle_N(s,t)}\prod_{p=1}^N V_{i_p}(s_p,\mathbf{0})ds_1\cdots ds_N,$$

with $\triangle_N(s,t):=\{(s_1,\cdots,s_N):t/\varepsilon^{2\delta}\ge s_1\ge\cdots\ge s_N\ge s/\varepsilon^{2\delta}\}$.

Proof. To avoid cumbersome expressions that may obscure the essence of the proof we consider only the special case of $M=1$ and $t_1=t$, $t_2=0$. The general case follows from exactly the same argument. We shall proceed with the induction argument on $p_1=P$. The case when $P=1$ is trivial because the stationarity of the relevant processes implies that the expression under the limit in (64) vanishes. In fact as a consequence of the remark made after the proof of Lemma 2 we can conclude that for any $q\ge 1$

$$\limsup_{\varepsilon\downarrow 0}\mathbf{E}|Z^{(1)}_{\varepsilon,\mathbf{i}}(0,t)|^q<+\infty.$$

Assume now that (64) has been proven for a certain $P-1\ge 1$ and that for any $q\ge 1$ we have

$$\limsup_{\varepsilon\downarrow 0}\mathbf{E}|Z^{(P-1)}_{\varepsilon,\mathbf{i}}(0,t)|^q<+\infty. \tag{65}$$

In analogy with (23) we write that

$$\mathbf{E}Z^{(P)}_{\varepsilon,\mathbf{i}}(0,t)=\sum_{n=0}^{N-1}\mathcal{I}_n(0,t)+\mathcal{R}_N(0,t) \tag{66}$$

with

$$\mathcal{I}_n(0,t):=\varepsilon^{P+n+1}\int\cdots\int_{\Delta^{(n)}_P(0,t)}\mathbf{E}\left\{\mathbf{E}_{s_2}W^n_{i_1}(\mathbf{s}^{(n)}_1,\varepsilon\boldsymbol{x}(s_2))\prod_{p=2}^P V_{i_p}(s_p,\varepsilon\boldsymbol{x}(s_p))\right\}d\mathbf{s}^{(n)}_1 ds_2\cdots ds_P \tag{67}$$

$$\mathcal{R}_N(0,t) := \varepsilon^{P+N+1} \int_{\Delta_P^{(N)}(0,t)} \cdots \int \mathbf{E}\left\{\mathbf{E}_{s_{1,N+1}} W_{i_1}^N(\mathbf{s}_1^{(N)}, \varepsilon\boldsymbol{x}(s_{1,N+1})) \prod_{p=2}^P V_{i_p}(s_p, \varepsilon\boldsymbol{x}(s_p))\right\} d\mathbf{s}_1^{(N)} ds_2 \cdots ds_P. \tag{68}$$

Here

$$\Delta_P^{(n)}(s,t) := \{(\mathbf{s}_1^{(n)}, s_2, \cdots, s_P) : t/\varepsilon^{2\delta} \geq \mathbf{s}_1^{(n)} \geq s_2 \cdots \geq s_P \geq s/\varepsilon^{2\delta}\}$$

with $\mathbf{s}_1^{(n)} := (s_{1,1}, \cdots, s_{1,n+1})$. We say that $t \geq \mathbf{s}_1 \geq s$, where $\mathbf{s} = (s_1, \cdots, s_n)$ is an ordered n-tuple i.e. $s_1 \geq \cdots \geq s_n$, when $t \geq s_1 \geq s_n \geq s$.

The argument used in the proof of Lemma 2 together with (65) shows that

$$\lim_{\varepsilon \downarrow 0} \mathcal{I}_n(0,t) = 0$$

for $n \geq 1$ and

$$\lim_{\varepsilon \downarrow 0} \mathcal{R}_N(0,t) = 0$$

provided that N is chosen sufficiently large. Asymptotically then, as $\varepsilon \downarrow 0$ the behavior of $\mathbf{E}Z_{\varepsilon,\mathbf{i}}^{(P)}(0,t)$ is the same as that of the term

$$\mathcal{I}_0(0,t) := \varepsilon^{P+1} \int_{\Delta_P^{(0)}(0,t)} \cdots \int \mathbf{E}\left\{V_{i_1}(s_1, \varepsilon\boldsymbol{x}(s_2)) \prod_{p=2}^P V_{i_p}(s_p, \varepsilon\boldsymbol{x}(s_p))\right\} ds_1 \cdots ds_P. \tag{69}$$

In order to deal with (69) we need a generalisation of the argument used in the proof of Lemma 2. Let us introduce some additional notation. For any multiindex $\mathbf{i} = (i_1, \cdots, i_p)$ and $p \geq 1$ we define $W_{\mathbf{i}}^{p,n}$ by induction as follows. We set

$$W_{i_1,\cdots,i_p}^{p,0}(s_1, \cdots, s_p, \boldsymbol{x}) := V_{i_1}(s_1, \boldsymbol{x}) \cdots V_{i_p}(s_p, \boldsymbol{x}) - \mathbf{E}\{V_{i_1}(s_1, \boldsymbol{x}) \cdots V_{i_p}(s_p, \boldsymbol{x})\}$$

and assuming that $W_{i_1,\cdots,i_p}^{p,n}(s_1, \cdots, s_{p-1}, \mathbf{s}_p^{(n)}, \boldsymbol{x})$ has been defined for any ordered $n+1$-tuple $\mathbf{s}_p^{(n)} = (s_{p,1}, \cdots, s_{p,n+1}) \leq s_{p-1}$ we set

$$W_{i_1,\cdots,i_p}^{p,n+1}(s_1, \cdots, s_{p-1}, \mathbf{s}_p^{(n+1)}, \boldsymbol{x}) := \nabla W_{i_1,\cdots,i_p}^{p,n}(s_1, \cdots, \mathbf{s}_p^{(n)}, \boldsymbol{x}) \cdot \mathbf{V}(s_{p,n+2}, \boldsymbol{x})$$

for any ordered $n+2$-tuple $\mathbf{s}_p^{(n+1)} = (s_{p,1}, \cdots, s_{p,n+1}, s_{p,n+2})$. Expanding the left hand side of (69) in analogy with (23) we obtain that

$$\mathcal{I}_0(0,t) =$$

$$\varepsilon^{P+1} \int_{\Delta_P^{(0)}(0,t)} \cdots \int \mathbf{E}\{V_{i_1}(s_1, \varepsilon\boldsymbol{x}(s_2)) V_{i_2}(s_2, \varepsilon\boldsymbol{x}(s_2))\} \mathbf{E}\left\{\prod_{p=3}^P V_{i_p}(s_p, \varepsilon\boldsymbol{x}(s_p))\right\} ds_1 ds_2 \cdots ds_P +$$

$$\sum_{n=0}^{N-1} \mathcal{I}_{1,n}(0,t) + \mathcal{R}_{1,N}(0,t)$$

where

$$\mathcal{I}_{1,n}(0,t) := \tag{70}$$

$$\varepsilon^{P+n+1} \int_{\Delta_P^{(1,n)}(0,t)} \cdots \int \mathbf{E}\left\{\mathbf{E}_{s_3} W_{i_1,i_2}^{2,n}(s_1, \mathbf{s}_2^{(n)}, \varepsilon\boldsymbol{x}(s_3)) \prod_{p=3}^P V_{i_p}(s_p, \varepsilon\boldsymbol{x}(s_p))\right\} ds_1 d\mathbf{s}_2^{(n)} ds_2 \cdots ds_P$$

$$\mathcal{R}_{1,N}(0,t) := \tag{71}$$

$$\varepsilon^{P+N+1}\int_{\Delta_P^{(1,N)}(0,t)}\cdots\int \mathbf{E}\left\{\mathbf{E}_{s_{2,N+1}}W^{2,N}_{i_1,i_2}(s_1,\mathbf{s}^{(N)},\varepsilon\boldsymbol{x}(s_{2,N+1}))\prod_{p=3}^{P}V_{i_p}(s_p,\varepsilon\boldsymbol{x}(s_p))\right\}ds_1 d\mathbf{s}_2^{(N)}ds_3\cdots ds_P,$$

$$\Delta_P^{(1,n)}(0,t) := \{(s_1,\mathbf{s}_2^{(n)},s_3,\cdots,s_P) : t/\varepsilon^{2\delta}\ge s_1\ge \mathbf{s}_2^{(n)}\ge\cdots\ge s_P\ge 0\}.$$

We represent the conditional expectations appearing in (70) and (71) using a generalization (Lemma 4) of Lemma 2.

To formulate it we need a generalized notion of a proper function, which we call a p-proper function. Let p be a positive integer. The p-proper function of order 1 is unique and is given by $\sigma(i)=0$, $i=1,\cdots,p$. Any p-proper function, σ', of order $n+1$ is generated from a p-proper function σ of order n as follows. For some $q\le p+n$,

$$\begin{aligned}\sigma'(p+n+1) &:= 0\\ \sigma'(k) &:= \sigma(k) \quad \text{for} \quad k\le n+p \text{ and } k\ne q\\ \sigma'(q) &:= 1-\sigma(q).\end{aligned}\tag{72}$$

We also distinguish a special class of Feynman diagrams $p\mathcal{G}_s(B)$: a diagram $\mathcal{F}$ of order $n+p$ belongs to $p\mathcal{G}_s(B)$ if $A_k(\mathcal{F})$ is not empty for all $k=p,...,n+p$.

Lemma 4 *For any positive integer p, $s_1\ge\cdots\ge s_{p-1}\ge \mathbf{s}_p^{(n-1)}\ge s$, a multiindex $\mathbf{i}=(i_1,\cdots,i_p)\in\{1,\cdots,d\}^p$ we have*

$$\mathbf{E}_s W^{p,n-1}_{\mathbf{i}}(s_1,\cdots,s_{p-1},\mathbf{s}_p^{(n-1)},\boldsymbol{x}) = \sum\int\cdots\int \varphi^{(p,n)}_{\mathbf{j},\sigma}(\mathbf{k}_1,\cdots,\mathbf{k}_{p+n}) \tag{73}$$

$$\exp\{-\sum_{m\in A_{n+p}(\mathcal{F})}|\mathbf{k}_m|^{2\beta}(s_{p,n}-s)\}P_{p,n-1}(\mathcal{F})Q(\mathcal{F})\prod_{m\in A_{n+p}(\mathcal{F})}\widehat{V}_{i_m,\sigma_m}(s,\boldsymbol{x},d\mathbf{k}_m),$$

where $\varphi^{(p,n)}_{\mathbf{j},\sigma}$ are functions satisfying $|\varphi^{(p,n)}_{\mathbf{j},\sigma}|\le 1$ and

$$P_{p,n}(\mathcal{F}) = \prod_{j=p}^{n+p-1}\left(\sum_{m\in A_j(\mathcal{F})}|\mathbf{k}_m|\right)\exp\{-\sum_{m\in A_j(\mathcal{F})}|\mathbf{k}_m|^{2\beta}(s_{p,j-p}-s_{p,j-p+1})\},$$

$$Q(\mathcal{F}) = \prod_{\widehat{mm'}\in E(\mathcal{F})}\mathbf{E}\left[\widehat{V}_{i_m,\sigma_m}(0,d\mathbf{k}_m)\widehat{V}_{i_{m'},\sigma_{m'}}(0,d\mathbf{k}_{m'})\right]. \tag{74}$$

The summation is over all multiindices $\mathbf{j}=(j_1,\cdots,j_{n+p})$, such that $\mathbf{j}_{|p}=\mathbf{i}$, all $\mathcal{F}\in p\mathcal{G}_s$ and all p-proper functions σ of order n. Here by a convention $s_{p,0}:=s_{p-1}$.

The proof of Lemma 4 is exactly the same as that of Lemma 2 and is omitted.

Using the above lemma and the argument presented in the foregoing to demonstrate that $\mathcal{I}_0(0,t)$ is asymptotically equal to $\mathbf{E}Z^{(P)}_{\varepsilon,\mathbf{i}}(0,t)$, as $\varepsilon\downarrow 0$, we can show that

$$\varepsilon^{P+1}\int_{\Delta_P(0,t)}\cdots\int\mathbf{E}\left\{V_{i_1}(s_1,\varepsilon\boldsymbol{x}(s_3))V_{i_2}(s_2,\varepsilon\boldsymbol{x}(s_3))\prod_{p=3}^{P}V_{i_p}(s_p,\varepsilon\boldsymbol{x}(s_p))\right\}ds_1ds_2\cdots ds_P$$

is asymptotically equal to $\mathbf{E}Z^{(P)}_{\varepsilon,\mathbf{i}}(0,t)$, as $\varepsilon\downarrow 0$. Repeating the above argument p-times we obtain (64). Finally we notice that hypercontractivity properties of the L^p norms over Gaussian measure space allow us to conclude that (65) holds with $P-1$ replaced by P □

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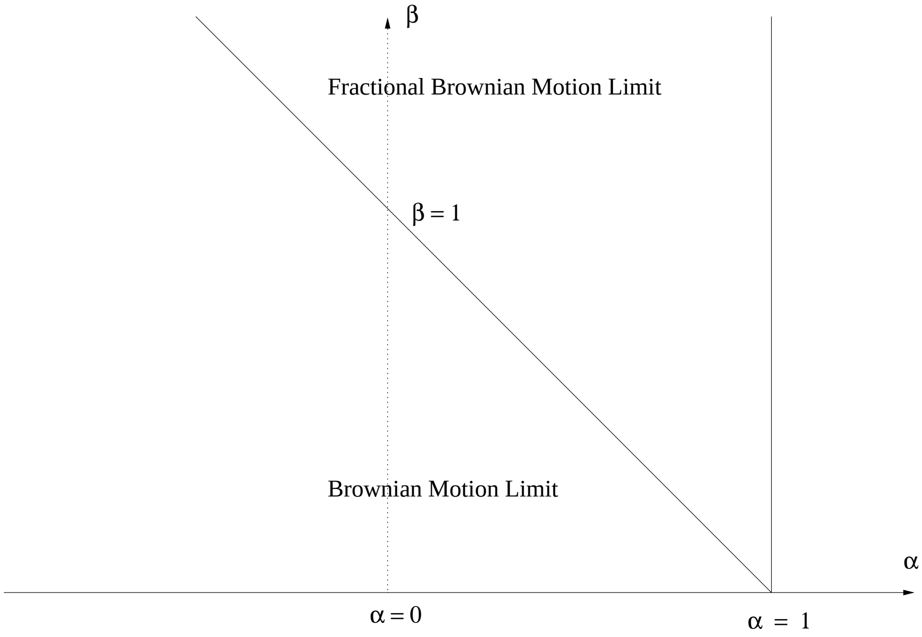

Figure 1: Phase diagram for scaling limit

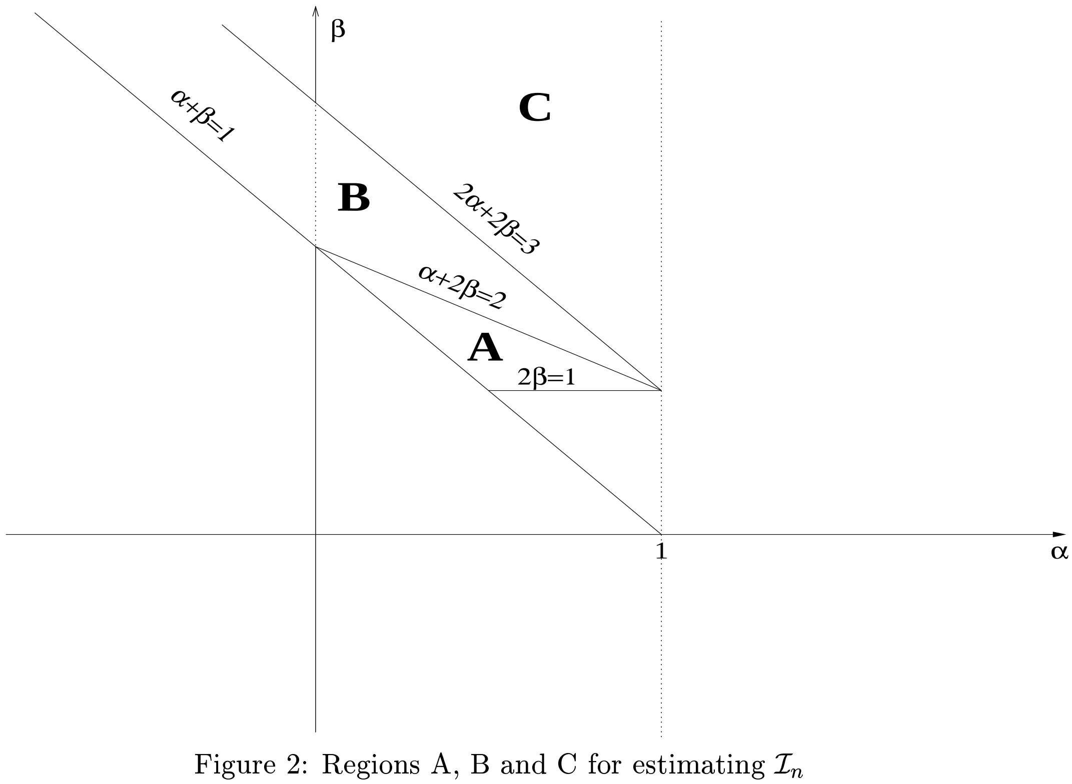

Figure 2: Regions A, B and C for estimating $\mathcal{I}_n$