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Information transfer in disordered media by broadband time-reversal: stability, resolution and capacity

Albert Fannjiang

Department of Mathematics, University of California, Davis, CA 95616, USA

E-mail: fannjiang@math.ucdavis.edu

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Abstract

We analyse rigorously the time reversal of a multiple-input-multiple-output system in a strongly fluctuating disordered medium described by the stochastic Schrödinger equation with a random potential under the sub-Gaussian assumption. We prove that in a broadband limit the conditions for stable super-resolution are the *packing* condition such that the spacing among the N transmitters and M receivers be more than the coherence length ℓ_c and the consecutive symbols in the datum streams are separated by more than the inverse of the bandwidth B^{-1} and the *multiplexing* (or stability) condition such that the number of the degrees of freedom per unit time at the transmitters ($\sim NB$) be much larger than the number of the degrees of freedom ($\sim MC$) per unit time in the ensemble of intended messages. Here C is the number of symbols per unit time in the datum streams intended for each receiver. When the two conditions are met, all receivers simultaneously receive streams of statistically stable, sharply focused signals intended for them, free of fading and interference. We show that an $O(P/\nu)$ information rate P/ν can be achieved with statistical stability under the condition $N \gg M \gg P/(\nu B)$ where P is the average total power constraint and ν the noise power per unit bandwidth. The packing condition then implies that $\gamma^{-d}\beta_c^{-d} \gg P/(\nu B)$ in the optimal transfer regime where γ is the Fresnel number, β_c the coherence bandwidth and d the transverse dimension. Our results should be valid for diffusive waves with $\beta_c =$ Thouless frequency. Therefore, under the ideal packing and multiplexing conditions time reversal communications result in a high signal-to-interference ratio and low probability of intercept and is an effective means for achieving the information capacity of disordered media in the presence of multiple users.

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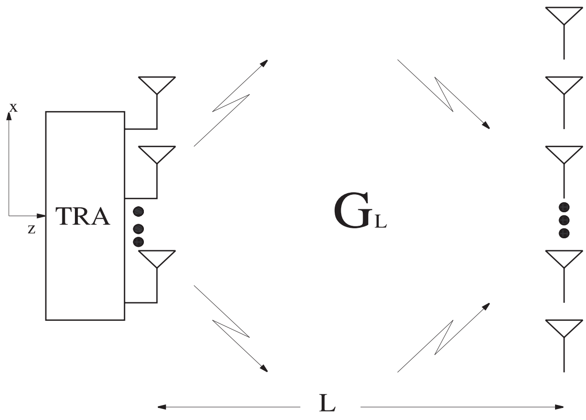

Figure 1. MIMO-TRA broadcast channel.

1. Introduction

Time reversal (TR) of waves is the process of recording the signal from a remote source and then retransmitting the signal in a time-reversed fashion to refocus on the source (see [4, 19, 20] and the references therein). The performance of TR depends on, among other factors, the reciprocity (or time symmetry) of the propagation channel. One of the most striking features of time reversal operation in a strongly scattering medium is super-resolution, the counterintuitive effect of scattering-enhancement of time reversal resolution [3, 7, 17]. It highlights the great potential of time reversal in technological applications such as communications where the ability of steering and pinpointing signals is essential for realizing the information carrying capacity of a multipath channel as well as achieving low probability of intercept [8, 27].

In order to take full advantage of the super-resolution effect in a random medium, one has to first achieve statistical stability which can be measured by the signal-to-interference ratio (SIR) and the signal-to-sidelobe ratio (SSR). Statistical stability and resolution are two closely related issues that should be analysed side-by-side; together, they are the basic measure of TR performance which depends on, but is not guaranteed by, the reciprocity (or time symmetry) of the propagation channel. It has been demonstrated experimentally that there are at least two routes to achieving statistical stability [7, 9]. One route is to use a time-reversal array (TRA) of sufficiently large aperture; the other is to use a broadband signal (even with one-element TRA of essentially zero aperture). There have been many advances in analytical understanding of the former situation (see [1, 12, 13] and references therein) which are, however, often too restrictive for TR applications. Compared with the case of a large aperture, the analytical understanding of the broadband case has been so far much less complete with the exception of a randomly *layered* medium [3].

In this paper we present the time reversal analysis for the broadband, multiple-input-multiple-output (MIMO) *broadcast* channel (see figure 1) whose k-component is described by the nondimensionalized stochastic Schrödinger equation

$$i\frac{\partial \Psi_z}{\partial z} + \frac{\gamma}{2k}\Delta_{\boldsymbol{x}}\Psi_z + \frac{k}{\gamma}\chi_z \circ \Psi_z = 0, \qquad \boldsymbol{x} \in \mathbb{R}^d, \tag{1}$$

in the so called paraxial (or parabolic) Markovian approximation. Here the refractive index fluctuation $\chi_z(\cdot)$ is a δ-correlated-in-z stationary random field with a power spectral density

$\Phi(\boldsymbol{p})$ such that $\mathbb{E}\left[\chi_z(\boldsymbol{x})\chi_{z'}(\boldsymbol{x}')\right] = \delta(z - z')\int \Phi(\boldsymbol{p})\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\mathrm{d}\boldsymbol{p}$ with $\mathbb{E}$ standing for the ensemble average; k is the (dimensionless) relative wavenumber to the centre wavenumber k_0; the Fresnel number $\gamma = L_z/(k_0 L_x^2)$ is a dimensionless number consisting of the centre wavenumber k_0 and the reference scales L_z and L_x in the longitudinal and transverse dimensions, respectively. The notation $\circ$ in equation (1) means the Stratonovich product (versus the Itô product). We have chosen the time scale such that the speed of propagation is unity (thus k= relative frequency ω). For simplicity of presentation we will assume isotropy, i.e. $\Phi(\boldsymbol{k}) = \Phi(|\boldsymbol{k}|), \forall \boldsymbol{k} \in \mathbb{R}^d$ and smoothness of Φ.

The stochastic parabolic wave equation (1) is a fundamental equation for wave propagation in a randomly inhomogeneous *continuum* as an approximation to the linear reduced wave equation with continuously varying random coefficients in a suitable scaling limit. Equation (1) arises in, e.g. modelling *long-distance* propagation of monochromatic light in turbulent atmosphere [24, 37]. In such a context, the δ-function in the z-correlation results from the large ratio between the propagation distance and the correlation length, both in the z-direction, and equation (1) is obtained via the scaling limit by holding the Fresnel number γ fixed [15]. Indeed, we shall consider the regime of small Fresnel number for which we can obtain precise error estimate. Equation (1) also models the cross-phase-modulation in nonlinear optical fibres in the wavelength-division-multiplexing scheme [25]. It has a certain degree of universality and encapsulates the *transverse* aspect of wave physics as the primary focus of the present work is on the multipath and multi-user effects. In this connection, we should mention the work [27] which is a numerical study of time-reversal in a stochastic wave-guide.

Our goal is to show that for the channel described by equation (1) the stability and super-resolution can be achieved simultaneously when there is sufficiently high number of degrees of freedom at the TRA. In order to concisely describe the degrees of freedom at TRA we assume the *packing* condition, namely that the spacing of the M receivers and N elements of the TRA is much more than the coherence length of the channel, and that the consecutive symbols in the M simultaneous datum streams are separated by more than B^{-1}, the inverse of the non-dimensionalized bandwidth $B(=$ frequency bandwidth$\times L_x^2/L_z)$. Our main theorem says that in the *saturated-fluctuation* regime (defined by (2) below) under the *broadband* limit (defined by (4) below) and *multiplexing* condition ($NB \gg MC$ where C is the number of symbols per unit time in each datum stream) the MIMO-TRA broadcast system achieves stable super-resolution in the sense that both the SIR and SSR tend to infinity and that the signal received by each receiver is focused to within a circle of the coherence length ℓ_c. Super-resolution refers to the fact that ℓ_c is essentially independent of the aperture of TRA in the saturated-fluctuation regime. Under the stability (packing and multiplexing) condition the streams of time-reversed signals from the multiple distributed transmitters refocus back to the multiple distributed receivers *independently* of the medium realization.

In the above result MC is the number of the degrees of freedom per unit time in the ensemble of input messages while NB is the total number of the degrees of freedom per unit time at the TRA. The multiplexing condition says that the number of degrees of freedom in the message ensemble must be smaller than the number of degrees of freedom available in the channel.

The main technical ingredient of our approach is the *exact*, *universal* low Fresnel number asymptotic obtained for the two-frequency mutual coherence function. The nearly exact calculation indicates that the multiplexing condition is sharp. The main assumption is the 4th order sub-Gaussianity property (13). The Gaussian-like behaviour for 4th order correlations is widely believed to occur in the saturated-fluctuation regime originally defined as $\sigma_*/\ell_c \gg 1$

which follows from the stronger condition

$$\alpha_*^2 = D_2 L \gg 1, \qquad \sigma_*^2 = D_2 L^3 \gg 1 \tag{2}$$

used in the present work [2]. Here L is the (longitudinal) distance between the TRA and the receivers and

$$D_2 = \frac{1}{d}\int |\boldsymbol{p}|^2 \Phi(\boldsymbol{p})\mathrm{d}\boldsymbol{p}$$

is the *angular* diffusion coefficient (hence $\alpha_* = \sqrt{D_2 L}$ is the tangent of the angular spread). In the saturated-fluctuation regime [17], $\gamma^{-1}\alpha_*$ is the spread in the so called *spatial frequency*, $\sigma_* = \sqrt{D_2 L^3}$ the spatial spread and their product $\gamma^{-1} D_2 L^2$ the *spatial-spread-bandwidth* product (SSB) which, as we will see, is exactly $\gamma^{-1}\beta_c^{-1}$, where β_c is the coherence bandwidth.

By the duality principle for the saturated-fluctuation regime, proved in [17], the *effective aperture* is 2π times the spatial spread σ_* (independent of the numerical aperture of TRA and hence super-resolution), and its dual quantity $\ell_c = \gamma L/(\omega\sigma_*) = \gamma/(\omega\alpha_*)$ is the coherence length (as well as the time reversal resolution) which is small for $\gamma \ll 1$ and $\alpha_* \gg 1$. Thus, the aperture of TRA is $O(N^{1/d}\ell_c)$ and may be also small compared with the $O(1)$ (transverse) correlation length of the medium fluctuation and yet sufficient for statistical stability in the usual TR refocusing experiments with $M = 1$ and $C = 1$ (the stability condition becomes $BN \gg 1$). Thus, our stability condition unifies and extends the previously observed condition of either large aperture or large bandwidth.

In what follows, we first formulate the problem and develop the essential tool for analyzing TR, the one- and two-frequency mutual coherence functions and then carry out the stability and resolution analysis for the single-input-single-output (SISO), multiple-output-single-output (MISO), single-input-multiple-output (SIMO) and the MIMO cases. Both MISO- and SIMO-TRA systems have been demonstrated experimentally to be feasible for ocean acoustic communication [11,26,32] and the MIMO-TRA system with $N > M$ has been shown to work well for ultrasound [8]. We then discuss the implications of our results on the channel capacity in section 5. Except in the discussion of information rate, we have by and large neglected the effect of noise in our analysis, assuming a high signal-to-noise ratio (SNR) as is the case for the experiments reported in [8, 11]. The robustness of TR in the presence of noises has been well documented (see, e.g. [33]).

2. TR MIMO-broadcast channel

We extend the time-reversal communication scheme [8] to the MIMO-broadcast channel. Let the M receivers located at $(L, \boldsymbol{r}_j)$, $j = 1, \ldots, M$ first send a pilot signal $\int \mathrm{e}^{\mathrm{i}(\omega t/\gamma)} g(\omega)\,\mathrm{d}\omega\delta(\boldsymbol{r}_j - \boldsymbol{a}_i)$ to the N-element TRA located at $(0, \boldsymbol{a}_i)$, $i = 1, \ldots, N$ which then use the time-reversed version of the received signals $\int \mathrm{e}^{\mathrm{i}(\omega t/\gamma)} g(\omega) G_L(\boldsymbol{r}_j, \boldsymbol{a}_i; \omega)\mathrm{d}\omega$ to modulate streams of symbols and send them back to the receivers. Here G_L is the Green function of equation (1) and $g^2(\omega)$ is the power density at ω. As shown in [3, 6], when the TRA has an infinite time-window (see the conclusion in section 6 for the case of a finite time-window), the signal arriving at the receiver plane with delay $L + t$ is given by

$$S(\boldsymbol{r}, t) = \sum_{l=1}^{T}\sum_{i=1}^{N}\sum_{j=1}^{M} m_j(\tau_l) \int \mathrm{e}^{-\mathrm{i}(\omega/\gamma)(t-\tau_l)} g(\omega) \times G_L(\boldsymbol{r}, \boldsymbol{a}_i; \omega) G_L^*(\boldsymbol{r}_j, \boldsymbol{a}_i; \omega)\mathrm{d}\omega, \tag{3}$$

where $m_j(\tau_l)$, $l = 1, \ldots, T \leqslant \infty$ are a stream of T symbols intended for the j-th receiver transmitted at times $\tau_1 < \tau_2 < \ldots < \tau_T$. We assume for simplicity that $|m_j(\tau_l)| = 1, \forall j, l$.

We assume that g is a smooth and rapidly decaying function with effective support of size $B\gamma$. For simplicity we take $g^2(\omega) = \exp(-(|\omega - 1|^2)/(2B^2\gamma^2))$. The broadband limit may be formulated as the double limit

$$\gamma \to 0, \qquad B \to \infty \quad \lim B\gamma = 0 \tag{4}$$

so that in the limit $g^2(\omega)$ becomes narrowly focused around $\omega = 1$. The idea underlying the definition is to view the broadband limit as a sequence of narrow-bands with indefinitely growing centre frequency and bandwidth. This is particularly well suited to the framework of parabolic approximation described by (1). Under the condition (4), the duality relation between the spatial spread and the time reversal resolution (or the coherence length) can be extended from the monochromatic case [17] to the broadband case.

The apparent narrow-banding of (4) is deceptive: the *delay-spread-bandwidth* product (DSB) turns out to be $B\beta_c^{-1}$ and is doubly divergent as $B \to \infty$ (the broadband limit) and $\beta_c \to 0$ (the saturated-fluctuation regime). Note that since ω is the *relative* frequency, the product $B\gamma$ should always be uniformly bounded between zero and unity, independent of $\gamma > 0$ (see the conclusion).

Packing condition. We assume that the spacing within the N TRA-elements and the M receivers be much larger than the coherence length ℓ_c and that the separation of the successive symbols be much larger than $(2B)^{-1}$. Though there is no technical limitation on M, N, T, it suffices to consider the case where all the N TRA-elements and all the M receivers are located within one circle of diameter σ_* (implying $M, N \ll \gamma^{-d}\beta_c^{-d}$), and all the T-datum streams are within one interval of the delay spread $\sim \beta_c^{-1}$ (implying $T \ll B\beta_c^{-1}$) since the signals separated by much more than one spatial spread σ_* or one delay spread δ_* are essentially uncorrelated.

For simplicity, we have assumed that all the receivers lie on the plane parallel to the TRA. When this is not the case, then the above spacing of antennas refers to the *transverse* separation parallel to the TRA.

Signal-to-interference-or-sidelobe ratio (SISR). Anticipating a singular limit we employ the coupling with smooth, compactly supported test functions. Denote the mean by $E(\boldsymbol{r}, t) = \gamma^{-d}\int \theta^*((\boldsymbol{x} - \boldsymbol{r})/\ell_c)\mathbb{E}S(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}$ where the coupling with the test function θ can be viewed as the averaging induced by measurement. Denote the variance by

$$V(\boldsymbol{r}, t) = \gamma^{-2d}\mathbb{E}\left[\int \theta^*((\boldsymbol{x} - \boldsymbol{r})/\ell_c)S(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x}\right]^2 - E^2(\boldsymbol{r}, t).$$

We have made the test function θ act on the scale of the coherence length ℓ_c, the smallest spatial scale of interest (the speckle size) in the present context. Different choices of scale would not affect the conclusion of our analysis.

The primary object of our analysis is

$$\rho(\boldsymbol{r}, t) = \frac{E^2(\boldsymbol{r}_j, \tau_l)}{V(\boldsymbol{r}, t)}, \qquad j = 1, \ldots, M, \quad l = 1, \ldots, T, \tag{5}$$

which is the SIR if $\boldsymbol{r} = \boldsymbol{r}_j, t = \tau_l$ and the SSR if $|\boldsymbol{r} - \boldsymbol{r}_j| \gg \ell_c, \forall j$ (spatial sidelobes) or $|t - \tau_l| \gg B^{-1}, \forall l$ (temporal sidelobes) (as $V(\boldsymbol{r}, \tau) \approx E^2(\boldsymbol{r}, \tau)$ as we will see below). We shall refer to it as the signal-to-interference-or-sidelobe ratio (SISR). In the special case of $\boldsymbol{r} = \boldsymbol{r}_j$ *and* $|t - \tau_l| \gg B^{-1}, \forall l$, ρ^{-1} is a measure of intersymbol interference. To show stability and resolution, we shall find the precise conditions under which $\rho \to \infty$ and $\mathbb{E}S(\boldsymbol{r}, t)$ is asymptotically $\sum_{l=1}^{T} \sum_{j=1}^{M} m_j(\tau_l) S_{jl}(\boldsymbol{r}, t)$, where

$$S_{jl}(\boldsymbol{r}, t) \approx \sum_{i=1}^{N} \int \mathrm{e}^{-\mathrm{i}(\omega(t-\tau_l)/\gamma)} g(\omega) \mathbb{E}\big[G_L(\boldsymbol{r}, \boldsymbol{a}_i; \omega) G_L^*(\boldsymbol{r}_j, \boldsymbol{a}_i; \omega)\big]\, \mathrm{d}\omega \tag{6}$$

is a sum of δ-like functions around $\boldsymbol{r}_j$ and $\tau_l = 0, \forall l$. In other words, we employ the TRA as a multiplexer to transmit the M scrambled datum streams to the receivers and we would like to turn the medium into a demultiplexer by employing the broadband time reversal technique (see the precise statement on page 2432).

3. Mutual coherence functions

A quantity repeatedly appearing in the subsequent analysis is the mutual coherence function Γ_z between the Green functions at two different frequencies $\omega_1 = \omega - \gamma\beta/2, \omega_2 = \omega + \gamma\beta/2$

$$\Gamma_z\left(\frac{\boldsymbol{x}+\boldsymbol{r}}{2}, \frac{\boldsymbol{x}-\boldsymbol{r}}{\gamma}; \omega, \beta\right) = \mathbb{E}\big[G_z(\boldsymbol{x}, \boldsymbol{a}; \omega - \gamma\beta/2) G_z^*(\boldsymbol{r}, \boldsymbol{a}'; \omega + \gamma\beta/2)\big].$$

We shall omit writing $\omega, \beta, \boldsymbol{a}, \boldsymbol{a}'$ when no confusion arises. Here we have chosen $\boldsymbol{x}, \boldsymbol{r}$ to be the pair of variables of concern and left out $\boldsymbol{a}, \boldsymbol{a}'$ as parameters. By the reciprocity of the Green function, we can choose one variable from $\{\boldsymbol{x}, \boldsymbol{a}\}$ and the other from $\{\boldsymbol{r}, \boldsymbol{a}'\}$ as the variables of Γ_z and leave the others as parameters.

One-frequency version. When $\beta = 0$, Γ_z satisfies

$$\frac{\partial}{\partial z}\Gamma_z - \frac{\mathrm{i}}{\omega}\nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{y}}\Gamma_z + \frac{\omega^2}{\gamma^2} D(\gamma \boldsymbol{y})\Gamma_z = 0, \tag{7}$$

where the structure function of the medium fluctuation $D(\boldsymbol{x})$ is given by

$$D(\boldsymbol{x}) = \int \Phi(\boldsymbol{k})[1 - \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}]\, \mathrm{d}\boldsymbol{k} \geqslant 0, \qquad \forall \boldsymbol{x} \in \mathbb{R}^d.$$

Equation (7) is exactly solvable by the Fourier transform in $\boldsymbol{x}$. For a sufficiently small Fresnel number such that

$$\alpha_*, \sigma_* \ll \gamma^{-1}$$

we can use the approximation

$$\omega^2\gamma^{-2}\int_0^L D\left(\gamma \boldsymbol{y} - \boldsymbol{p} z\gamma/\omega\right)\, \mathrm{d}z \approx \int_0^L D_2|\omega \boldsymbol{y} - \boldsymbol{p} z|^2\, \mathrm{d}z$$

to obtain

$$\begin{aligned}\Gamma_L(\boldsymbol{x}, \boldsymbol{y}; \omega, 0) &\approx \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\Gamma}_0\left(\boldsymbol{p}, \boldsymbol{y} - \frac{L\boldsymbol{p}}{\omega}; \omega, 0\right)\mathrm{e}^{-\int_0^L D_2|\omega\boldsymbol{y}+\boldsymbol{p}z|^2\mathrm{d}z}\mathrm{d}\boldsymbol{p} \\ &= \int \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}\hat{\Gamma}_0\left(\boldsymbol{p}, \boldsymbol{y} - \frac{L\boldsymbol{p}}{\omega}; \omega, 0\right)\mathrm{e}^{-\int_0^1 |\tilde{\boldsymbol{y}}+\tilde{\boldsymbol{p}}z|^2\mathrm{d}z}\mathrm{d}\boldsymbol{p}\end{aligned} \tag{8}$$

where $\tilde{\boldsymbol{y}} = \boldsymbol{y}\omega\alpha_*$ and $\tilde{\boldsymbol{p}} = \boldsymbol{p}\sigma_*$. It is clear from (8) that Γ_L has a Gaussian-tail in $\boldsymbol{y}$ (the difference coordinates) and, by rescaling, an effective support $\sim \alpha_*^{-1}$ and hence $\ell_c = \gamma/(\omega\alpha_*)$ (recall that $\boldsymbol{y}$ is the coordinate on the scale γ^{-1}).

Two-frequency version. The two-frequency mutual coherence function is not exactly solvable except for some special cases. Fortunately the asymptotic for $\gamma \ll 1$ has a universal form and can be calculated exactly. Without loss of generality we assume $\beta > 0$ in what follows.

Using the so called two-frequency Wigner distributions we have proved in [14] that in the limit $\gamma \to 0$, Γ_z satisfies the universal equation

$$\frac{\partial \Gamma_z}{\partial z} - \frac{\mathrm{i}}{\omega}\nabla_{\boldsymbol{y}} \cdot \nabla_{\boldsymbol{x}} \Gamma_z = -D_2 \left| -\omega \boldsymbol{y} + \frac{\beta}{2}\boldsymbol{x}\right|^2 \Gamma_z - \frac{\beta^2}{2} D_0 \Gamma_z, \tag{9}$$

where $D_0 = \int \Phi(\boldsymbol{k})\mathrm{d}\boldsymbol{k}$. The key to understanding equation (9) is the rescaling:

$$\tilde{\boldsymbol{x}} = \boldsymbol{x}/\sigma_*, \qquad \tilde{\boldsymbol{y}} = \gamma \boldsymbol{y}/\ell_c, \qquad \tilde{z} = z/L, \qquad \tilde{\beta} = \beta/\beta_c \tag{10}$$

with $\beta_c = D_2^{-1} L^{-2}$ which transforms equation (9) into the form

$$\frac{\partial \tilde{\Gamma}_{\tilde{z}}}{\partial \tilde{z}} - \mathrm{i}\nabla_{\tilde{\boldsymbol{y}}} \cdot \nabla_{\tilde{\boldsymbol{x}}} \tilde{\Gamma}_{\tilde{z}} = -\left|\tilde{\boldsymbol{y}} + \frac{\tilde{\beta}}{2}\tilde{\boldsymbol{x}}\right|^2 \tilde{\Gamma}_{\tilde{z}} - \frac{\tilde{\beta}^2 D_0 \tilde{\Gamma}_{\tilde{z}}}{2\sigma_*^2}. \tag{11}$$

By another change in variables equation (11) is then transformed into that of the *quantum harmonic oscillator* and solved exactly, see appendix A. The solution is given by

$$\Gamma_L(\boldsymbol{x}, \boldsymbol{y}; 1, \beta) = \left(\frac{\sigma_*}{\alpha_*}\right)^d \tilde{\Gamma}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}; \tilde{\beta}) \tag{12}$$

with

$$\begin{aligned}
\tilde{\Gamma}_1(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}; \tilde{\beta}) = {} & \frac{(1+\mathrm{i})^{d/2}\tilde{\beta}^{d/4}}{(2\pi)^d \sin^{d/2}\left[\tilde{\beta}^{1/2}(1+\mathrm{i})\right]} \mathrm{e}^{-(\tilde{\beta}^2 D_0/2\sigma_*^2)} \\
& \times \int \mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{y}' \mathrm{e}^{\mathrm{i}(|\tilde{\boldsymbol{y}}-\boldsymbol{y}'|^2/2\tilde{\beta})} \mathrm{e}^{\mathrm{i}((\tilde{\boldsymbol{y}}-\boldsymbol{y}')\cdot(\tilde{\boldsymbol{x}}-\boldsymbol{x}')/2)} \mathrm{e}^{\mathrm{i}(\tilde{\beta}|\tilde{\boldsymbol{x}}-\boldsymbol{x}'|^2/8)} \\
& \times \mathrm{e}^{((1-\mathrm{i})/2\sqrt{\tilde{\beta}})\cot(\sqrt{\tilde{\beta}}(1+\mathrm{i}))|\tilde{\boldsymbol{y}}-\tilde{\beta}\tilde{\boldsymbol{x}}/2-(\boldsymbol{y}'-\tilde{\beta}\boldsymbol{x}'/2/\cos(\sqrt{\tilde{\beta}}(1+\mathrm{i})))|^2} \\
& \times \mathrm{e}^{-((1-\mathrm{i})/2\sqrt{\tilde{\beta}})|\boldsymbol{y}'-\tilde{\beta}\boldsymbol{x}'/2|^2 \tan(\sqrt{\tilde{\beta}}(1+\mathrm{i}))} \Gamma_0\left(\sigma_* \boldsymbol{x}', \frac{\boldsymbol{y}'}{\omega\alpha_*}\right).
\end{aligned}$$

The prescaled version of (13) has been obtained in [14, 18].

Several remarks are in order: (i) the Green function for Γ_L is of the Gaussian form in $\boldsymbol{x}, \boldsymbol{y}$; (ii) in the vanishing fluctuation limit $D_0, D_2 \to 0$ the free-space two-frequency mutual coherence function is recovered; (iii) the apparent singular nature of the limit $\beta \to 0$ in (12) is deceptive. Indeed, the small β limit is regular and yields the result obtained from equation (9) with $\beta = 0$; (iv) in the saturated-fluctuation regime (2), D_0 is typically much smaller than $D_2^2 L^3 \gg 1$ so the factor $\exp\left(-\tilde{\beta}^2 D_0/(2\sigma_*^2)\right)$ is negligible in the saturated-fluctuation regime. On the other hand, the rapidly decaying factor $\sin^{-d/2}\left(\tilde{\beta}^{1/2}(1+\mathrm{i})\right)$ is crucial for the stability argument below; (v) $\Gamma_L(\boldsymbol{x}, \boldsymbol{y}; \omega, \beta)$ is slowly varying in $\boldsymbol{x}$ on the scale σ_* for $\beta \sim \beta_c$ and more rapidly varying in $\boldsymbol{x}$ for $\beta \gg \beta_c$; (vi) the coherence bandwidth β_c vanishes in the regime of saturated-fluctuation.

Fourth-order sub-Gaussianity. The saturated-fluctuation regime (2) can result from either long-distance propagation and/or large medium fluctuation. It is widely accepted that, in this regime, the statistics of the wave fields (for at least lower moments) become Gaussian-like resulting in the exponential distribution for the intensity [22, 23, 34, 35, 37, 38]. The Gaussian statistics follow heuristically from central-limit-theorem as the number of uncorrelated sub-channels (paths) per transverse dimension in the cross section of diameter σ_* increases linearly with the spatial-spread-bandwidth product, as explained in the introduction. This is consistent with the experimental finding of the saturation of intensity fluctuation with the scintillation index approaching unity [24].

In what follows we shall make the 4th order *sub-Gaussianity* hypothesis, namely that the fourth moments of the Green function at different frequencies $\{G_L(\omega)\}$ can be estimated by those of the Gaussian process of the same covariance. More specifically, we assume that

$$\begin{aligned}
&|\mathbb{E}[G_L(\omega_1)\otimes G_L^*(\omega_1)\otimes G_L(\omega_2)\otimes G_L^*(\omega_2)]\\
&\qquad -\mathbb{E}[G_L(\omega_1)\otimes G_L^*(\omega_1)]\otimes\mathbb{E}[G_L(\omega_2)\otimes G_L^*(\omega_2)]|\\
&\quad\leqslant K|\mathbb{E}[G_L(\omega_1)\otimes G_L(\omega_2)]\otimes\mathbb{E}[G_L^*(\omega_1)\otimes G_L^*(\omega_2)]|\\
&\qquad +K|\mathbb{E}[G_L(\omega_1)\otimes G_L^*(\omega_2)]\otimes\mathbb{E}[G_L^*(\omega_1)\otimes G_L(\omega_2)]| \qquad (13)
\end{aligned}$$

for some constant K independent of $\gamma\to 0, |\omega_1-1| = O(B\gamma), |\omega_2-1| = O(B\gamma)$ and all the variables. For a jointly Gaussian process, the constant $K = 1$. Note that in view of the scaling in the two-frequency mutual coherence the first term on the RHS of (13) is much smaller than the second term due to the frequency difference as $G_L(\omega) = G_L^*(-\omega)$.

The sub-Gaussianity assumption will be used to estimate the 4th order correlations of Green functions appearing in the calculation for the variance V by the two-frequency mutual coherence function in the saturated-fluctuation regime.

4. SISO to MIMO

Our first application of the mutual coherence functions is the estimate for the delay spread. Consider the band-limited impulse response $u(\boldsymbol{x},t) = \int g(\omega)\mathrm{e}^{(\mathrm{i}\omega(L-t)/\gamma)}G_L(\boldsymbol{x},0;\omega)\mathrm{d}\omega$. It follows easily using the preceding results that the mean delay is L and the asymptotic for the delay spread δ_*, when $B\gg\beta_c$, is given by

$$\begin{aligned}
\delta_* &= \sqrt{\int (t-L)^2\mathbb{E}|u(\boldsymbol{x},t)|^2\mathrm{d}t\Big/\int \mathbb{E}|u(\boldsymbol{x},t)|^2\mathrm{d}t}\\
&\approx \sqrt{-\frac{\mathrm{d}^2}{\mathrm{d}\beta^2}\Big|_{\beta=0}\Gamma_L(\boldsymbol{x},0;1,\beta)/\Gamma_L(\boldsymbol{x},0;1,0)}\sim\beta_c^{-1}, \qquad (14)
\end{aligned}$$

which is slowly varying in $\boldsymbol{x}$ on the scale σ_*. As commented before it suffices to consider the case with a finite T such that $|\tau_1-\tau_T|\sim\beta_c^{-1}$, implying the number of symbols in each datum stream $T\ll B\beta_c^{-1}$, the DSB. In what follows, due to $\beta_c\ll 1$ the temporal component of the signals is essentially decoupled from the spatial component and determined by the power distribution g^2.

SISO. This case corresponds to $N=1, M=1$. Let $\boldsymbol{a}_1=0$. In the calculation of $E(\boldsymbol{x},t)$, the expression

$$\langle\theta,\Gamma_L\rangle(\boldsymbol{r})\equiv\int\theta^*\Big(\frac{\boldsymbol{r}_1-\boldsymbol{r}}{\ell_c}+\frac{\boldsymbol{y}\gamma}{\ell_c}\Big)\Gamma_L\Big(\boldsymbol{r}_1+\frac{\boldsymbol{y}\gamma}{2},\boldsymbol{y};\omega,0\Big)\mathrm{d}\boldsymbol{y}$$

arises and involves only the one-frequency mutual coherence. Using (8) with $\Gamma_0(\boldsymbol{x},\boldsymbol{y}) = \delta(\boldsymbol{x}+(\gamma\boldsymbol{y}/2))\delta(\boldsymbol{x}-(\gamma\boldsymbol{y}/2))$ and making the necessary rescaling of variables we obtain the following asymptotic

$$\langle\theta,\Gamma_L\rangle(\boldsymbol{r})\approx C_0(\boldsymbol{r},\boldsymbol{r}_1)\beta_c^d, \qquad (15)$$

$$C_0=\int\mathrm{d}\boldsymbol{p}\theta^*\Big(\boldsymbol{p}+\frac{\boldsymbol{r}_1-\boldsymbol{r}}{\ell_c}\Big)e^{-(\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{r}_1/\sigma_*)}\mathrm{e}^{-|\boldsymbol{p}|^2/3}. \qquad (16)$$

To derive (15) we have used the condition (2). Note that the transfer function in (16) is Gaussian in $\boldsymbol{p}$ and that $C_0(\boldsymbol{r}, \boldsymbol{r}_1)$ has a Gaussian-tail in $|\boldsymbol{r} - \boldsymbol{r}_1|/\ell_c$ and $C_0(\boldsymbol{r}_1, \boldsymbol{r}_1)$ is bounded away from zero and slowly varying in $\boldsymbol{r}_1$ on the scale σ_*. That is, after proper normalization $C_0(\boldsymbol{r}, \boldsymbol{r}_1)$ behaves like a δ-function centred at $\boldsymbol{r}_1$. By (15) and (16) we obtain the mean field asymptotic $E(\boldsymbol{r}, t) \approx 0$ for $|\boldsymbol{r} - \boldsymbol{r}_1| \gg \ell_c$ (spatial sidelobes) or $|t - \tau_l| \gg B^{-1}, \forall l$ (temporal sidelobes) and $E(\boldsymbol{r}_1, \tau_l) \approx \sqrt{4\pi} C_0(\boldsymbol{r}_1, \boldsymbol{r}_1)\beta_c^d B\gamma$.

The calculation for the variance V involves the four-point correlation of the Green functions at different frequencies. Under the sub-Gaussianity condition (13) the calculation reduces to that of two-frequency mutual coherence functions.

Using (12) with $\Gamma_0(\boldsymbol{x}, \boldsymbol{y}) = \gamma^d \delta(\boldsymbol{x} + (\gamma \boldsymbol{y}/2))\delta(\boldsymbol{x} - (\gamma \boldsymbol{y}/2))$ we obtain the asymptotic for the dominant term in the calculation for the variance $V(\boldsymbol{x}, \tau)$ prior to the ω-integration

$$\begin{aligned}&\Gamma_L(\boldsymbol{r}_1, 0; \omega, \beta) \int \Gamma_L\left(\frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2}, \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\gamma}; \omega, \beta\right)\\ &\times \theta^*(\frac{\boldsymbol{x}_1 - \boldsymbol{r}}{\ell_c})\theta(\frac{\boldsymbol{x}_2 - \boldsymbol{r}}{\ell_c}) d\frac{\boldsymbol{x}_1}{\gamma} d\frac{\boldsymbol{x}_2}{\gamma} \approx C_{\tilde{\beta}} \beta_c^{2d}\end{aligned} \tag{17}$$

with the constant $C_{\tilde{\beta}}$ given by

$$\begin{aligned}C_{\tilde{\beta}} &= (2\pi)^{-2d}(1+\mathrm{i})^d \tilde{\beta}^{d/2} \sin^{-d}(\sqrt{\tilde{\beta}}(1+\mathrm{i}))e^{-(\tilde{\beta}^2 D_0/\sigma_*^2)}\\ &\times \mathrm{e}^{((1-\mathrm{i})/2\sqrt{\tilde{\beta}})\cot(\sqrt{\tilde{\beta}}(1+\mathrm{i}))(\tilde{\beta}^2|\boldsymbol{r}_1|^2/4\sigma_*^2)} \mathrm{e}^{\mathrm{i}(\tilde{\beta}/8\sigma_*^2)(|\boldsymbol{r}_1|^2+|\boldsymbol{r}|^2)}\\ &\times \int \theta^*(\tilde{\boldsymbol{y}} + \frac{\tilde{\boldsymbol{y}}'}{2})\theta(\tilde{\boldsymbol{y}} - \frac{\tilde{\boldsymbol{y}}'}{2})\mathrm{e}^{(\mathrm{i}/2\tilde{\beta}|\tilde{\boldsymbol{y}}'|^2} \mathrm{e}^{\mathrm{i}(\tilde{\boldsymbol{y}}'\cdot \boldsymbol{r}/2\sigma_*)}\\ &\times \mathrm{e}^{((1-\mathrm{i})/2\sqrt{\tilde{\beta}})\cot\left(\sqrt{\tilde{\beta}}(1+\mathrm{i})\right)\left|-(\tilde{\beta}/2\sigma_*)\boldsymbol{r}+\tilde{\boldsymbol{y}}'\right|^2} \mathrm{d}\tilde{\boldsymbol{y}}\mathrm{d}\tilde{\boldsymbol{y}}'.\end{aligned} \tag{18}$$

Note that $C_{\tilde{\beta}}$ depends only slowly on $\boldsymbol{r}, \boldsymbol{r}_1$ for $\sigma_* \gg 1$. Due to the rapidly decaying factor $\sin^{-d}\left(\sqrt{\tilde{\beta}}(1+\mathrm{i})\right)$ the $\tilde{\beta}$-integration of $C_{\tilde{\beta}}$ is convergent as $B \to \infty$. As $\beta_c \ll 1$, the ω_1 and ω_2 integrals essentially decouple after the change in variables: $(\omega_1, \omega_2) = (\omega - \beta\gamma/2, \omega + \beta\gamma/2)$. We conclude that $V(\boldsymbol{x}, t) \leqslant 2\sqrt{2\pi} K\gamma^2 \beta_c^{2d+1} BT \int C_{\tilde{\beta}} \mathrm{d}\tilde{\beta}$. Note that the variance increases linearly with the number T of symbols in each datum stream.

The asymptotic SISR for the SISO-TRA is given by $\rho = O(B\beta_c^{-1}T^{-1})$. Note that the SISR is slowly varying in the test point $\boldsymbol{r}$ and the receiver location $\boldsymbol{r}_1$ on the scale of σ_*.

SIMO. Let us turn to the SIMO case with $N = 1$ element TRA located at $\boldsymbol{a}_1 = 0$.

The mean field calculation is analogous to the SISO case. Namely, $E(\boldsymbol{r}_j, \tau_l) \approx \sqrt{4\pi} C_0(\boldsymbol{r}_j, \boldsymbol{r}_j)\beta_c^d B\gamma$ and zero in the temporal or spatial sidelobe.

In view of the remark following (13) the variance of S is dominated by the contribution from the diagonal terms in the summation over receivers given by

$$\begin{aligned}&\sum_{l,l'=1}^{T} \int \mathrm{e}^{-(\omega(\tau_l - \tau_{l'})/\gamma)} g^2(\omega)\mathrm{d}\omega \sum_{j=1}^{M} \int \theta^*\left(\frac{\boldsymbol{x}_1 - \boldsymbol{r}}{\ell_c}\right)\theta\left(\frac{\boldsymbol{x}_2 - \boldsymbol{r}}{\ell_c}\right)\\ &\qquad \times \int \mathrm{d}\beta \Gamma_L(\boldsymbol{r}_j, 0; \omega, \beta)\Gamma_L\left(\frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2}, \frac{\boldsymbol{x}_1 - \boldsymbol{x}_2}{\gamma}; \omega, \beta\right) \mathrm{d}\frac{\boldsymbol{x}_1}{\gamma} \mathrm{d}\frac{\boldsymbol{x}_2}{\gamma}\\ &\qquad \approx \sqrt{2\pi} B\gamma^2 TM \int C_{\tilde{\beta}} \mathrm{d}\tilde{\beta}\beta_c^{2d},\end{aligned}$$

because $|\boldsymbol{r}_i - \boldsymbol{r}_j| \gg \ell_c$ regardless of whether the test point is near or away from any receiver. Therefore, we have the estimate: $\rho = O(B\beta_c^{-1}M^{-1}T^{-1})$.

MISO. The case corresponds to $M = 1$. Each term in the summation over the N TRA-elements has the same asymptotic as that of the SISO case. Hence, $E(\boldsymbol{r}_j, \tau_l) \approx \sqrt{4\pi} N C_0(\boldsymbol{r}_j, \boldsymbol{r}_j)\beta_c^d B\gamma$ and zero in the spatial or temporal sidelobes.

For the variance calculation, let us first note that the correlations of two Green functions starting with two TRA-elements located at $\boldsymbol{a}_i, \boldsymbol{a}_j$ satisfy equation (9) in the variables $(\boldsymbol{a}_i, \boldsymbol{a}_j)$, by the reciprocity of the time-invariant channel and hence vanish as $|\boldsymbol{a}_i - \boldsymbol{a}_j| \gg \ell_c$. The variance of the signal at $\boldsymbol{r}$ (whether at $\boldsymbol{r}_1$ or away from it) before performing the ω-integration is then dominated by the following diagonal terms in the summation over receivers

$$\sum_{j=1}^{N} \mathbb{E}\left[G_L^*(\boldsymbol{r}_1, \boldsymbol{a}_j; \omega_1) G_L(\boldsymbol{r}_1, \boldsymbol{a}_j; \omega_2)\right] \int \theta^*\left(\frac{\boldsymbol{x}_1 - \boldsymbol{r}}{\ell_c}\right)$$
$$\times \theta\left(\frac{\boldsymbol{x}_2 - \boldsymbol{r}}{\ell_c}\right) \mathbb{E}[G_L(\boldsymbol{x}_1, \boldsymbol{a}_j; \omega_1) G_L^*(\boldsymbol{x}_2, \boldsymbol{a}_j; \omega_2)] \mathrm{d}\frac{\boldsymbol{x}_1}{\gamma} \mathrm{d}\frac{\boldsymbol{x}_2}{\gamma}$$
$$\approx N C_{\tilde{\beta}} \beta_c^{2d}.$$

The ω-integration induces the additional factor of $\sqrt{2\pi} B\gamma^2 T$. Hence

$$V(\boldsymbol{r}, t) \leqslant 2\sqrt{2\pi} K\gamma^2 \beta_c^{2d+1} BT \int C_{\tilde{\beta}} \mathrm{d}\tilde{\beta}$$

since $|\boldsymbol{r} - \boldsymbol{r}_j| \ll \sigma_*, \forall j$. We conclude that $\rho = O(NB\beta_c^{-1}T^{-1})$.

MIMO. The analysis for the MIMO case combines all the previous cases. The mean signal has the same asymptotic as that of the MISO case, i.e. linearly proportional to BN. The variance of the signal prior to performing the ω-integration is dominated by

$$\sum_{i,j=1}^{M,N} \mathbb{E}\left[G_L^*(\boldsymbol{r}_i, \boldsymbol{a}_j; \omega_1) G_L(\boldsymbol{r}_i, \boldsymbol{a}_j; \omega_2)\right] \int \theta^*\left(\frac{\boldsymbol{x}_1 - \boldsymbol{r}}{\ell_c}\right)$$
$$\times \theta\left(\frac{\boldsymbol{x}_2 - \boldsymbol{r}}{\ell_c}\right) \mathbb{E}\left[G_L(\boldsymbol{x}_1, \boldsymbol{a}_j; \omega_1) G_L^*(\boldsymbol{x}_2, \boldsymbol{a}_j; \omega_2)\right] \mathrm{d}\frac{\boldsymbol{x}_1}{\gamma} \mathrm{d}\frac{\boldsymbol{x}_2}{\gamma}$$
$$\approx NMC_{\tilde{\beta}} \beta_c^{2d}$$

and therefore $V \leqslant TMN2K\gamma^2 B\beta_c^{2d+1} \int C_{\tilde{\beta}} \mathrm{d}\tilde{\beta}$. We conclude that $\rho = O(NB\beta_c^{-1}T^{-1}M^{-1})$.

Although we have omitted the details of preceding calculations, they can be fully justified. We summarize the results in the following.

Theorem 1. *Let the N-element TRA, M receivers and the number of symbols T satisfy the packing condition. Assume the 4th order sub-Gaussianity condition (13) in the saturated-fluctuation regime (2) and let $1 \ll \alpha_* \ll \gamma^{-1}$, $1 \ll \sigma_* \ll \gamma^{-1}$.*

Then in the broadband limit (4) the asymptotic SISR $\sim NM^{-1}T^{-1}B\beta_c^{-1}$ *is valid uniformly for all $\boldsymbol{r}_j$, $j = 1, \ldots, M$, with the constant of proportionality $2^{-1}(2\pi)^{-1/2}K^{-1}(\int C_{\tilde{\beta}} \mathrm{d}\tilde{\beta})^{-1}|C_0|^2$ where C_0 and $C_{\tilde{\beta}}$ are given by (16) and (18), respectively.*

The asymptotic signal at the receiver plane within the distance much less than σ_* from the receivers is $\sum_{l=1}^{T}\sum_{j}^{M} m_j(\tau_l) S_{jl}(\boldsymbol{x}, t)$ where $S_{jl}(\boldsymbol{x}, t)$ is given by (6).

Note here that $T\beta_c \sim C$ is roughly the number of symbols per unit time in each datum stream since T symbols are packed within a delay spread $\delta_* \sim \beta_c^{-1}$.

5. Information rate

In this section, following [16], we derive the information rate for a memoryless channel which is constructed out of the time-invariant channel model analysed above. The temporal dependence is introduced by drawing an independent realization from the medium ensemble after each use of the medium realization. This is a widely used model in communications literature for time-varying channels with the coherence time much longer than one delay spread of the realization [21, 39]. We assume as in standard practice that in addition to the random channel fluctuations additive-white-Gaussian-noise (AWGN) is present at each receiver, that the input signal vector is multivariate Gaussian and that the channel, the noise and the input signal are mutually independent. We further assume the probability distribution of the wave field to be exactly Gaussian (instead of being just 4th-order sub-Gaussian) to simplify our discussion.

Prior to adding noise, each frequency component of the time reversed signal

$$\sum_{i=1}^{M}\sum_{n=1}^{N} m_i(\tau_l)g(\omega)G_L(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)G_L^*(\boldsymbol{r}_i,\boldsymbol{a}_n;\omega)$$
$$= \overbrace{\sum_{n=1}^{N} m_i(\tau_l)g(\omega)G_L(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)G_L^*(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)}^{N\text{-degree central }\chi^2\text{ r.v.}}$$
$$+\underbrace{\sum_{i\neq j}\sum_{n=1}^{N} m_i(\tau_l)g(\omega)G_L(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)G_L^*(\boldsymbol{r}_i,\boldsymbol{a}_n;\omega)}_{N(M-1)\text{ i.i.d. zero-mean r.v.s}}$$

is a sum of a central χ^2 random variable with N degrees of freedom and $N(M-1)$ i.i.d. mean-zero random variables. This is due to the Gaussianity assumption which implies that uncorrelated entries are statistically independent. Therefore, for $N \gg 1$ the interference statistic is approximately Gaussian, by the central limit theorem. More generally, after synthesizing all the available frequencies, the interference statistic becomes approximately Gaussian if $NB\beta_c^{-1} \gg 1$ which is always the case for broadband signals. In a broadband channel $NB\beta_c^{-1}$ is the number of independent subchannels from TRA to each receiver.

For the discussion of power constraint below let us normalize the Green function G_L as $\sum_{n=1}^{N}\mathbb{E}\left|G_L(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)\right|^2 \approx 1, \forall j$. As a consequence, each frequency component has the mean

$$\mathbb{E}\Big[\sum_{i=1}^{M}\sum_{n=1}^{N} m_i(\tau_l)g(\omega)G_L(\boldsymbol{r}_j,\boldsymbol{a}_n;\omega)G_L^*(\boldsymbol{r}_i,\boldsymbol{a}_n;\omega)\Big] \approx g(\omega)m_j(\tau_l), \tag{19}$$

which exhibits the simple input–output relation: the ω-component of the input signal for the jth receiver is $m_j g(\omega)$ and the received signal component is $m_j g(\omega)$ corrupted by the noise and interference which for $N \gg 1$ is approximately Gaussian. Since the M receivers operate independently of one another, the total time-reversal broadcast channel consists of M independent subchannels in parallel each of which has the above input–output relation. Thus, the total information rate is the sum of those of the M subchannels from TRA to individual receivers. And, in view of the simple input–output relation, each subchannel can be viewed as a SISO linear filter channel corrupted by (approximately) Gaussian noise/interference for which Shannon's theorem is applicable.

According to Shannon's theorem [5] the ergodic capacity (in nats per unit time and bandwidth) of a SISO linear filter channel is $\ln(1+\mathrm{SINR})$ where SINR, the signal-to-interference-and-noise ratio at each receiver, is given by the harmonic sum of the SIR, the SIR and SNR, the SNR, i.e. $\mathrm{SINR} = (\mathrm{SIR}^{-1}+\mathrm{SNR}^{-1})^{-1}$. For the extension of Shannon's result to the MIMO setting, see [21, 39].

We set the symbol rate to be $2B$. Hence $\mathrm{SIR}(\omega) \sim N/M$, independent of ω. Let ν be the noise level, per unit bandwidth, at each receiver. Suppose the *average* transmission power is constrained to P and all the transmit and receive antennas are identical. In view of (19), $\mathrm{SNR}(\omega) = \mu^2/\nu$ where $\mu = |m_j|$ can be related to the average total power P as $\mu^2 M \sim P/B$ since the average input power per unit bandwidth is

$$\sum_{n=1}^{N}\sum_{i=1}^{M} |m_i(\tau_l)|^2 |g|^2(\omega)\mathbb{E}\big|G_L(\boldsymbol{x}_i, \boldsymbol{a}_n; \omega)\big|^2 \sim M\mu^2.$$

Thus, $\mathrm{SNR}(\omega) \sim P/(\nu BM)$. Therefore the total channel capacity (in nats per unit time) is roughly given by

$$BM \ln\left[1 + \frac{1}{M}\left(\frac{1}{N} + \frac{\nu B}{P}\right)^{-1}\right]. \tag{20}$$

What is the maximal rate at which a TRA, with fixed number of elements N, fixed average total power P and fixed noise level (per unit bandwidth) ν, can transfer information if there is no limitation to the number of receivers M and the bandwidth B? Expression (20) can be optimized at the low SNR limit $M \gg P/(\nu B)$ to yield the optimal information rate of P/ν which is the power-to-noise ratio. We see that the simplest strategy for optimizing the information rate of a given TRA under the power and noise constraints is to enlarge the bandwidth B as much as possible. And if we can satisfy $N \gg M \gg P/(\nu B)$ then we can achieve stability as well as the optimal information rate. An $O(P/\nu)$ information rate is consistent with the classical result of minimum energy $k_B T$ requirement for transmitting one nat information where k_B is the Boltzmann constant and T the temperature [28, 31].

Since the number of receivers (and TRA-elements) in the area of one spatial spread cannot exceed $\sigma_*^d/\ell_c^d = \gamma^{-d}\beta_c^{-d}$ the regime with the optimal information rate thus requires $\gamma^{-d}\beta_c^{-d} \gg P/(\nu B)$ which can be satisfied with $\gamma \ll 1$ (high frequency) and/or $\beta_c \ll 1$ (saturated-fluctuation).

It may be worthwhile to compare the information rate of the MIMO-TR channel with that of the reciprocal, non-TR channel in which there are M transmitters, without channel knowledge, and N receivers, with channel knowledge. The information rate for the non-TR channel with *narrowband* signals has been extensively studied in the literature [21, 29, 36, 39] and scales like $BM \ln \mathrm{SNR}$ for $M \leqslant N$ at high $\mathrm{SNR} \sim P/(\nu BM) \gg 1$ (thus $M \ll P/(\nu B)$). According to (20), the information rate of the TR channel also scales the same way provided that $N \gg P/(\nu B)$. We know, however, from the above discussion that the optimal rate cannot be achieved in this regime.

6. Conclusion

The saturated-fluctuation regime (2) constitutes the so called space-frequency-selective multipath fading channels in wireless communications [30]. In such a channel, TR has the super-resolution given by $\ell_c = \gamma/\sqrt{D_2 L}$. We have established firmly the packing and multiplexing conditions for stable super-resolution for the MIMO-TRA communication system under the 4th order sub-Gaussianity assumption. The stability in the case $M = 1$ has been experimentally demonstrated in [7, 8] and analysed in [32, 40]. We show that an $O(P/\nu)$ information rate can be achieved by broadband time-reversal, resulting in statistically stable, sharply focused signals at the receiver end. Our results should be valid for other multiply scattered waves such as diffusive waves for which β_c is the Thouless frequency and ℓ_c is on the scale of the wavelength.

Let us point out several possible extensions of our results. First, the case of even broader bandwidth of $0 < \lim B\gamma \leqslant 1$ can easily be treated by partitioning the full bandwidth into many sub-bands with their own B and γ satisfying (4). Since the self-averaging takes place in each sub-band and the whole process is linear, stable super-resolution is valid in the full passband. Second, in the case of a finite time-window, the out-put signals, unlike (3), involve a coupling of neighbouring wavenumbers [18]. If the time window is sufficiently large ($\gg \beta_c^{-1}$) then the coupling takes place only between wavenumbers of separation much smaller than β_c and our result carries over without major adjustment. Finally, our results may also be extended to time-varying channels, prevalent in mobile wireless communications, with a low spread factor $T_c^{-1}\delta_* \ll 1$ where T_c is the coherence time of the time-varying channels [30].

Appendix A. Derivation of (12)

In terms of the variables $\boldsymbol{y}_1 = \tilde{\boldsymbol{y}} + \tilde{\beta}\tilde{\boldsymbol{x}}/2, \boldsymbol{y}_2 = \tilde{\boldsymbol{y}} - \tilde{\beta}\tilde{\boldsymbol{x}}/2$ equation (9) can be written as

$$\frac{\partial}{\partial \tilde{z}}\tilde{\Gamma}_{\tilde{z}} = \frac{\mathrm{i}\tilde{\beta}}{2}\left(\nabla_1^2 - \nabla_2^2\right)\tilde{\Gamma}_{\tilde{z}} - |\boldsymbol{y}_2|^2\tilde{\Gamma}_{\tilde{z}} - \frac{\tilde{\beta}^2 D_0}{2\sigma_*^2}\tilde{\Gamma}_{\tilde{z}}, \tag{A.1}$$

where ∇_1, ∇_2 are the gradients with respect to $\boldsymbol{y}_1, \boldsymbol{y}_2$, respectively. Consider the function

$$W(\tilde{z}, \boldsymbol{p}_1, \boldsymbol{y}_2) = \mathrm{e}^{(\tilde{\beta}^2 D_0/2\sigma_*^2)\tilde{z}}\mathrm{e}^{-\mathrm{i}(\tilde{z}\tilde{\beta}/2)|\boldsymbol{p}_1|^2}\frac{1}{(2\pi)^d}\int \tilde{\Gamma}_{\tilde{z}}\left(\frac{\boldsymbol{y}_1 - \boldsymbol{y}_2}{2\tilde{\beta}}, \frac{\boldsymbol{y}_1 + \boldsymbol{y}_2}{2}\right)\mathrm{e}^{-\mathrm{i}\boldsymbol{y}_1\cdot\boldsymbol{p}_1}\,\mathrm{d}\boldsymbol{y}_1$$

which satisfies the equation

$$\frac{\partial}{\partial \tilde{z}}W = -\frac{\mathrm{i}\tilde{\beta}}{2}\nabla_2^2 W - |\boldsymbol{y}_2|^2 W. \tag{A.2}$$

Equation (A.2) is just the Schrödinger equation with an imaginary, quadratic potential and can be solved by separation of variables. Consider the one-dimensional version of the equation:

$$\frac{\partial}{\partial \tilde{z}}W_j = -\frac{\mathrm{i}\tilde{\beta}}{2}\frac{\partial^2}{\partial y_j^2}W_j - y_j^2 W_j, \qquad j = 1, 2, \ldots, d, \tag{A.3}$$

and forming tensor product $\prod_{j=1}^d W_j(y_j)$.

We begin by searching for solutions of the Gaussian form

$$W_j = \mathrm{e}^{-A(\tilde{z}) - B(\tilde{z})|y_j - C(\tilde{z})|^2}, \tag{A.4}$$

where A, B, C are complex-valued functions of $\tilde{z}$, parametrized by $\boldsymbol{p}_1$. Substituting (A.4) into equation (A.2) and comparing the coefficients we obtain the ODEs governing A, B, C:

$$B' = 1 + \mathrm{i}2\tilde{\beta}B^2, \tag{A.5}$$

$$C' = -\frac{C}{B}, \tag{A.6}$$

$$A' = C^2 - \mathrm{i}\tilde{\beta}B, \tag{A.7}$$

which can be solved in the order of B, C, A and yield

$$B(\tilde{z}) = \frac{1}{(1-\mathrm{i})\sqrt{\tilde{\beta}}}\frac{K\mathrm{e}^{2(1-\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}} + 1}{K\mathrm{e}^{2(1-\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}} - 1},$$

$$C(\tilde{z}) = C(0)\exp\left[-\int_0^{\tilde{z}} B(s)^{-1}\,\mathrm{d}s\right], \tag{A.8}$$

where the constant K is given by

$$K = \frac{2B(0)\sqrt{\tilde{\beta}} + 1 + \mathrm{i}}{2B(0)\sqrt{\tilde{\beta}} - 1 - \mathrm{i}}.$$

We set $K = 1$ corresponding to $B(0) = +\infty$, $j = 1, 2, \ldots, d$ and write

$$A(\tilde{z}) = -\mathrm{i}\tilde{\beta}\int_1^{\tilde{z}} B(s)\mathrm{d}s + \int_0^{\tilde{z}} C^2(s)\mathrm{d}s, \qquad \tilde{z} > 0$$

with

$$B(\tilde{z}) = \frac{1}{(1-\mathrm{i})\sqrt{\tilde{\beta}}}\frac{\mathrm{e}^{2(1-\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}} + 1}{\mathrm{e}^{2(1-\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}} - 1}$$
$$= \frac{-1}{(1+\mathrm{i})\sqrt{\tilde{\beta}}}\cot\left[(1+\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}\right].$$

Then a straightforward calculation leads to the Green function

$$G_{\tilde{\Gamma}}(\tilde{z}, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_1', \boldsymbol{y}_2') \sim \mathrm{e}^{-(\tilde{\beta}^2 D_0/2\sigma_*^2)\tilde{z}}\mathrm{e}^{\mathrm{i}d\tilde{\beta}\int_1^{\tilde{z}} B(s)\mathrm{d}s - \int_\infty^{\tilde{z}} |\boldsymbol{C}|^2(s)\mathrm{d}s}\mathrm{e}^{-B(\tilde{z})|\boldsymbol{y}_2 - \boldsymbol{C}(\tilde{z})|^2}$$
$$\times \int \mathrm{e}^{-\mathrm{i}\tilde{\beta}|\boldsymbol{p}_1|^2\tilde{z}/2}\mathrm{e}^{\mathrm{i}\boldsymbol{p}_1\cdot(\boldsymbol{y}_1 - \boldsymbol{y}_1')}\mathrm{d}\boldsymbol{p}_1$$
$$\sim \mathrm{e}^{-(\tilde{\beta}^2 D_0/2\sigma_*^2)\tilde{z}}(1+\mathrm{i})^{d/2}\tilde{z}^{d/2}\tilde{\beta}^{d/4}(2\pi)^d\tilde{z}^d\tilde{\beta}^d\sin^{d/2}\left[(1+\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}\right]$$
$$\times \mathrm{e}^{\mathrm{i}(|\boldsymbol{y}_1 - \boldsymbol{y}_1'|^2/2\tilde{\beta}\tilde{z})}\mathrm{e}^{-(|\boldsymbol{y}_2'|^2/(1+\mathrm{i})\sqrt{\tilde{\beta}})\tan\left((1+\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}\right)}$$
$$\times \mathrm{e}^{(1/(1+\mathrm{i})\sqrt{\tilde{\beta}})\cot\left((1+\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}\right)\left|\boldsymbol{y}_2 - \left(\boldsymbol{y}_2'/\cos\left((1+\mathrm{i})\tilde{z}\sqrt{\tilde{\beta}}\right)\right)\right|^2},$$

where $\boldsymbol{C}(z) = (C_j(\tilde{z}))$ is given by the formula (A.8) with the initial data $\boldsymbol{C}(0) = \boldsymbol{y}_2'$.

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