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INTEGRAL EQUATIONS WITH HYPERSINGULAR KERNELS – THEORY AND APPLICATIONS TO FRACTURE MECHANICS

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Abstract

Hypersingular integrals of the type

$$I_\alpha(T_n, m, r) = \neq\!\!\!\!\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha} ds \ , \quad |r| < 1$$

and

$$I_\alpha(U_n, m, r) = \neq\!\!\!\!\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha} ds \ , \quad |r| < 1$$

are investigated for general integers α (positive) and m (non-negative), where $T_n(s)$ and $U_n(s)$ are the Tchebyshev polynomials of the 1st and 2nd kinds, respectively. Exact formulas are derived for the cases $\alpha = 1, 2, 3, 4$ and $m = 0, 1, 2, 3$; most of them corresponding to new solutions derived in this paper. Moreover, a systematic approach for evaluating these integrals when $\alpha > 4$ and $m > 3$ is provided. The integrals are also evaluated as $|r| > 1$ in order to calculate stress intensity factors (SIFs). Examples involving crack problems are given and discussed with emphasis on the linkage between mathematics and mechanics of fracture. The examples include classical linear elastic fracture mechanics (LEFM), functionally graded materials (FGM), and gradient elasticity theory. An appendix, with closed form solutions for a broad class of integrals, supplements the paper.

1 Introduction

Finite and boundary element methods are two of the most frequently used numerical approaches for solving crack problems in fracture mechanics. An alternative approach is the integral equation method, which is more efficient (*e.g.* it reduces a partial differential equation (PDE) in two dimensions to an one-dimensional integral equation) and, in general, is more accurate than the aforementioned methods. The accuracy of the integral equation method relies on the analytical

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evaluation of singular kernels to cancel the singularity (regularization). In general, the cancellation of singularity is not trivial, in particular, in the case of hypersingular integrals. This is the main concern of this paper.

Integral equations arising in static crack problems in fracture mechanics are typically Fredholm integral equations of the form

$$\int_c^d \mathsf{kernel}(x,t)\ D(t)\ dt \quad = \quad p(x)\ , \qquad c < x < d\ , \tag{1}$$

where $\mathsf{kernel}(x,t)$ is, in general, a singular function of (x,t); $D(t)$ is the unknown, called density function; $p(x)$ is some known (input) function corresponding to the loading on the crack faces; and the interval (c,d) refers to the crack surfaces where $2a = d - c$ denotes the crack length. By the Fourier transform, we write

$$\mathsf{kernel}(x,t) \quad = \quad \int_{-\infty}^{\infty} K(\xi) \mathrm{e}^{i(t-x)\xi} d\xi\ . \tag{2}$$

The singular part of the kernel can be separated from the regular part, by decomposing the Fourier transform as

$$K(\xi) = \underbrace{K_\infty(\xi)}_{\text{singular}} + \underbrace{[K(\xi) - K_\infty(\xi)]}_{\text{nonsingular}}\ , \tag{3}$$

which can be accomplished through asymptotic analysis (discussed later in this paper). Such an analysis is difficult for complicated $K(\xi)$. This is another issue to be addressed in this paper.

Once the decomposition (3) is accomplished, the integral equation (1) can be rewritten as

$$⨍_c^d \frac{c_\alpha\ D(t)}{(t-x)^\alpha}\ dt \ + \ \int_c^d k(x,t)\ D(t)\ dt \ + \ f(x) \ = \ p(x)\ , \qquad c < x < d\ , \tag{4}$$

where $⨍$ denotes an improper integral; c_α is a constant associated to the singular kernel $1/(t-x)^\alpha$; $k(x,t)$ is the nonsingular (regular) kernel; $f(x)$ is a function standing for the free term; and α is a positive integer which determines the degree of the singularity. If $\alpha = 1$, the integral equation (4) is called a Cauchy singular Fredholm integral equation, and the singular term is evaluated as a Cauchy principal-value (CPV) integral. If $\alpha \geq 2$, it is called a hypersingular Fredholm integral equation and the singular term is evaluated as a Hadamard finite-part (HFP) integral [19, 24, 26, 28]. The notation $⨍$ and $⨎$ refer to CPV and HFP, respectively. Most of the works in the literature involve either $\alpha = 1$ [3, 4, 6, 7, 20, 30] or $\alpha = 2$ [18, 19, 34]. Thus another focus of this paper is to deal with hypersingular integral equations with $\alpha \geq 3$ which arise naturally in gradient elasticity theories (see Example 3 in Section 7).

The singular and hypersingular integrals which involve Tchebyshev polynomials (T_n, first kind; U_n, second kind) and weight function $(1-s^2)^{m-\frac{1}{2}}$, $m \geq 0$, with singularity $\alpha \geq 1$ are defined by

$$I_\alpha(T_n, m, r) = ⨎_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha} ds\ , \quad |r| < 1\ , \tag{5}$$

and

$$I_\alpha(U_n, m, r) = \fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha} ds \ , \quad |r| < 1 \ . \tag{6}$$

The scope of this paper is as follows. First, Cauchy singular integrals, *i.e.* $\alpha = 1$, are evaluated and exact formulas are derived for general m. The new results here are closed form analytical solutions for $I_1(T_n, m, r)$, $m \geq 1$ and $I_1(U_n, m, r)$, $m \geq 2$. Once $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ are known, hypersingular integrals $I_\alpha(T_n, m, r)$ and $I_\alpha(U_n, m, r)$, $\alpha \geq 2$, can be found by successive differentiation (with respect to r) in the sense of finite-part integrals; formulas for $I_2(T_n, m, r)$, $m \geq 1$ and $I_2(U_n, m, r)$, $m \geq 2$ are derived in this manner. In the cases where $\alpha \geq 3$, evaluation of hypersingular integrals becomes tedious and the formulas are lengthy. Thus $I_\alpha(T_n, m, r)$ and $I_\alpha(U_n, m, r)$ are provided only for $\alpha = 3, 4$ and general m.

2 Related Work

Singular integral equations have played an active role in the field of solid mechanics, particularly in the solution of fracture mechanics problems. For instance, the Journal *Integral Equations and Operator Theory* has dedicated a special issue to "Integral Equations Methods in Engineering and Physics" (Volume 5, No. 4, 1982). Also, in June 1984, IMACS (International Association for Mathematics and Computers in Simulation) has held a Symposium [11] devoted to "Numerical Solution of Singular Integral Equations".

According to the notation introduced in equation (4), singular integral equations can be classified by the order of singularity α. The case $\alpha = 1$ has been widely used and well developed [6, 7]. A rich field of application of singular integral equations is fracture mechanics of bi-material and nonhomogeneous materials. For instance, the investigation of crack behavior in nonhomogeneous materials has found many applications to functionally graded materials (FGMs) [3, 5, 16]. Another use of singular integral equations involves FGMs for high temperature applications, so that thermal stress intensity factors can be numerically calculated [15, 29]. Application of hypersingular Fredholm integral equations for $\alpha \geq 2$ can be found in references [18, 19, 34].

Quadrature formulas which involve hypersingular integrals have been drawing a considerable amount of concentration [14, 18, 19, 21, 22, 25, 26, 36] after Kutt first introduced the Hadamard finite-part (HFP) idea in his work [24]. Based on some previous work, Kaya [18] has presented a very nice interpretation about HFP integrals. One of the key steps in the derivations involves the fact that higher order singular integrals can be obtained from lower order ones by exchangeability of differentiation and integration [19, 26], *e.g.*

$$\fint_{-1}^{1} \frac{D(s)}{(s-r)^{\alpha+1}} ds = \frac{1}{\alpha} \fint_{-1}^{1} \frac{\partial}{\partial r} \left[\frac{D(s)}{(s-r)^\alpha} \right] ds = \frac{1}{\alpha} \frac{d}{dr} \fint_{-1}^{1} \frac{D(s)}{(s-r)^\alpha} ds \ , \quad |r| < 1, \tag{7}$$

where $D(s)$ is the normalized density function. For instance, in order to find

$$\fint_{-1}^{1} \frac{D(s)}{(s-r)^2} ds \ , \quad |r| < 1 \ ,$$

it suffices to know how to evaluate

$$\oint_{-1}^{1} \frac{D(s)}{s-r} ds \ , \quad |r| < 1 \ .$$

This concept is applied later in this paper.

Another main motivation for numerical evaluations of hypersingular integrals is due to the boundary element method, and the reader is directed to the review paper by Tanaka *et al* [35]. Most recent work has been focused on singularity with $\alpha = 2$ in two-dimensional problems.

3 Theoretical Aspects

First, relevant concepts involving integration and approximation are given. These concepts position the contribution of the work with respect to the available literature. Next, a discussion on the influence of the density function on the corresponding singular integral equation formulation is presented. Afterwards, basic properties of the Tchebyshev polynomials are provided. These properties are heavily used in the analytical derivations that follow.

3.1 Integration and Approximation

As far as the integration and numerical procedures are concerned, the integral equation (4) may be normalized through the following change of variables

$$s = \frac{2}{d-c}\left(t - \frac{c+d}{2}\right) \quad \text{and} \quad r = \frac{2}{d-c}\left(x - \frac{c+d}{2}\right) \quad , \tag{8}$$

which leads to the normalized version of the integral equation (4) written as[1]

$$\oint_{-1}^{1} \frac{D(s)}{(s-r)^{\alpha}} \, ds + \int_{-1}^{1} \mathcal{K}(r,s) D(s) \, ds + F(r) \ = \ P(r) \ , \quad -1 < r < 1 \ . \tag{9}$$

The density function $D(s)$ is further assumed to have the representation

$$D(s) \ = \ R(s) W(s) \ . \tag{10}$$

The weight function $W(s)$ determines the singular behavior of the solution $D(s)$ and has the form

$$W(s) \ = \ (1-s)^{m_1} (1+s)^{m_2} \quad . \tag{11}$$

In general, $m_1 \neq m_2$, and the corresponding integrals, which involve Jacobi polynomials $P_n^{(m_1,m_2)}(s)$, are of the type

$$\oint_{-1}^{1} \frac{(1-s)^{m_1}(1+s)^{m_2} P_n^{(m_1,m_2)}(s)}{s-r} ds \ , \tag{12}$$

[1] The notations in this paper have been chosen as following: x and t refer to the physical quantities and have dimension of "Length"; r and s are normalized (dimensionless) variables, corresponding to x and t, respectively.

and can be expressed in terms of gamma and hypergeometric functions [13, 18, 19, 23]. In this paper, only the case $m_1 = m_2$ is considered and m_1, m_2 are set to be

$$m_1 = m_2 = m - \frac{1}{2} \ . \tag{13}$$

Thus $W(s)$ can be expressed as

$$W(s) \ = \ \left(1 - s^2\right)^{m-\frac{1}{2}} \quad m = 0, 1, 2, \cdots \quad . \tag{14}$$

According to function-theoretic method [4, 7, 18, 19, 27] , the value of m is determined by the order of singularity α. For example, if $\alpha = 1$, then $m = 0$ and the fundamental solution $D(s)$ to the Cauchy singular integral equation (9) takes the form

$$D(s) \ = \ R(s)(1 - s^2)^{-\frac{1}{2}} \ = \ \frac{R(s)}{\sqrt{1 - s^2}} \ . \tag{15}$$

In this case, which consists of the majority of the work involving applications of integral equations to fracture mechanics [3, 4, 6, 7, 20, 30] , $R(s)$ is chosen to be

$$R(s) \ = \ \sum_{n}^{\infty} a_n T_n(s) \ \ ; \tag{16}$$

and because of that, the CPV integral $I_1(T_n, 0, r)$ can be evaluated exactly [13, 19] :

$$I_1(T_n, 0, r) \ = \ \fint_{-1}^{1} \frac{T_n(s)}{(s - r)\sqrt{1 - s^2}} \, ds \ = \ \pi U_{n-1}(r) \ , \qquad n \geq 1 \ . \tag{17}$$

Another reason for choosing the approximation (15) is that with respect to the weight function $W(s) = 1/\sqrt{1 - s^2}$, the class of the Tchebyshev polynomials of first kind $T_n(s)$ is an orthogonal family [13, 19]:

$$\int_{-1}^{1} \frac{T_m(s) T_n(s)}{\sqrt{1 - s^2}} ds = \begin{cases} \pi & m = n = 0 \\ \pi/2 & m = n; \quad m, n = 1, 2, 3, \cdots \\ 0 & m \neq n; \quad m, n = 0, 1, 2, \cdots \end{cases} \tag{18}$$

With this orthogonal property a Galerkin-type method [23] may be applied to find the coefficients a_n in equation (16).

If $\alpha = 2$, then $m = 1$, and the solution $D(s)$ to the hypersingular integral equation (9) is characterized by

$$D(s) \ = \ R(s)(1 - s^2)^{\frac{1}{2}} \ = \ R(s)\sqrt{1 - s^2} \ . \tag{19}$$

Correspondingly, $R(s)$ is chosen to be

$$R(s) \ = \ \sum_{n}^{\infty} b_n U_n(s) \ \ , \tag{20}$$

because of the same reasons for the case $\alpha = 1$, namely, analytical evaluation and orthogonal property. With respect to the first reason, the HFP integral $I_2(U_n, 1, r)$ can be evaluated analytically [19] :

$$I_2(U_n, 1, r) \;=\; \fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2} ds = -(n+1)\pi U_n(r) \;, \qquad n \geq 0 \;. \tag{21}$$

According to the second reason, by orthogonality,

$$\int_{-1}^{1} U_m(s) U_n(s) \sqrt{1-s^2} ds = \begin{cases} \pi/2 & m = n; \quad m, n = 0, 1, 2, \cdots \\ 0 & m \neq n; \quad m, n = 0, 1, 2, \cdots \end{cases} , \tag{22}$$

and one may apply Galerkin-type methods [23] to find the coefficients b_n in equation (20).

When $m = 3$, then $W(s) = (1-s^2)^{5/2}$, and neither $T_n(s)$ nor $U_n(s)$ is an orthogonal family. *However, if collocation method is applied, one does not need the orthogonal property, as long as the expansion function $R(s)$ is chosen such that*

$$\fint_{-1}^{1} \frac{R(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^\alpha} ds$$

can be evaluated analytically. For example, if $R(s)$ is expandeded as a Tchebyshev polynomial of the 1st kind $T_n(s)$ or the 2nd kind $U_n(s)$, *i.e.*

$$R(s) \;=\; \sum_n^\infty a_n T_n(s) \quad \text{or} \quad R(s) \;=\; \sum_n^\infty b_n U_n(s) \;\; , \tag{23}$$

then the evaluation of

$$I_\alpha(T_n, m, r) = \fint_{-1}^{1} \frac{T_n(s)\,(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha}\, ds \;\; \text{or} \;\; I_\alpha(U_n, m, r) = \fint_{-1}^{1} \frac{U_n(s)\,(1-s^2)^{m-\frac{1}{2}}}{(s-r)^\alpha}\, ds$$

for general $m = 0, 1, 2, \cdots$ and $\alpha = 1, 2, 3, \cdots$ is a necessary step for the numerical approach to the integral equation (9). This is the one of main tasks in this paper and is addressed in Sections 4 and 5 .

3.2 Selection of the Density Function

Usually the unknown function $D(t)$ in equation (1) can be chosen as the displacement profile (*e.g.* $u(t)$ – a displacement function), the (first) derivative of the displacement function ($du(t)/dt$, denoted by $\phi(t)$ – the slope function), or a higher derivative of $u(t)$. The choice of the unknown function $D(t)$ will affect the degree of singularity in the formulation. For example, consider the standard mode III crack problem in a free space [10] and a linear elastic fracture mechanics (LEFM) setting. If $D(t)$ is chosen to be the slope function $\phi(t)$, then the governing integral equation is the Cauchy singular integral equation

$$D(t) \equiv \phi(t) \;\; , \quad \fint_c^d \frac{\phi(t)}{t-x} dt = p(x) \;, \quad c < x < d \;. \tag{24}$$

However, if $D(t)$ is chosen to be the displacement function $w(t)$, then the hypersingular integral equation with $\alpha = 2$ is obtained,

$$D(t) \equiv w(t) \quad , \quad \oint_c^d \frac{w(t)}{(t-x)^2} dt = p(x) \ , \quad c < x < d \ . \tag{25}$$

The differences between the above two formulations are discussed next.

In general, the (numerical) solution of a Cauchy singular integral equation, *e.g.* equation (24), is easier than the corresponding hypersingular integral equation, *e.g.* equation (25). A quick observation is that if equation (24) is used, then the actual crack surface displacements are obtained directly; while if equation (25) is used, an extra step of integration is needed to recover $w(t)$. However, integration is not an unpleasant thing to do.

So, what can be gained from a more singular equation such as (25)? A formulation with more singular integral may lead to a simpler kernel function and, thus, simplify the kernel evaluation and decomposition described in equation (3). Issues regarding the differences between formulations of the type given in equations (24) and (25) will be discussed in detail in the example section of this paper.

3.3 Properties of Tchebyshev Polynomials

The evaluation of Cauchy singular and hypersingular integrals which involve the Tchebyshev polynomials $T_n(s)$ and $U_n(s)$ highly depends on the special properties of these polynomials. They are listed here for the sake of completeness and because they will be of much use later in the development of this work. Most of them (but not all) can be found in Hochstrasser [13] and Kaya and Erdogan [19].

- Definition of Tchebyshev polynomials of the first kind:

$$T_n(s) = \cos[n \cos^{-1}(s)] \ , \qquad n = 0, 1, 2, \cdots \tag{26}$$

- Definition of Tchebyshev polynomials of the second kind:

$$U_n(s) = \frac{\sin[(n+1)\cos^{-1}(s)]}{\sin[\cos^{-1}(s)]} \ , \qquad n = 0, 1, 2, \ldots \tag{27}$$

- Iterative (recursive) properties:

$$sT_n(s) = \frac{1}{2}[T_{n+1}(s) + T_{n-1}(s)] \ , \qquad n \geq 1 \tag{28}$$

$$sU_n(s) = \frac{1}{2}[U_{n+1}(s) + U_{n-1}(s)] \ , \qquad n \geq 1 \tag{29}$$

$$T_n(s) = \frac{1}{2}[U_n(s) - U_{n-2}(s)] \ , \qquad n \geq 2 \tag{30}$$

$$U_n(s)(1-s^2) = sT_{n+1}(s) - T_{n+2}(s) \ , \quad n \geq 0 \tag{31}$$

By means of equation (28), one may rewrite equation (31) above as

$$U_n(s) = \frac{1}{2(1-s^2)}[T_n(s) - T_{n+2}(s)] \ , \quad n \geq 0 \tag{32}$$

Thus an additional equality, which is useful in handling cubic hypersingular integrals can be derived[2]:

$$\begin{aligned} U_n(s)(1-s^2)^{\frac{3}{2}} &\overset{(32)}{=} \frac{1}{2}\left[T_n(s) - T_{n+2}(s)\right]\sqrt{1-s^2} \\ &\overset{(30)}{=} -\frac{1}{2}\left[\frac{1}{2}U_{n+2}(s) - U_n(s) + \frac{1}{2}U_{n-2}(s)\right]\sqrt{1-s^2} \ , \quad n \geq 2 \\ &= -\frac{1}{4}\left[U_{n+2}(s) - 2U_n(s) + U_{n-2}(s)\right]\sqrt{1-s^2} \ , \quad n \geq 2 \end{aligned} \tag{33}$$

- Derivatives:

$$\frac{dT_n(s)}{ds} = nU_{n-1}(s) \ , \quad n \geq 1 \tag{34}$$

$$\frac{dU_n(s)}{ds} = \frac{1}{1-s^2}\left[\frac{n+2}{2}U_{n-1}(s) - \frac{n}{2}U_{n+1}(s)\right] \ , \quad n \geq 1 \tag{35}$$

4 Cauchy Singular Integral Formulas ($\alpha = 1$)

This section mainly evaluates $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$, which are defined in equations (5) and (6). The new result here is that the singular integral formulas are found for general m. In order to obtain this new result, two well known Cauchy singular integral formulas are introduced [13, 19]: one is already stated in equation (17), and the other one is

$$I_1(U_n, 1, r) = \fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}ds = -\pi T_{n+1}(r) \ , \quad n \geq 0 \ , \tag{36}$$

which can be obtained as follows

$$\begin{aligned} I_1(U_n, 1, r) &= \fint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}ds \\ &\overset{(32)}{=} \frac{1}{2}\fint_{-1}^{1} \frac{T_n(s) - T_{n+2}(s)}{\sqrt{1-s^2}(s-r)}ds \\ &\overset{(17)}{=} \frac{\pi}{2}[U_{n-1}(r) - U_{n+1}(r)] \\ &\overset{(30)}{=} -\pi T_{n+1}(r) \ . \end{aligned}$$

[2]The equation number is stacked above the equal sign to show how the equations are being derived and connected.

The integral formulas for $m = 0, 1, 2, 3, \cdots$ are derived below. The general formulas have the restriction of minimum n. The lower n terms can not be derived by general formulas, and are given in Appendix A.

4.1 $I_1(T_n, m, r),\ m = 0, 1, 2, 3$

- $I_1(T_n, 0, r)$:
  This is equation (17).

- $I_1(T_n, 1, r)$:

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r} ds &\overset{(30)}{=} \frac{1}{2} \int_{-1}^{1} \frac{[U_n(s) - U_{n-2}(s)]\sqrt{1-s^2}}{s-r} ds \\ &\overset{(36)}{=} \frac{\pi}{2}[T_{n-1}(r) - T_{n+1}(r)] , \quad n \geq 2 . \end{aligned} \tag{37}$$

- $I_1(T_n, 2, r)$:

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds &\overset{(30)}{=} \int_{-1}^{1} \frac{\frac{1}{2}[U_n(s) - U_{n-2}(s)](1-s^2)^{\frac{3}{2}}}{s-r} ds \\ &\overset{(40)}{=} \frac{\pi}{8} \{[T_{n-1}(r) - 2T_{n+1}(r) + T_{n+3}(r)] - [T_{n-3}(r) - 2T_{n-1}(r) + T_{n+1}(r)]\} \\ &= -\frac{\pi}{8}[T_{n-3}(r) - 3T_{n-1}(r) + 3T_{n+1}(r) - T_{n+3}(r)] , \quad n \geq 4 \end{aligned} \tag{38}$$

- $I_1(T_n, 3, r)$:

$$\begin{aligned} \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{5}{2}}}{s-r} ds &\overset{(30)}{=} \int_{-1}^{1} \frac{\frac{1}{2}[U_n(s) - U_{n-2}(s)](1-s^2)^{\frac{5}{2}}}{s-r} ds \\ &\overset{(41)}{=} \frac{\pi}{32} \{[T_{n-5}(r) - 4T_{n-3}(r) + 6T_{n-1}(r) - 4T_{n+1}(r) + T_{n+3}(r)] - \\ &\qquad [T_{n-3}(r) - 4T_{n-1}(r) + 6T_{n+1}(r) - 4T_{n+3}(r) + T_{n+5}(r)]\} , \quad n \geq 6 \\ &= \frac{\pi}{32}[T_{n-5}(r) - 5T_{n-3}(r) + 10T_{n-1}(r) \\ &\qquad -10T_{n+1}(r) + 5T_{n+3}(r) - T_{n+5}(r)] , \quad n \geq 6 \end{aligned} \tag{39}$$

4.2 $I_1(U_n, m, r),\ m = 1, 2, 3$

- $I_1(U_n, 1, r)$:
  This is equation (36).

- $I_1(U_n, 2, r)$:

$$\begin{aligned} \int_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds &\overset{(32)}{=} \frac{1}{2} \int_{-1}^{1} \frac{[T_n(s) - T_{n+2}(s)]\sqrt{1-s^2}}{s-r} ds \\ &\overset{(37)}{=} \frac{\pi}{4} \{[T_{n-1}(r) - T_{n+1}(r)] - [T_{n+1}(r) - T_{n+3}(r)]\} , \quad n \geq 2 \\ &= \frac{\pi}{4}[T_{n-1}(r) - 2T_{n+1}(r) + T_{n+3}(r)] , \quad n \geq 2 \end{aligned} \tag{40}$$

- $I_1(U_n, 3, r)$:

$$\begin{aligned}
\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{5}{2}}}{s-r} ds &\overset{(32)}{=} \frac{1}{2} \fint_{-1}^{1} \frac{[T_n(s) - T_{n+2}(s)](1-s^2)^{\frac{3}{2}}}{s-r} ds \\
&\overset{(38)}{=} \frac{\pi}{16} \{[T_{n-1}(r) - 3T_{n+1}(r) + 3T_{n+3}(r) - T_{n+5}(r)] - \\
&\qquad [T_{n-3}(r) - 3T_{n-1}(r) + 3T_{n+1}(r) - 3T_{n+3}(r)]\} \ , \quad n \geq 4 \\
&= -\frac{\pi}{16} [T_{n-3}(r) - 4T_{n-1}(r) + 6T_{n+1}(r) - 4T_{n+3}(r) + T_{n+5}(r)] \ , \quad n \geq 4
\end{aligned} \tag{41}$$

4.3 $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$

At this point one may easily see the procedural steps above, which take advantage of recursive properties (30) and (32) between the Tchebyshev polynomials $T_n(s)$ and $U_n(s)$. For instance, evaluation of $I_1(T_n, 4, r) = \fint_{-1}^{1} T_n(s)(1-s^2)^{7/2}/(s-r)ds$ can be reduced to evaluation of $I_1(U_n, 4, r) = \fint_{-1}^{1} U_n(s)(1-s^2)^{7/2}/(s-r)ds$, which, in turn, can be reduced to evaluation of $I_1(T_n, 3, r) = \fint_{-1}^{1} T_n(s)(1-s^2)^{5/2}/(s-r)ds$. After a suitable number of steps, this reduction leads to either (17) or (36). This procedure is summarized in Figure 1.

Thus, by induction, one obtains the following formulas for $m \geq 1$.

- $I_1(T_n, m, r)$, where $m \geq 1$, and $n \geq 2m$

$$\boxed{\fint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds = \pi(-1)^{m+1} \left(\frac{1}{2}\right)^{2m-1} \sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} T_{n+1-2m+2j}(r) \ .} \tag{42}$$

- $I_1(U_n, m, r)$, where $m \geq 2$, and $n \geq 2m-2$

$$\boxed{\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds = \pi(-1)^{m} \left(\frac{1}{2}\right)^{2m-2} \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} T_{n+3-2m+2j}(r) \ .} \tag{43}$$

The usual notation

$$\binom{m}{j} = \frac{(m)!}{j!\ (m-j)!}$$

denotes the binomial coefficients.

5 Hypersingular Integral Formulas ($\alpha \geq 2$)

Once a Cauchy singular integral formula has been reached, all other hypersingular integral formulas may be obtained successively by taking differentiation with respect to r, and making use of the finite-part integral formula (7).

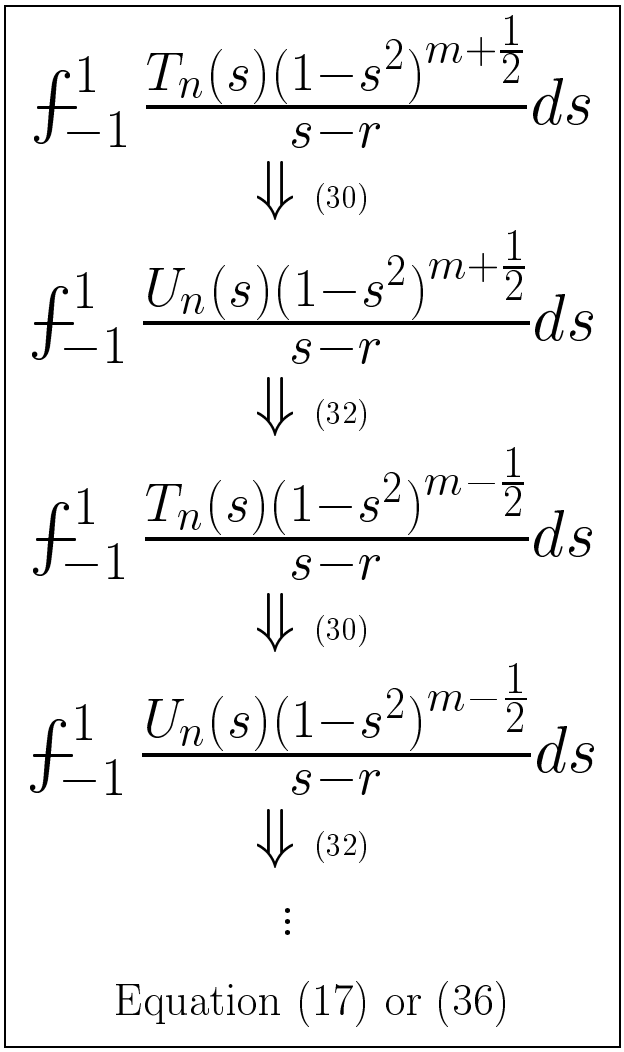

Figure 1: Evaluation of $I_1(T_n, m, r)$ and $I_1(U_n, m, r)$ for general m. The procedure reduces integrals with higher m to those with lower m.

5.1 $I_2(T_n, m, r)$

By means of

$$\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^2} ds = \frac{d}{dr} \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds \ ,$$

one readily obtains:

- $I_2(T_n, 0, r)$ [19]:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(s)}{\sqrt{1-s^2}(s-r)^2} ds &= \pi \frac{dU_{n-1}(r)}{dr} \\ &\overset{(35)}{=} \frac{\pi}{1-r^2} \left[\frac{n+1}{2} U_{n-2}(r) - \frac{n-1}{2} U_n(r) \right] \ , \quad n \geq 2 \end{aligned} \tag{44}$$

- $I_2(T_n, 1, r)$:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^2} ds &= \frac{\pi}{2}\frac{d}{dr}[T_{n-1}(r) - T_{n+1}(r)] \\ &\overset{(34)}{=} \frac{\pi}{2}\left[(n-1)U_{n-2}(r) - (n+1)U_n(r)\right] \ , \quad n \geq 2 \end{aligned} \tag{45}$$

- $I_2(T_n, 2, r)$:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^2} ds &= -\frac{\pi}{8}\frac{d}{dr}[T_{n-3}(r) - 3T_{n-1}(r) + 3T_{n+1}(r) - T_{n+3}(r)] \\ &\overset{(34)}{=} -\frac{\pi}{8}\left[\ (n-3)U_{n-4}(r) - 3(n-1)U_{n-2}(r) + 3(n+1)U_n(r) - \right. \\ &\qquad \left. (n+3)U_{n+2}(r)\ \right] \ , \quad n \geq 4 \end{aligned} \tag{46}$$

- $I_2(T_n, 3, r)$:

$$\begin{aligned} &\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds \\ &= \frac{\pi}{32}\frac{d}{dr}[T_{n-5}(r) - 5T_{n-3}(r) + 10T_{n-1}(r) - 10T_{n+1}(r) + 5T_{n+3}(r) - T_{n+5}(r)] \\ &\overset{(34)}{=} \frac{\pi}{32}\left[(n-5)U_{n-6}(r) - 5(n-3)U_{n-4}(r) + 10(n-1)U_{n-2}(r) - 10(n+1)U_n(r)\right. \\ &\qquad \left. +5(n+3)U_{n+2}(r) - (n+5)U_{n+4}(r)\ \right] \ , \quad n \geq 6 \end{aligned} \tag{47}$$

- $I_2(T_n, m, r)$, where $m \geq 1$, and $n \geq 2m+1$:

$$\boxed{\begin{array}{l} \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{(s-r)^2} ds = \\ \pi(-1)^{m+1}\left(\frac{1}{2}\right)^{2m-1} \sum_{j=0}^{2m-1}(-1)^j \binom{2m-1}{j} (n+1-2m+2j)U_{n-2m+2j}(r) \end{array}} \tag{48}$$

5.2 $I_2(U_n, m, r)$

The following equality

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^2} ds = \frac{d}{dr}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{s-r} ds$$

leads to:

- $I_2(U_n, 1, r)$ [19]:

$$\oint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^2} ds = -\pi\frac{dT_{n+1}(r)}{dr} \overset{(34)}{=} -\pi(n+1)U_n(r) \ , \quad n \geq 0 \ , \tag{49}$$

  which is the same as (21).

- $I_2(U_n, 2, r)$:

$$\begin{aligned} \oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^2} ds &= \frac{\pi}{4}\frac{d}{dr}[T_{n-1}(r) - 2T_{n+1}(r) + T_{n+3}(r)] \\ &\overset{(34)}{=} \frac{\pi}{4}\left[\ (n-1)U_{n-2}(r) - 2(n+1)U_n(r) + (n+3)U_{n+2}(r)\right]\ ,\ n \geq 2 \end{aligned} \tag{50}$$

- $I_2(U_n, 3, r)$:

$$\begin{aligned} &\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds \\ &= -\frac{\pi}{16}\frac{d}{dr}[T_{n-3}(r) - 4T_{n-1}(r) + 6T_{n+1}(r) - 4T_{n+3}(r) + T_{n+5}(r)] \\ &\overset{(34)}{=} -\frac{\pi}{16}[(n-3)U_{n-4}(r) - 4(n-1)U_{n-2}(r) + 6(n+1)U_n(r) - 4(n+3)U_{n+2}(r) \\ &\qquad +(n+5)U_{n+4}(r)\ ]\ ,\quad n \geq 4 \end{aligned} \tag{51}$$

- $I_2(U_n, m, r)$, where $m \geq 2$, and $n \geq 2m-1$:

$$\boxed{\begin{array}{l} \oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^2} ds = \\ \pi(-1)^m \left(\frac{1}{2}\right)^{2m-2} \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} (n+3-2m+2j) U_{n+2-2m+2j}(r) \end{array}} \tag{52}$$

5.3 $I_3(T_n, m, r)$

By means of

$$\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3} ds = \frac{1}{2}\frac{d}{dr} \oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^2} ds\ ,$$

one obtains:

- $I_3(T_n, 0, r)$:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^3} ds &= \frac{\pi}{8(1-r^2)^2}\Big[(n+1)(n+2)U_{n-3}(r) \\ &\quad -2(n^2-3)U_{n-1}(r) + (n-1)^2 U_{n+1}(r)\Big]\ ,\quad n \geq 3 \end{aligned} \tag{53}$$

- $I_3(T_n, 1, r)$:

$$\begin{aligned} \oint_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{(s-r)^3} ds &= \frac{\pi}{8(1-r^2)}\Big[(n^2-n)U_{n-3}(r) \\ &- \ (2n^2+2)U_{n-1}(r) + (n^2+n)U_{n+1}(r)\Big]\ ,\quad n \geq 3 \end{aligned} \tag{54}$$

- $I_3(T_n,2,r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds &= \frac{\pi}{32(1-r^2)}\Big\{-(n+3)(n+2)U_{n+3}(r)\\ &+ [(n+3)(n+4)+3n(n+1)]U_{n+1}(r)-[3(n+1)(n+2)+3(n-1)(n-2)]U_{n-1}(r)\\ &+ [3n(n-1)+(n-3)(n-4)]U_{n-3}(r)-(n-3)(n-2)U_{n-5}(r)\Big\}\ ,\quad n\geq 5\end{aligned} \tag{55}$$

- $I_3(T_n,3,r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds &= \frac{\pi}{128(1-r^2)}\Big[(n^2+9n+20)U_{n+5}(r)\\ &- 6(n^2+6n+10)U_{n+3}(r)+15(n^2+3n+4)U_{n+1}(r)\\ &- 20(n^2+2)U_{n-1}(r)+15(n^2-3n+4)U_{n-3}(r)-6(n^2-6n+10)U_{n-5}(r)\\ &+ (n^2-9n+2)U_{n-7}(r)\Big]\ ,\quad n\geq 7\end{aligned} \tag{56}$$

- $I_3(T_n,m,r)$, where $m\geq 1$, and $n\geq 2m+2$:

$$\boxed{\begin{array}{l}\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3}ds =\\ (-1)^{m+1}\left(\frac{1}{2}\right)^{2m+1}\frac{\pi}{1-r^2}\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}(n+1-2m+2j)\times\\ [(n+2-2m+2j)U_{n-1-2m+2j}(r)-(n-2m+2j)U_{n+1-2m+2j}(r)]\end{array}} \tag{57}$$

5.4 $I_3(U_n,m,r)$

By means of

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3}ds = \frac{1}{2}\frac{d}{dr}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^2}ds\ ,$$

one gets:

- $I_3(U_n,1,r)$:

$$\begin{aligned}&\oint_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{(s-r)^3}ds\\ &= \frac{\pi}{4(1-r^2)}\left\{-(2n^2+3n+2)U_{n-1}(r)+(n^2+n)U_{n+1}(r)\right\}\ ,\ n\geq 1\end{aligned} \tag{58}$$

- $I_3(U_n,2,r)$:

$$\begin{aligned}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds &= \frac{\pi}{16(1-r^2)}\Big[-(n^2+5n+6)U_{n+3}(r)\\ &+ (3n^2+9n+12)U_{n+1}(r)-(3n^2+3n+6)U_{n-1}(r)\\ &+ (n^2-n)U_{n-3}(r)\Big]\ ,\quad n\geq 3\end{aligned} \tag{59}$$

- $I_3(U_n,3,r)$:

$$\begin{aligned}\oint_{-1}^{1}\frac{U_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds &= \frac{\pi}{64(1-r^2)}\Big[(n^2+9n+20)U_{n+5}(r)\\ &- (5n^2+31n+54)U_{n+3}(r)+(10n^2+34n+48)U_{n+1}(r)-(10n^2+6n+20)U_{n-1}(r)\\ &+ (5n^2-11n+12)U_{n-3}(r)-(n^2-5n+6)U_{n-5}(r)\Big]\ , \quad n\geq 5\end{aligned} \tag{60}$$

- $I_3(U_n,m,r)$, where $m\geq 2$, and $n\geq 2m$:

$$\boxed{\begin{array}{l}\oint_{-1}^{1}\frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3}ds = \\ (-1)^m\left(\frac{1}{2}\right)^{2m}\frac{\pi}{1-r^2}\sum_{j=0}^{2m-2}(-1)^j\binom{2m-2}{j}(n+3-2m+2j)\times \\ [(n+4-2m+2j)U_{n+1-2m+2j}(r)-(n+2-2m+2j)U_{n+3-2m+2j}(r)]\end{array}} \tag{61}$$

5.5 $I_4(T_n,m,r)$

By means of

$$\oint_{-1}^{1}\frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^4}ds = \frac{1}{3}\frac{d}{dr}\oint_{-1}^{1}\frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3}ds\ ,$$

one reaches the following results:

- $I_4(T_n,0,r)$:

$$\begin{aligned}\oint_{-1}^{1}\frac{T_n(s)}{(s-r)^4\sqrt{1-s^2}}ds &= \frac{\pi}{48(1-r^2)^3}\Big[(n^3+6n^2+11n+6)U_{n-4}(r)\\ &-(3n^3+6n^2-25n-44)U_{n-2}(r)+(3n^3-5n^2-19n+37)U_n(r)\\ &-(n^3-5n^2+7n-3)U_{n+2}(r)\Big]\ , \qquad n\geq 4\end{aligned} \tag{62}$$

- $I_4(T_n,1,r)$:

$$\begin{aligned}\oint_{-1}^{1}\frac{T_n(s)\sqrt{1-s^2}}{(s-r)^4}ds &= \frac{\pi}{48(1-r^2)^2}\Big[\\ &(n^3-n)U_{n-4}(r)-(3n^3+9n+12)U_{n-2}(r)+(3n^3+9n-12)U_n(r)\\ &-(n^3-n)U_{n+2}(r)\Big]\ , \qquad n\geq 4\end{aligned} \tag{63}$$

- $I_4(T_n,2,r)$:

$$\begin{aligned}\oint_{-1}^{1}\frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^4}ds &= \frac{\pi}{192(1-r^2)^2}\Big[(n^3+6n^2+11n+6)U_{n+4}(r)\\ &- (5n^3+18n^2+43n+30)U_{n+2}(r)+(10n^3+12n^2+134n-36)U_n(r)\\ &- (10n^3-12n^2+134n+36)U_{n-2}(r)+(5n^3-18n^2+43n-30)U_{n-4}(r)\\ &- (n^3-6n^2+11n-6)U_{n-6}(r)\Big], \qquad n\geq 6\end{aligned} \tag{64}$$

- $I_4(T_n,3,r)$:

$$\begin{aligned}
\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds &= \frac{\pi}{384(1-r^2)^2}\Big[-(\frac{1}{2}n^3+6n^2+\frac{47}{2}n+30)U_{n+6}(r)\\
&+ (\frac{7}{2}n^3+30n^2+\frac{197}{2}n+120)U_{n+4}(r)-(\frac{21}{2}n^3+54n^2+\frac{327}{2}n+180)U_{n+2}(r)\\
&+ (\frac{35}{2}n^3+30n^2+\frac{325}{2}n+90)U_{n}(r)-(\frac{35}{2}n^3-30n^2+\frac{325}{2}n-90)U_{n-2}(r)\\
&+ (\frac{21}{2}n^3-54n^2+\frac{327}{2}n-180)U_{n-4}(r)-(\frac{7}{2}n^3-30n^2+\frac{197}{2}n-120)U_{n-6}(r)\\
&+ (\frac{1}{2}n^3-6n^2+\frac{47}{2}n-30)U_{n-8}(r)\Big], \qquad n\geq 8
\end{aligned} \tag{65}$$

- $I_4(T_n,m,r)$, where $m\geq 1$, and $n\geq 2m+3$:

$$\boxed{\begin{aligned}
&\oint_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^4}ds = \\
&(-1)^{m+1}\left(\tfrac{1}{2}\right)^{2m+2}\tfrac{1}{3}\tfrac{\pi}{(1-r^2)^2}\textstyle\sum_{j=0}^{2m-1}(-1)^j\binom{2m-1}{j}(n+1-2m+2j)\times\\
&\Big\{[(n+2-2m+2j)(n+3-2m+2j)]U_{n-2-2m+2j}(r)-\\
&\quad[2(n-2m+2j)^2+4(n-2m+2j)-6]U_{n-2m+2j}(r)+\\
&\quad[(n-2m+2j)(n-1-2m+2j)]U_{n+2-2m+2j}(r)\Big\}
\end{aligned}} \tag{66}$$

5.6 $I_4(U_n,m,r)$:

By means of

$$\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^4}ds = \frac{1}{3}\frac{d}{dr}\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^3}ds\ ,$$

one obtains

- $I_4(U_n,1,r)$:

$$\begin{aligned}
\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{1}{2}}}{(s-r)^4}ds &= \frac{\pi}{24(1-r^2)^2}\Big[\\
&-(2n^3+9n^2+11n+6)U_{n-2}(r)+(3n^3+3n^2-2n-6)U_n(r)\\
&-(n^3-n)U_{n+2}(r)\Big]\ , \qquad n\geq 2
\end{aligned} \tag{67}$$

- $I_4(U_n,2,r)$:

$$\begin{aligned}
\oint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^4}ds &= \frac{\pi}{96(1-r^2)^2}\Big[(n^3+6n^2+11n+6)U_{n+4}(r)\\
&-(4n^3+18n^2+44n+30)U_{n+2}(r)+(6n^3+18n^2+54n+42)U_n(r)\\
&-(4n^3+6n^2+20n+18)U_{n-2}(r)+(n^3-n)U_{n-4}(r)\Big]\ , \qquad n\geq 4
\end{aligned} \tag{68}$$

- $I_4(U_n, 3, r)$:

$$\begin{aligned}
\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = \frac{\pi}{192(1-r^2)^2}\Big[& -(\frac{1}{2}n^3+6n^2+\frac{47}{2}n+320)U_{n+6}(r) \\
+ \; & (3n^3+27n^2+93n+117)U_{n+4}(r) - (\frac{15}{2}n^3+45n^2+\frac{285}{2}n+165)U_{n+2}(r) \\
+ \; & (10n^3+30n^2+110n+90)U_n(r) - (\frac{15}{2}n^3+\frac{105}{2}n)U_{n-2}(r) \\
+ \; & (3n^3-9n^2+21n-15)U_{n-4}(r) \\
- \; & (\frac{1}{2}n^3+3n^2+\frac{11}{2}n-3)U_{n-6}(r)\Big], \qquad n \geq 6
\end{aligned} \tag{69}$$

- $I_4(U_n, m, r)$, where $m \geq 2$, and $n \geq 2m+1$:

$$\boxed{\begin{array}{l}
\fint_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-\frac{1}{2}}}{(s-r)^4} ds = \\
(-1)^m \left(\frac{1}{2}\right)^{2m+1} \frac{1}{3}\frac{\pi}{(1-r^2)^2} \sum_{j=0}^{2m-2} (-1)^j \binom{2m-2}{j} (n+3-2m+2j) \times \\
\Big\{ [(n+4-2m+2j)(n+5-2m+2j)] U_{n-2m+2j}(r) - \\
\quad [2(n-2m+2j)^2 + 10(n-2m+2j)+10] U_{n+2-2m+2j}(r) + \\
\quad [(n+2-2m+2j)(n-1-2m+2j)] U_{n+4-2m+2j}(r) \Big\}
\end{array}} \tag{70}$$

6 Evaluation of Stress Intensity Factors (SIFs)

An important task is to evaluate the stress intensity factors (SIFs) at both crack tips, since the propagation of a crack starts around its tips. In mode III crack problems, SIFs can be calculated from

$$K_{III}(d) = \lim_{x \to d^+} \sqrt{2\pi(x-d)}\sigma_{yz}(x,0) \quad , \quad (x > d) \tag{71}$$

and

$$K_{III}(c) = \lim_{x \to c^-} \sqrt{2\pi(c-x)}\sigma_{yz}(x,0) \quad , \quad (x < c) \; . \tag{72}$$

Note that the limit is taken from outside of the crack surfaces and towards both tips. Usually the left hand side of integral equation (4) is the expression for $\sigma_{yz}(x,0)$ which is valid for x is inside the crack surfaces (c,d) as well as outside of (c,d). Thus to calculate SIFs, the key is to evaluate the following integrals which are obtained after proper normalization and the change of variables described in equation (8),

$$S_\alpha(T_n, m, r) = \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-(1/2)}}{(s-r)^\alpha} ds \; , \quad r \notin (-1,1) \tag{73}$$

and

$$S_\alpha(U_n, m, r) = \int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-(1/2)}}{(s-r)^\alpha} ds \; , \quad r \notin (-1,1) \; . \tag{74}$$

Note that the above integrals are not singular as $x \neq t$ for $t \in (c,d)$ and $x \notin (c,d)$.

The strategy to evaluate $S_\alpha(T_n,m,r)$ and $S_\alpha(U_n,m,r)$ for general integers α (positive) and m (non-negative) is similar to the process for evaluating $I_\alpha(T_n,m,r)$ and $I_\alpha(U_n,m,r)$. It consists of evaluating the integrals $S_1(T_n,m,r)$ and $S_1(U_n,m,r)$ by means of the reduction procedure described in Section 4.3, and taking differentiation (with respect to r) to obtain $S_\alpha(T_n,m,r)$ and $S_\alpha(U_n,m,r)$ for $\alpha \geq 2$. The relevant derivations are provided below where the range of r is restricted to $|r| > 1$. These formulas are used in calculating SIFs for the examples presented in Section 7

6.1 $S_1(T_n,m,r)$ and $S_1(U_n,m,r)$

- $S_1(T_n,0,r)$ (This is a well known integral [8]):

$$\int_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}} ds = -\pi \frac{\left(r - \sqrt{r^2-1}|r|/r\right)^n}{\sqrt{r^2-1}|r|/r} \ , \quad n \geq 0 \ . \tag{75}$$

- $S_1(T_n,1,r)$:

$$\begin{aligned} &\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r}\, ds \\ &\overset{(30)}{=} \frac{1}{2}\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r} ds - \frac{1}{2}\int_{-1}^{1} \frac{U_{n-2}(s)\sqrt{1-s^2}}{s-r}\, ds \\ &\overset{(80)}{=} \pi \frac{|r|}{r}\sqrt{r^2-1}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^n , \quad n \geq 2 \ . \end{aligned} \tag{76}$$

- $S_1(T_0,2,r)$:

$$\int_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = \pi(r^2-1)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right) \ . \tag{77}$$

- $S_1(T_1,2,r)$:

$$\int_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{3}{2}}}{s-r}\, ds = \frac{\pi}{2}(r^2-1)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^2 \ . \tag{78}$$

- $S_1(T_n,2,r)$:

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds \overset{(30),(81)}{=} -\frac{\pi|r|}{r}(r^2-1)^{\frac{3}{2}}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^n \ , \ n \geq 2 \ . \tag{79}$$

- $S_1(U_n, 1, r)$:

$$\begin{aligned}
&\int_{-1}^{1} \frac{U_n(s)\sqrt{1-s^2}}{s-r}\, ds \\
&\overset{(32)}{=} \frac{1}{2}\int_{-1}^{1} \frac{T_n(s)}{(s-r)\sqrt{1-s^2}} ds - \frac{1}{2}\int_{-1}^{1} \frac{T_{n+2}(s)}{(s-r)\sqrt{1-s^2}}\, ds \\
&\overset{(75)}{=} -\pi\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geq 0\ . \qquad (80)
\end{aligned}$$

- $S_1(U_n, 2, r)$:

$$\begin{aligned}
&\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{s-r}\, ds \\
&\overset{(32)}{=} \frac{1}{2}\int_{-1}^{1} \frac{T_n(s)\sqrt{1-s^2}}{s-r} ds - \frac{1}{2}\int_{-1}^{1} \frac{T_{n+2}(s)\sqrt{1-s^2}}{s-r}\, ds \\
&\overset{(76)}{=} \pi(r^2-1)\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \quad n \geq 2\ . \qquad (81)
\end{aligned}$$

The formulas for $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$ with general m can be deduced by the same procedure described in Figure 1, and are listed below.

$$\begin{aligned}
S_1(T_n, m, r) &= \int_{-1}^{1} \frac{T_n(s)(1-s^2)^{m-1/2}}{s-r} ds \\
&= \pi(-1)^{m+1}\frac{|r|}{r}(r^2-1)^{m-1/2}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \ m \geq 0 \text{ and } \ n \geq 2m\ . \qquad (82)
\end{aligned}$$

$$\begin{aligned}
S_1(U_n, m, r) &= \int_{-1}^{1} \frac{U_n(s)(1-s^2)^{m-1/2}}{s-r} ds \\
&= \pi(-1)^{m}(r^2-1)^{m-1}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \ m \geq 1 \text{ and } \ n \geq 2m-2\ . \qquad (83)
\end{aligned}$$

6.2 $S_2(T_n, m, r)$ and $S_2(U_n, m, r)$

Differentiating (with respect to r) the formulas for $S_1(T_n, m, r)$ and $S_1(U_n, m, r)$, we obtain the formulas for $S_2(T_n, m, r)$ and $S_2(U_n, m, r)$.

$$\begin{aligned}
&\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^2}\, ds \\
&= -\pi(n+1)\frac{|r|}{r}\sqrt{r^2-1}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1} + 2\pi r\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \ n \geq 0 \qquad (84)
\end{aligned}$$

and

$$\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = -\frac{\pi|r|}{r}(r^2-1)^{\frac{3}{2}}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n}, \ n \geq 2\ . \qquad (85)$$

6.3 $S_3(T_n, m, r)$ and $S_3(U_n, m, r)$

The following formulas are obtained by differentiating twice (with respect to r) the corresponding formulas obtained in Subsection 6.1.

$$\int_{-1}^{1} \frac{U_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3} ds = \frac{\pi}{2}\left[(n^2+2n+3) - 3(n+1)\frac{|r|}{\sqrt{r^2-1}}\right]\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}, \; n \geq 0 \; . \tag{86}$$

$$\int_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3} ds = \frac{3\pi}{2}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^2\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right) \; . \tag{87}$$

$$\int_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3} ds = \frac{3\pi}{2}\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)\left(1 - \frac{|r|}{\sqrt{r^2-1}}\right) \; . \tag{88}$$

$$\begin{aligned}\int_{-1}^{1} \frac{T_n(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}\, ds &= \frac{\pi}{4}\left\{\left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n+1}\left[(n^2+2n+3) - 3(n+1)\frac{|r|}{\sqrt{r^2-1}}\right]\right. \\ &\left. - \left(r - \frac{|r|}{r}\sqrt{r^2-1}\right)^{n-1}\left[(n^2-2n+3) - 3(n-1)\frac{|r|}{\sqrt{r^2-1}}\right]\right\}, \; n \geq 2 \; .\end{aligned} \tag{89}$$

7 Examples

Three examples are presented here, which emphasize various aspects of Fredholm singular integral equation formulations and their linkage to fracture mechanics. These examples are:

1. Internal mode I crack in an infinite strip.
2. Mode III crack problem in nonhomogeneous materials.
3. Gradient elasticity theory applied to a mode III crack.

The first and last examples consider homogeneous materials, and the second example considers nonhomogeneous materials, which has relevant applications to the field of functionally graded materials [2, 32]. The first two examples are from classical elasticity, and the last one is from gradient elasticity theory. The first example involves mode I cracks and the last two examples involve mode III cracks. All the examples are formulated by using hypersingular integral equations. For the first two examples the order of singularity α is 2, and for the last example α is 3. A detailed comparison between U_n and T_n representations is given in the first example. A discussion on the influence of the density function on the order of singularity of the integral equation is presented in the second and third examples. The description of the examples is summarized in Table 1

Table 1: Description of the examples.

Description	**Example 1**	**Example 2**	**Example 3**
Homogeneous material	$\surd$		$\surd$
Nonomogeneous material		$\surd$	
Classical elasticity	$\surd$	$\surd$	
Gradient elasticity			$\surd$
Crack mode	I	III	III
Density function	displacement (v)	displacement (w)	slope (ϕ)
Degree of singularity α	2	2	3
Weight function exponent, $m-(1/2)$	1/2	1/2	3/2
Representation	U_n, T_n	U_n, T_n	T_n

7.1 Internal Mode I Crack in an Infinite Strip [19]

Consider a crack in an infinite strip of homogeneous material, as illustrated by Figure 2. The governing partial differential equations (PDEs) and boundary conditions are:

$$\begin{aligned}
&\nabla^2 u(x,y) + \tfrac{2}{\kappa-1}\left(\tfrac{\partial^2 u(x,y)}{\partial x^2} + \tfrac{\partial^2 u(x,y)}{\partial x \partial y}\right) = 0 \ , && -\infty < x,\ y < \infty \ , \\
&\nabla^2 v(x,y) + \tfrac{2}{\kappa-1}\left(\tfrac{\partial^2 v(x,y)}{\partial y^2} + \tfrac{\partial^2 v(x,y)}{\partial x \partial y}\right) = 0 \ , && -\infty < x,\ y < \infty \ , \\
&\sigma_{xx}(0,y) = \sigma_{xy}(0,y) = \sigma_{xx}(h,y) = \sigma_{xy}(h,y) = 0 \ , && -\infty < y < \infty \ , \\
&\sigma_{xy}(x,0) = 0 \ , && 0 < x < h \ , \\
&\sigma_{yy}(x,0) = -p(x) \ , && x \in (c,d) \ , \\
&v(x,0) = 0 \ , && x \notin [c,d] \ ,
\end{aligned} \tag{90}$$

where u and v are the x and y components of the displacement vector; σ_{ij} is the stress tensor; κ is an elastic constant ($\kappa = 3-4\nu$ for plane strain, $\kappa = (3-\nu)/(1+\nu)$ for plane stress, and ν is the Poisson's ratio.) This problem has been studied by Kaya and Erdogan [19] by means of a U_n representation, and it has also been used as a benchmark problem by Kabir *et. al.* [17]. Here both U_n and T_n are employed and compared.

The governing integral equation can be written in the form given by equation (4) as [19]

$$\text{⨍}_c^d \ \frac{\triangle v(t)}{(t-x)^2} dt + \int_c^d k(x,t) \triangle v(t) dt = -\pi \left(\frac{1+\kappa}{2\mu}\right) p(x) \ , \quad c < x < d \ , \tag{91}$$

where the primary variable is the crack opening displacement $\triangle v$ given by

$$\triangle v(x) = v(x,0^+) - v(x,0^-) \ , \quad c < x < d \ ,$$

and the kernel $k(x,t)$ is given in Kaya and Erdogan [19], equations (51) – (54c), page 112. It is worth noting that as $h \to \infty$ (see Figure 2), the integral equation for the half plane is recovered and the kernel $k(x,t)$ is reduced to a much simpler form[3]

$$k(x,t) = \frac{-1}{(t+x)^2} + \frac{12x}{(t+x)^3} - \frac{12x^2}{(t+x)^4} \ .$$

[3]To be consistent with the notation adopted in this paper, we have used symbols different from reference [19]. For instance, in [19] upper case $K(t,x)$ is used for $k(t,x)$.

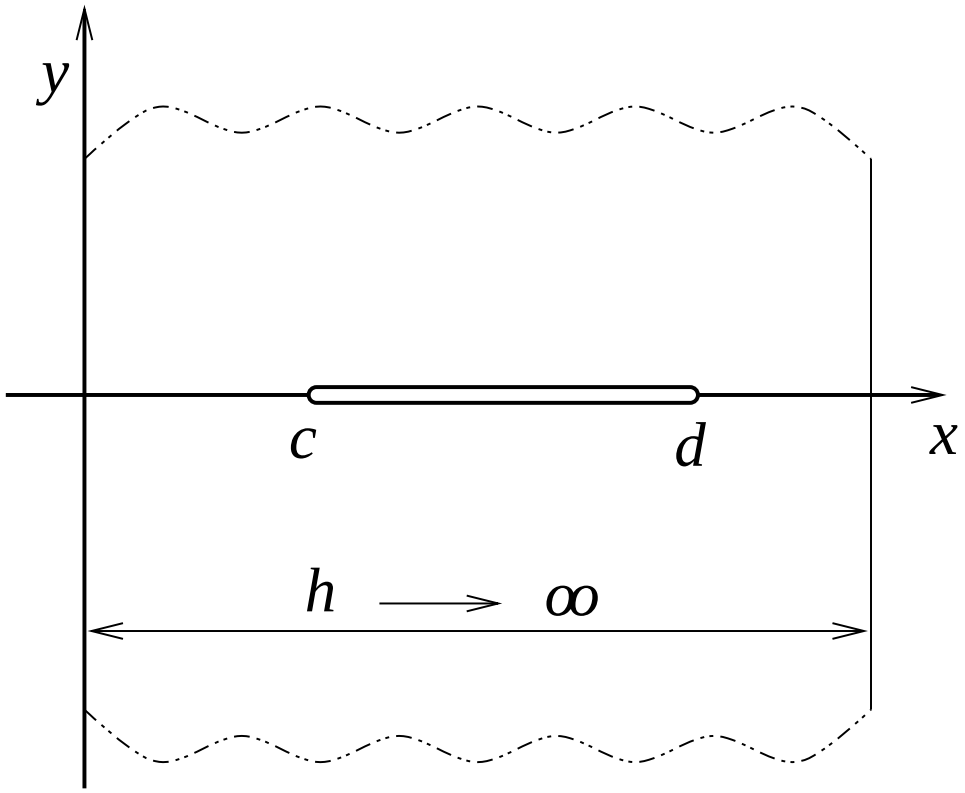

Figure 2: A mode I crack in an infinite strip.

After normalization, the corresponding integral equation can be written in a fashion similar to equation (9), *i.e.*[4]

$$\fint_{-1}^{1} \frac{D(s)}{(s-r)^2} ds + \int_{-1}^{1} \mathcal{K}(r,s) D(s) ds = P(r) \ , \quad -1 < r < 1 \ , \tag{92}$$

where $D(s)$ is the unknown displacement function, the regular kernel is

$$\mathcal{K}(r,s) = \frac{-1}{\left[(r+s) + 2\left(\frac{d+c}{d-c}\right)\right]^2} + \frac{12\left[s + \left(\frac{d+c}{d-c}\right)\right]}{\left[(r+s) + 2\left(\frac{d+c}{d-c}\right)\right]^3} - \frac{12\left[s + \left(\frac{d+c}{d-c}\right)\right]^2}{\left[(r+s) + 2\left(\frac{d+c}{d-c}\right)\right]^4} \ ,$$

and the loading function is

$$P(r) = -\pi \left(\frac{1+\kappa}{2\mu}\right) p\left(\left(\frac{d-c}{2}\right) s + \frac{d+c}{2}\right) \ .$$

The case $c > 0$ represents an internal crack, which is the case of interest in this work. Based on the dominant behavior of the singular kernels of the integral equation (92), the solution takes the form

$$D(s) = R(s)\sqrt{1-s^2} \ .$$

[4]Again, the notation is different from [19].

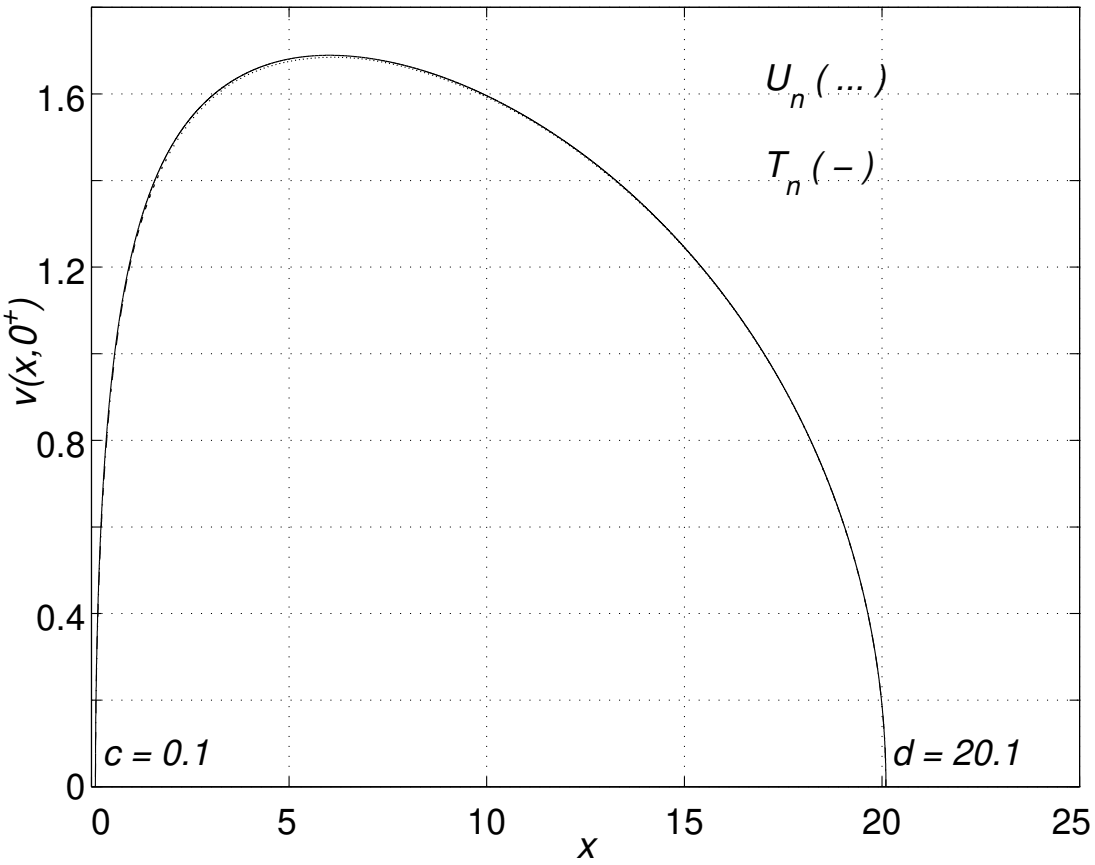

Figure 3: Displacement profiles for a mode I crack in an infinite strip obtained by means of U_n and T_n representations ($N+1 = 15$). Here $c = 0.1$, $d = 20.1$, $2a = 20$, and $(c+d)/(d-c) = 1.01$. The crack is tilted to the left because of the "edge effect".

Here the representation function $R(s)$ is approximated in terms of Tchebyshev polynomials of 1st and 2nd kinds, *i.e.*

$$R(s) = \sum_{n=0}^{N} a_n U_n(s) \quad \text{and} \quad R(s) = \sum_{n=0}^{N} b_n T_n(s) \ .$$

The unknown coefficients a_n and b_n are determined by selecting an appropriate set of collocation points

$$r_j = \cos\left(\frac{(2n-1)\pi}{2(N+1)}\right) \ , \ j = 1, 2, \cdots, N+1; \quad \text{for } U_n \text{ representation} \ .$$

$$r_j = \cos\left(\frac{n\pi}{N+2}\right) \ , \ j = 1, 2, \cdots, N+1; \quad \text{for } T_n \text{ representation} \ .$$

Once the solution is obtained, the SIFs can be calculated from[5]

$$\begin{aligned} K_I(c) &= \lim_{x\to c^-} \sqrt{2\pi(c-x)}\sigma_{yy}(x,0) \ , \quad (x<c) \\ &= \left(\frac{2\mu}{1+\kappa}\right) \lim_{x\to c^+} \frac{D(x)}{\sqrt{2\pi(x-c)}} \ , \quad (x>c) \\ &= \left(\frac{2\mu}{1+\kappa}\right) \sqrt{\frac{d-c}{2\pi}} R(-1) \end{aligned} \tag{93}$$

and

$$\begin{aligned} K_I(d) &= \lim_{x\to d^+} \sqrt{2\pi(x-d)}\sigma_{yy}(x,0) \ , \quad (x>d) \\ &= \left(\frac{2\mu}{1+\kappa}\right) \lim_{x\to d^-} \frac{D(x)}{\sqrt{2\pi(d-x)}} \ , \quad (x<d) \\ &= \left(\frac{2\mu}{1+\kappa}\right) \sqrt{\frac{d-c}{2\pi}} R(+1) \end{aligned} \tag{94}$$

which are obtained from equation (91) by observing that its left-hand-side gives the stress component $\sigma_{yy}(x,0)$ outside the crack interval (c,d).

Table 2: Normalized stress intensity factors (SIFs) for an internal crack in a half-plane. $N+1$ terms are used in approximating the primary variable.

$\frac{d+c}{d-c}$	$N+1$	U_n Representation		T_n Representation		Kaya and Erdogan [19]	
		$\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_I(c)}{p_0\sqrt{\pi(d-c)/2}}$	$\frac{K_I(d)}{p_0\sqrt{\pi(d-c)/2}}$
1.01	15	3.6437	1.3292	3.8037	1.3313	3.6387	1.3298
1.05	10	2.1541	1.2535	2.1920	1.2543	2.1547	1.2536
1.1	10	1.7583	1.2108	1.7655	1.2111	1.7587	1.2108
1.2	6	1.4637	1.1625	1.4728	1.1632	1.4637	1.1626
1.3	6	1.3316	1.1331	1.3346	1.1335	1.3316	1.1331
1.4	6	1.2544	1.1123	1.2556	1.1125	1.2544	1.1123
1.5	4	1.2036	1.0966	1.2066	1.0969	1.2035	1.0967
2.0	4	1.0913	1.0539	1.0916	1.0540	1.0913	1.0539
3.0	4	1.0345	1.0246	1.0346	1.0246	1.0345	1.0246
4.0	4	1.0182	1.0141	1.0182	1.0141	1.0182	1.0141
5.0	4	1.0112	1.0092	1.0112	1.0092	1.0112	1.0092
10.0	4	1.0026	1.0024	1.0026	1.0024	1.0026	1.0024
20.0	4	1.0006	1.0006	1.0006	1.0006	1.0006	1.0006

Table 2 presents the SIFs at both tips of an internal crack in a half-plane ($h\to\infty$) under uniform load ($p(x)=p_0$) obtained with both U_n and T_n representations. First, it is worth noting that the present SIF results for the U_n representation compare well with those reported in Table

[5]Kaya and Erdogan [19] do not consider the factor $\sqrt{\pi}$ in the definition of SIFs, equations (93) and (94). Note that this does not affect the normalized SIFs (*e.g.* see Table 2).

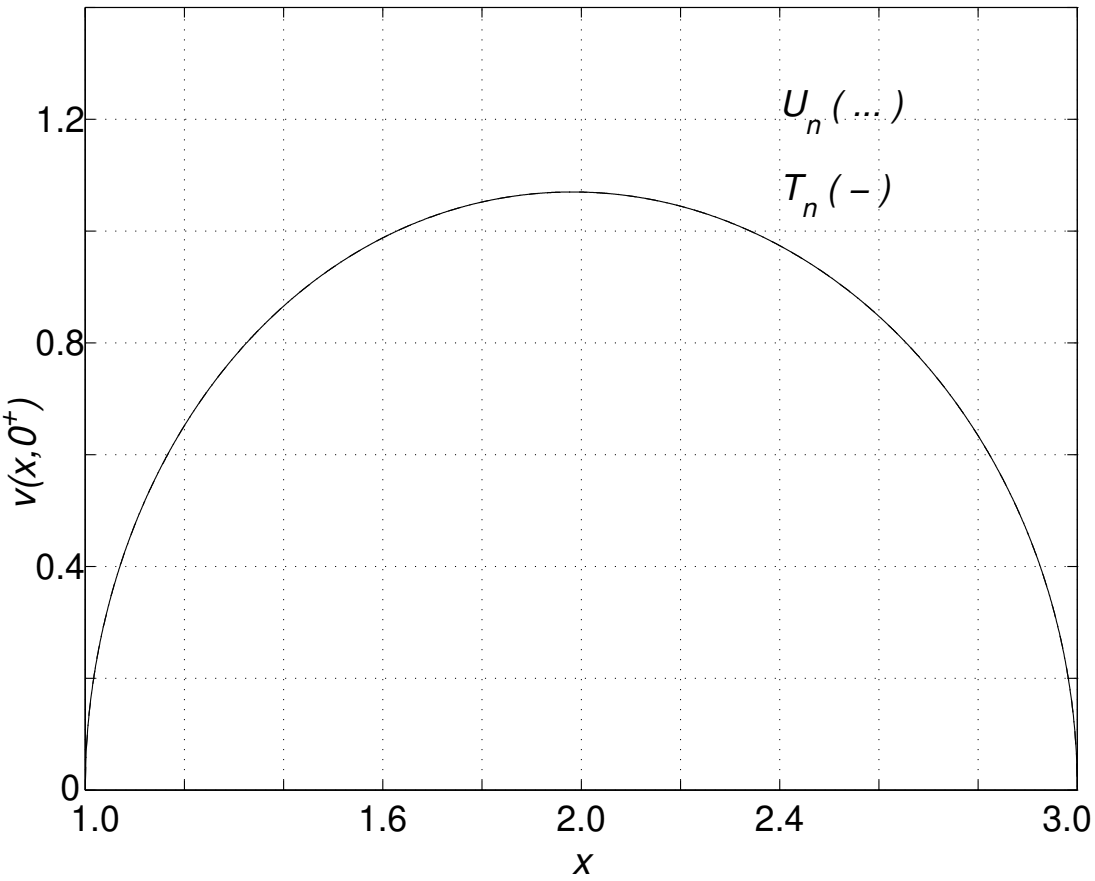

Figure 4: Displacement profiles for a mode I crack in an infinite strip obtained by means of U_n and T_n representations ($N+1=8$). Here $c=1$, $d=3$, $2a=2$, and $(c+d)/(d-c)=2$.

1 (page 114) of the paper by Kaya and Erdogan [19] for the entire range of values describing the relative position of the crack, *i.e.* $1.01 < (d+c)/(d-c) < 20$. Next, comparing the SIFs obtained with the U_n and T_n representations in Table 2, we note that the results compare quite well, except when $(d+c)/(d-c) \approx 1.0$, and the discrepancy is bigger at the left-hand-side (LHS) than at the right-hand-side (RHS) crack tip. This occurs because of the "edge effect" [33]. If 42 terms (*i.e.* $N+1=42$) and T_n representation are considered for the case $(d+c)/(d-c)=1.01$, then the normalized SIFs at the LHS and RHS crack tips are 3.6437 and 1.3302, respectively. Thus, when there is an "edge effect", the results are sensitive to the discretization adopted. Moreover, for the same number of collocation points, the level of accuracy attained with the U_n representation is slightly different from that with the T_n representation.

Figures 3 and 4 compare the crack profiles for U_n and T_n representations. One may observe that the displacement profiles obtained from both representations practically agree within plotting accuracy, especially in Figure 4. Note that the displacement profile in Figure 3 is tilted to the left because of the "edge effect". Such effect is negligible in Figure 4.

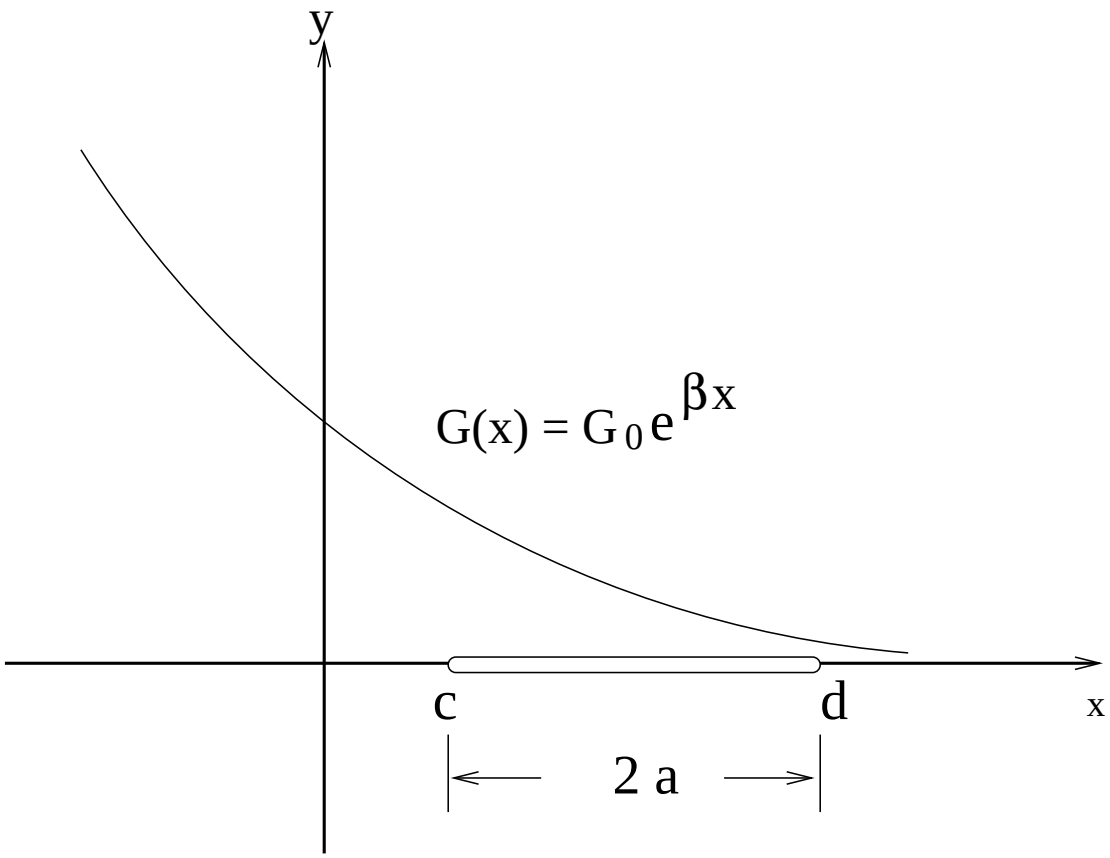

Figure 5: The half plane of the antiplane shear problem for nonhomogeneous material with shear modulus $G(x) = G_0 \mathrm{e}^{\beta x}$.

7.2 Mode III Crack Problem in Nonhomogeneous Materials [5]

Consider the antiplane shear problem for the nonhomogeneous material shown in Figure 5 with shear modulus variation given by

$$G(x) = G_0 \mathrm{e}^{\beta x} \ , \tag{95}$$

where G_0 and β are material constants. Erdogan [5] has studied this problem in order to investigate the singular nature of the crack-tip stress field in bonded nonhomogeneous materials under antiplane shear loading. He uses a slope formulation[6], while here we use a displacement formulation. To understand what can be gained through the displacement formulation, we first state the governing PDE and the boundary conditions for the crack problem:

$$\begin{array}{lll} \nabla^2 w(x,y) + \beta \frac{\partial w(x,y)}{\partial x} = 0 \ , & -\infty < x < \infty \ , \quad y \geq 0 \ , \\ w(x,0) = 0 \ , & x \notin [c,d] \ , \\ \sigma_{yz}(x,0^+) = p(x) \ , & x \in (c,d) \ , \end{array} \tag{96}$$

[6]The governing integral equations are described by relationships (20), (21), and (22) of Reference [5].

where $p(x)$ is the traction function along the crack surfaces (c, d); and because of symmetry, only the upper half plane $y > 0$ is considered. By the Fourier transform we write $w(x, y)$ as

$$w(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[A(\xi) \mathrm{e}^{\lambda(\xi) y}\right] \mathrm{e}^{ix\xi} d\xi \ , \tag{97}$$

where $A(\xi)$ is to be determined by the boundary conditions, and

$$[\lambda(\xi)]^2 = \xi^2 + i\beta\xi \ . \tag{98}$$

Because of the far field boundary condition, $\lim_{y\to\infty} w(x, y) = 0$, $\lambda(\xi)$ is found to have a non-negative real part which can be expressed as

$$\lambda(\xi) = \frac{-1}{\sqrt{2}} \sqrt{\sqrt{\xi^4 + \beta\xi^2} + \xi^2} - \frac{i}{\sqrt{2}} \mathsf{S}(\beta\xi) \sqrt{\sqrt{\xi^4 + \beta\xi^2} - \xi^2} \ , \tag{99}$$

where the sign function $\mathsf{S}(\cdot)$ is defined as

$$\mathsf{S}(\eta) = \begin{cases} 1 \ , & \eta > 0 \\ 0 \ , & \eta = 0 \\ -1 \ , & \eta < 0 \ . \end{cases} \tag{100}$$

By applying the inverse Fourier transform to equation (97), one finds

$$A(\xi) = \frac{1}{\sqrt{2\pi}} \int_{c}^{d} w(t, 0) \mathrm{e}^{it\xi} dt \ , \tag{101}$$

which leads to the following integral equation[7]:

$$\sigma_{yz}(x, 0) = p(x) = \frac{G(x)}{2\pi} \int_{c}^{d} k(x, t) w(t, 0) dt \ , \tag{102}$$

where

$$k(x,t) = \lim_{y\to 0^+} \int_{-\infty}^{\infty} \left[\lambda(\xi) \mathrm{e}^{\lambda(\xi) y}\right] \mathrm{e}^{i(t-x)\xi} d\xi \ . \tag{103}$$

The trade-off between a displacement-based and a slope-based formulation can be seen here if one recalls the issue regarding decomposition of the $K(\xi)$ described in (3). In summary,

$$K(\xi) = \lim_{y\to 0^+} \frac{\lambda(\xi)}{\xi} \mathrm{e}^{\lambda(\xi) y} = \frac{\lambda(\xi)}{\xi} \ , \quad \text{if slope formulation is used,} \tag{104}$$

and

$$K(\xi) = \lim_{y\to 0^+} \lambda(\xi) \mathrm{e}^{\lambda(\xi) y} = \lambda(\xi) \ , \quad \text{if displacement formulation is adopted.} \tag{105}$$

[7]Note that equations (102) and (103) correspond to equations (20) and (21) in reference [5], respectively. However, the present notation is different from [5], in which the dummy variable used for the Fourier transform is α, (a, b) stands for the crack surfaces, and $m(\alpha)$ corresponds to our $\lambda(\xi)$.

The decomposition of $K(\xi)$ in (104) is difficult because of the term ξ in the denominator. On the other hand, the decomposition of $K(\xi)$ in (105) can be achieved through a simple asymptotic analysis:

$$\text{Real part of } \lambda(\xi) = \frac{-1}{\sqrt{2}}\sqrt{\sqrt{\xi^4+\beta\xi^2}+\xi^2} \quad \stackrel{|\xi|\to\infty}{\longrightarrow} \quad -|\xi| \ , \tag{106}$$

$$i \times \text{Imaginary part of } \lambda(\xi) = \frac{-i}{\sqrt{2}}\mathsf{S}(\beta\xi)\sqrt{\sqrt{\xi^4+\beta\xi^2}-\xi^2} \quad \stackrel{|\xi|\to\infty}{\longrightarrow} \quad -\frac{i\beta}{2}\frac{|\xi|}{\xi} \ . \tag{107}$$

Thus the governing hypersingular integral equation is found to be

$$\frac{G(x)}{2\pi}\int_c^d \left[\frac{2}{(t-x)^2}+\frac{\beta}{t-x}+N(x,t)\right] D(t)dt \ = \ p(x) \ , \quad c<x<d \ , \tag{108}$$

where we have let

$$D(t) \ = \ w(t,0) \ , \tag{109}$$

and the nonsingular kernel is

$$\begin{aligned} N(x,t) \ = \ \int_0^\infty \Bigg\{ & \frac{-\beta^2\sqrt{\xi}\cos[(t-x)\xi]}{\left(\sqrt{\xi}+\frac{1}{\sqrt{2}}\sqrt{\sqrt{\xi^2+\beta^2}+\xi}\right)\left(\xi+\sqrt{\xi^2+\beta^2}\right)} + \\ & \frac{-\beta^4\sin[(t-x)\xi]}{\left(\beta+\sqrt{2}\mathsf{S}(\beta)\sqrt{\sqrt{\xi^4+\beta^2\xi^2}-\xi^2}\right)\left(2\xi^2+\beta^2+2\sqrt{\xi^4+\beta^2\xi^2}\right)} \Bigg\} d\xi \ . \end{aligned} \tag{110}$$

Recall that the function $\mathsf{S}(\cdot)$ is defined by equation (100). As a consistency check, note that if $\beta = 0$, then both the Cauchy singular kernel $\beta/(x-t)$ and the nonsingular kernel $N(x,t)$ will be dropped from equation (108) so that equation (25) is recovered.

Figure 6 shows numerical results for displacement profiles considering a crack with uniformly applied shear tractions $\sigma_{yz}(x,0) = -p_0$ ($|x| < a$), and various values of the material parameter β. Note that the cracks are tilted to the right because of material nonhomogeneity. Further numerical results, including SIFs at both tips of the crack and corresponding displacement profiles, are given by Chan *et. al.* [2]. From a numerical point of view, they have shown that essentially the same results are obtained either by U_n or T_n representations for this specific problem [2].

7.3 Gradient Elasticity Applied to Mode III Cracks [31]

One of the most relevant aspects of the formulas derived in Sections 4 and 5 is the evaluation of hypersingular integrals such as $I_\alpha(T_n, m, r)$ for $m \geq 2$ in the weight function $W(s)$ given by equations (10) and (14). This example illustrates this point for the case $m = 2$.

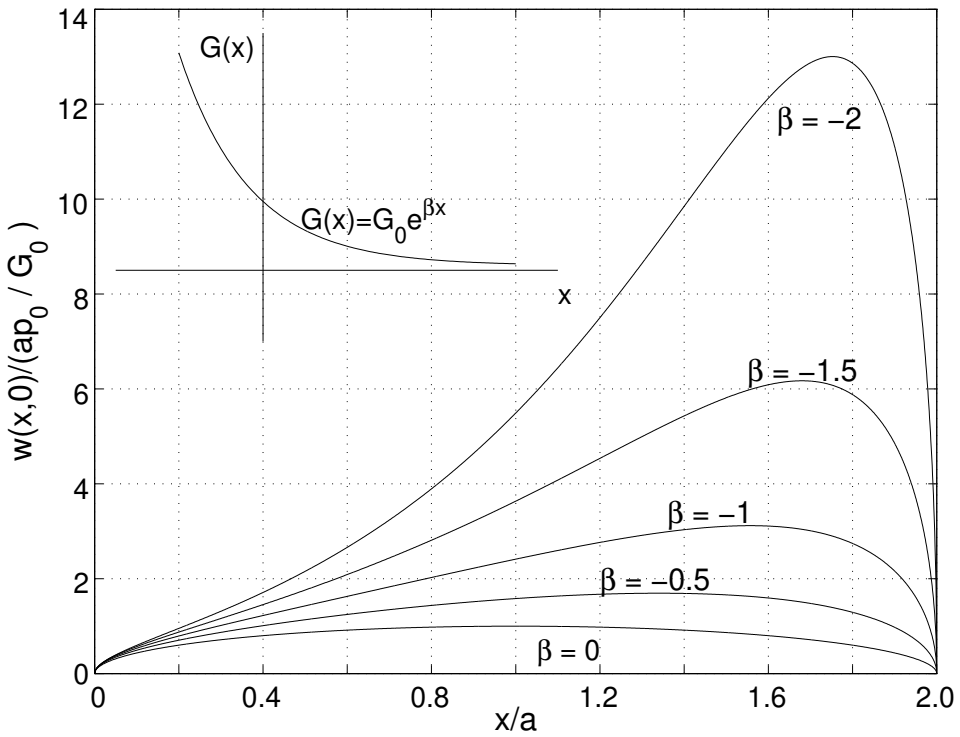

Figure 6: The half plane of the antiplane shear problem for nonhomogeneous material with shear modulus $G(x) = G_0 \mathrm{e}^{\beta x}$. The cracks are tilted to the right because of material nonhomogeneity.

Paulino *et. al.* [31] have presented a hypersingular integral equation formulation for a mode III crack in a material described by constitutive equations of gradient elasticity with both volumetric and surface energy gradient dependent terms. A similar study, using a different approach, has been conducted by Vardoulakis *et. al.* [37] and Exadaktylos *et. al.* [9]. For this problem, the governing PDE is

$$-\ell^2 \nabla^4 w + \nabla^2 w = 0 \ , \tag{111}$$

where ℓ is the characteristic length of the material associated to volumetric strain-gradient terms, and w is the antiplane shear displacement. The boundary conditions are

$$\begin{array}{ll} \sigma_{yz}(x,0) = p(x) \ , & |x| < a \\ \mu_{yyz}(x,0) = 0 \ , & -\infty < x < \infty \\ w(x,0) = 0 \ , & |x| > a \ , \end{array} \tag{112}$$

where the notation of references [9, 31, 37] is adopted here. Enforcing the governing equation (111), imposing the boundary conditions (112), taking account of symmetry along the x-axis, and using Fourier transform method, Paulino *et. al.* [31] have obtained the following governing Fredholm hypersingular integral equation

$$\frac{G}{\pi} \int_{-a}^{a} \left\{ \frac{-2\ell^2}{(t-x)^3} + \frac{1 - (\frac{\ell'}{2\ell})^2}{t-x} + k(x,t) \right\} \phi(t) \, dt + \frac{G\ell'}{2} \phi'(x) = p(x), \ |x| < a \tag{113}$$

with singularity $\alpha = 3$, where the slope function $\phi(x)$ is defined to be

$$\phi(x) = \frac{\partial w(x,0)}{\partial x} \tag{114}$$

which satisfies the single-valuedness condition

$$\int_{-a}^{a} \phi(t)dt = 0 \ , \tag{115}$$

for the solution of the fracture mechanics problem. Here, $k(x,t)$ is the nonsingular kernel given by

$$k(x,t) = \int_0^\infty \frac{\frac{\ell'}{2}\xi\left(\sqrt{\frac{\ell^2\xi^2+1}{\ell^2}} - \xi\right) - \frac{1}{4}(\frac{\ell'}{\ell})^2\left(\sqrt{\frac{\ell^2\xi^2+1}{\ell^2}} - \xi\right) + \frac{1}{4}\frac{(\ell')^3}{\ell^4}}{\frac{\ell'}{\ell^2} - \left(\sqrt{\frac{\ell^2\xi^2+1}{\ell^2}} + \xi\right)} \sin[\xi(t-x)]d\xi \ ; \tag{116}$$

where $(-a,\ a)$ stands for the crack surfaces; ℓ' is the characteristic length responsible for surface strain-gradient terms; G is the shear modulus of the material; $p(x)$ is the known loading function; t is the integration variable, and x is the collocation variable.

The behavior of the solution in terms of the density function $\phi(t)$, can be expressed as

$$\phi(t) \ = \ R(t)(a^2 - t^2)^{\frac{3}{2}} \ , \tag{117}$$

where $R(t)$ is taken to be an expansion of Tchebyshev polynomials of first kind (T_n). This example motivates the whole work of this paper because the analytical evaluation of the hypersingular integral $I_3(T_n, 2, r)$ is needed for successfully solving the governing integral equation (113). The expression for $I_3(T_n, 2, r)$ is given by (55).

Note that the unknown density function is taken to be the first derivative of the displacement function, as described by equation (114). For this particular example, the decomposition of the original kernel into singular and nonsingular parts, stated by equation (3), can be accomplished by means of partial fractions [31]. In general, this step of decomposition is not an easy task, as discussed in Example 2. An alternative is to consider the displacement function $w(x,0)$ as the unknown density function. In this case, a hypersingular integral equation with $\alpha = 4$ is obtained. With order of singularity $\alpha = 4$, the behavior of the density function $D(t)$ in equation (4) can be expressed by [31]

$$D(t) \ = \ R(t)(a^2 - t^2)^{\frac{5}{2}} \ . \tag{118}$$

Thus one needs to evaluate the hypersingular integral $I_4(T_n, 3, r)$ in order to implement the numerical approximation. The expression for $I_4(T_n, 3, r)$ is given by equation (56).

The numerical results are presented in terms of SIFs (Table 3) and displacement profiles (Figure 7) by considering the slope function $\phi(x)$ as unknown in equation (113) and the T_n expansion to $R(t)$ in equation (117). Table 3 shows the convergence of the SIFs by choosing

Table 3: Stress intensity factors $K_{III}(a) = 3\sqrt{\pi a}(\frac{\ell}{a})^2 G \sum_{n=0}^{N} a_n$, where a_n are the coefficients of the T_n expantion to $R(t)$ in equation (117). Here $\ell' = 0$

N	ℓ=0.8	ℓ=0.5	ℓ=0.2	ℓ=0.1	ℓ=0.05	ℓ=0.01	ℓ=0.005
11	20.3131	15.8292	7.4396	4.5116	2.6342	0.1282	0.0319
21	11.8757	9.5632	4.4791	2.1538	0.9541	0.0898	0.1602
31	11.6607	9.3937	4.3878	2.0856	0.9204	0.1649	0.0404
41	11.6665	9.3983	4.3902	2.0878	0.9246	0.1378	0.0682
51	11.6667	9.3984	4.3903	2.0878	0.9247	0.1399	0.0658
61	11.6667	9.3984	4.3903	2.0878	0.9247	0.1400	0.0653
71	11.6667	9.3984	4.3903	2.0878	0.9247	0.1399	0.0654
81	11.6667	9.3984	4.3903	2.0878	0.9247	0.1399	0.0654
91	11.6667	9.3984	4.3903	2.0878	0.9247	0.1399	0.0654
101	11.6667	9.3984	4.3903	2.0878	0.9247	0.1399	0.0654

different values of volumetric gradient dependent term, ℓ, and letting surface energy gradient dependent term $\ell' = 0$. The displacement profile in Figure 7 shows the "cusping" phenomenon at the crack tips, which is also described by Barenblatt's "cohesive zone" theory [1]. The interesting point is that the cusping crack obtained here is a natural outcome of the higher order gradient elasticity theory. Further numerical results and discussions are provided in reference [31].

8 Concluding Remarks

Closed form analytical solutions are provided here for a broad class of improper integrals with hypersingular kernels and density functions approximated by means of Tchebyshev polynomials. Whenever possible, the symbolical and numerical tools of the computer algebra software MAPLE [8] [12, 38, 39] have been used to verify the proposed solutions. A systematic approach for evaluating integrals when higher order singularities is also given in the present paper.

The examples involve crack problems and aspects such as LEFM, nonhomogeneous materials, and gradient elasticity theory (see Table 1). All these problems are solved by means of Fredholm hypersingular integral equation formulations. When classical elasticity is used, both T_n and U_n representations lead to essentially the same numerical results. For a crack problem in nonhomogeneous material, the difficulty that arises in splitting the singular and nonsingular parts from the original kernels can be circumvented by means of displacement-based, rather than slope-based, formulation.

As a closing remark, we note that as material property variation in space and higher order gradient continuum theories are considered, the formulation of the crack problem and the associated kernels become quite involved. Thus, better analytical and numerical techniques are needed to successfully solve the governing singular integral equations. This paper is a combination in this sense.

[8] Maple can evaluate some CPV integrals. However, in general, computer algebra systems are very limited with respect to hypersingular integrals.

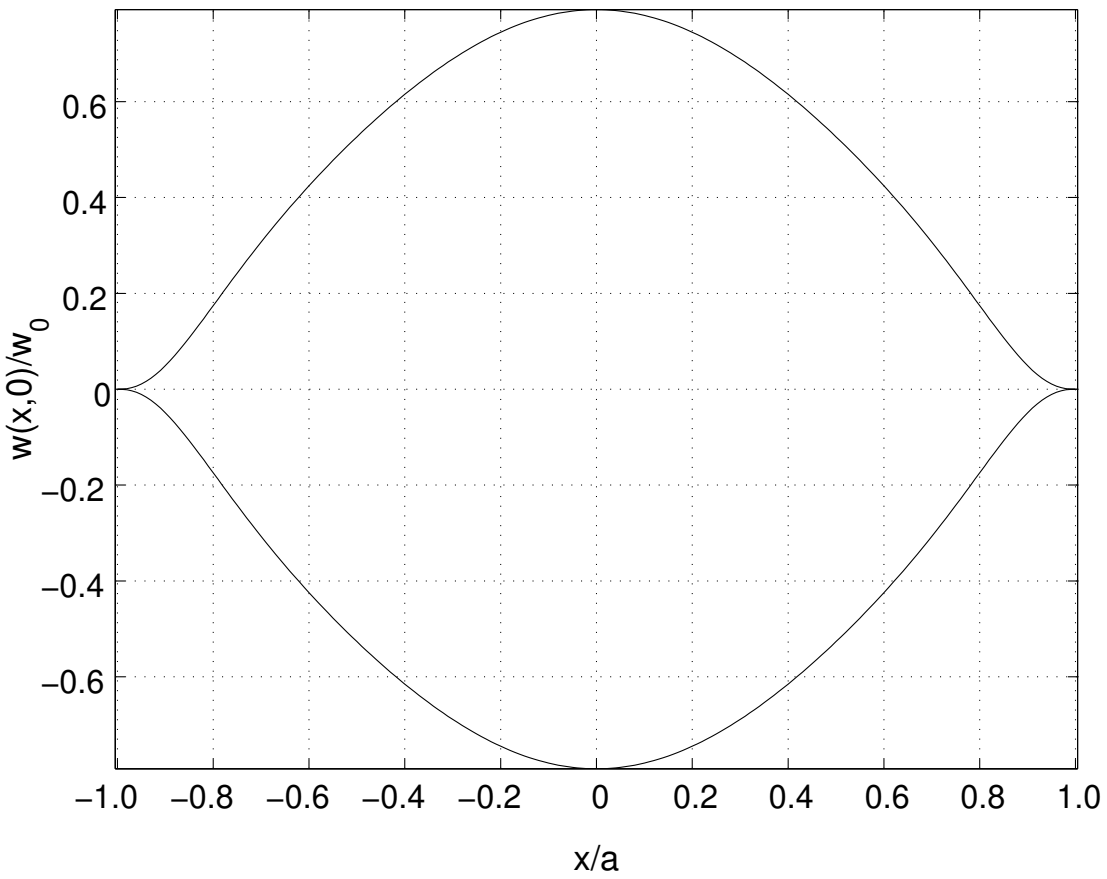

Figure 7: Full crack displacement profile at $\ell/a = 0.2$ and $\ell'/a = 0.1$ under uniform crack surface antiplane loading $\sigma_{yz}(x,0) = -\sigma_0$; $w_0 = a\sigma_0/G_0$.

Appendix

A Integrals associated with lower order n

The general formula given in the text, *e.g.* equations (42), (43), (48), (52), (57), (61), (66), and (70) are only valid above certain values of n. Thus the goal of this Appendix is to provide the expressions for integrals associated with lower order n.

A.1 $I_1(T_n, 1, r),\ n = 0,\ 1$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_0(s)\sqrt{1-s^2}}{s-r} ds = -r \ , \quad |r| < 1 \tag{119}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_1(s)\sqrt{1-s^2}}{s-r} ds = -r^2 + \frac{1}{2} \ , \quad |r| < 1 \tag{120}$$

A.2 $I_1(T_n, 2, r),\ n = 0 \cdots 5$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = r^3 - \frac{3}{2} r \ , \quad |r| < 1 \tag{121}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = r^4 - \frac{3}{2} r^2 + \frac{3}{8} \ , \quad |r| < 1 \tag{122}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_2(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = 2r^5 - 4r^3 + \frac{9}{4} r \ , \quad |r| < 1 \tag{123}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_3(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = 4r^6 - 9r^4 + 6r^2 - \frac{7}{8} \ , \quad |r| < 1 \tag{124}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_4(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = 8r^7 - 20r^5 + 16r^3 - 4r \ , \quad |r| < 1 \tag{125}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_5(s)(1-s^2)^{\frac{3}{2}}}{s-r} ds = 16r^8 - 44r^6 + 41r^4 - 14r^2 + 1 \ , \quad |r| < 1 \tag{126}$$

A.3 $I_1(T_n, 3, r),\ n = 0 \cdots 5$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -\frac{15}{8}r + \frac{5}{2}r^3 - r^5 \tag{127}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = \frac{5}{16} - \frac{15}{8}r^2 + \frac{5}{2}r^4 - r^6 \tag{128}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_2(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = \frac{5}{12}r - \frac{25}{4}r^3 + 6r^5 - 2r^7 \tag{129}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_3(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -\frac{25}{32} + \frac{55}{8}r^2 - 15r^4 + 13r^6 - 4r^8 \tag{130}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_4(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -\frac{65}{16}r + 20r^3 - 36r^5 + 28r^7 - 8r^9 \tag{131}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{T_5(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = \frac{31}{32} - 15r^2 + 55r^4 - 85r^6 + 60r^8 - 16r^{10} \tag{132}$$

A.4 $I_1(U_n, 3, r),\ n = 0 \cdots 5$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_0(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -\frac{15}{8}r + \frac{5}{2}r^3 - r^5 \tag{133}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_1(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = \frac{5}{8} - \frac{15}{4}r^2 + 5r^4 - 2r^6 \tag{134}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_2(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = \frac{25}{8}r - 10r^3 + 11r^5 - 4r^7 \tag{135}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_3(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -\frac{15}{16} + 10r^2 - 25r^4 + 24r^6 - 8r^8 \tag{136}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_4(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = -5r + 30r^3 - 61r^5 + 52r^7 - 16r^9 \tag{137}$$

$$\frac{1}{\pi}\fint_{-1}^{1} \frac{U_5(s)(1-s^2)^{\frac{5}{2}}}{s-r}ds = 1 - 20r^2 + 85r^4 - 146r^6 + 112r^8 - 32r^{10} \tag{138}$$

A.5 $I_2(T_n, 3, r),\ n = 0 \cdots 6$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -5r^4 + \frac{15}{2} r^2 - \frac{15}{8} \ , \quad |r| < 1 \tag{139}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -6r^5 + 10r^3 - \frac{15}{4} r \ , \quad |r| < 1 \tag{140}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -14r^6 + 30r^4 - \frac{75}{4} r^2 + \frac{5}{2} \ , \quad |r| < 1 \tag{141}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -32r^7 + 78r^5 - 60r^3 + \frac{55}{4} r \ , \quad |r| < 1 \tag{142}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -72r^8 + 196r^6 - 180r^4 + 60r^2 - \frac{65}{16} \ , \quad |r| < 1 \tag{143}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -160r^9 + 480r^7 - 510r^5 + 220r^3 - 30r \ , \quad |r| < 1 \tag{144}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{T_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -352r^{10} + 1152r^8 - 1386r^6 + 730r^4 - 150r^2 + 6 \ , \quad |r| < 1 \tag{145}$$

A.6 $I_2(U_n, 3, r),\ n = 0 \cdots 6$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{U_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -5r^4 + \frac{15}{2} r^2 - \frac{15}{8} \ , \quad |r| < 1 \tag{146}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{U_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -12r^5 + 20r^3 - \frac{15}{2} r \ , \quad |r| < 1 \tag{147}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{U_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -28r^6 + 55r^4 - 30r^2 + \frac{25}{8} \ , \quad |r| < 1 \tag{148}$$

$$\frac{1}{\pi} \fint_{-1}^{1} \frac{U_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2} ds = -64r^7 + 144r^5 - 100r^3 + 20r \ , \quad |r| < 1 \tag{149}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2}ds = -144r^8+364r^6-305r^4+90r^2-5\ ,\quad |r|<1 \tag{150}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2}ds = -320r^9+896r^7-876r^5+340r^3-40r\ ,\quad |r|<1 \tag{151}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^2}ds = -704r^{10}+2160r^8-2408r^6+1155r^4-210r^2+7\ ,\quad |r|<1 \tag{152}$$

A.7 $I_3(T_n, 2, r),\ n = 0\cdots 4$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_0(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds = 3r\ ,\quad |r|<1 \tag{153}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_1(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds = 6r^2-\frac{3}{2}\ ,\quad |r|<1 \tag{154}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_2(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds = 20r^3-12r\ ,\quad |r|<1 \tag{155}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_3(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds = 60r^4-54r^2+6\ ,\quad |r|<1 \tag{156}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_4(s)(1-s^2)^{\frac{3}{2}}}{(s-r)^3}ds = 168r^5-200r^3+48r\ ,\quad |r|<1 \tag{157}$$

A.8 $I_3(T_n, 3, r),\ n = 0\cdots 7$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -10r^3+\frac{15}{2}r\ ,\quad |r|<1 \tag{158}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -15r^4+15r^2-\frac{15}{8}\ ,\quad |r|<1 \tag{159}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -42r^5+60r^3-\frac{75}{4}r\ ,\quad |r|<1 \tag{160}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -112r^6+195r^4-90r^2+\frac{55}{8}\ ,\quad |r|<1 \tag{161}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -288r^7+588r^5-360r^3+60r\ ,\quad |r|<1 \tag{162}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -720r^8+1680r^6-1275r^4+330r^2-15\ ,\quad |r|<1 \tag{163}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -1760r^9+4608r^7-4158r^5+1460r^3-150r\ ,\quad |r|<1 \tag{164}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{T_7(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -4224r^{10}+12240r^8-12768r^6+5655r^4-930r^2+27\ ,\quad |r|<1 \tag{165}$$

A.9 $I_3(U_n, 3, r),\ n = 0\cdots 6$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -10r^3+\frac{15}{2}r\ ,\quad |r|<1 \tag{166}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -30r^4+30r^2-\frac{15}{4}\ ,\quad |r|<1 \tag{167}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -84r^5+110r^3-30r\ ,\quad |r|<1 \tag{168}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -224r^6+360r^4-150r^2+10\ ,\quad |r|<1 \tag{169}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -576r^7+1092r^5-610r^3+90r\ ,\quad |r|<1 \tag{170}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -1440r^8+3136r^6-2190r^4+510r^2-20\ ,\quad |r|<1 \tag{171}$$

$$\frac{1}{\pi}\fint_{-1}^{1}\frac{U_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^3}ds = -3520r^9+8640r^7-7224r^5+2310r^3-210r\ ,\quad |r|<1 \tag{172}$$

A.10 $I_4(T_n, 3, r),\ n = 0 \cdots 8$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -10r^2 + \frac{5}{2}\ , \quad |r| < 1 \tag{173}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -20r^3 + 10r\ , \quad |r| < 1 \tag{174}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -70r^4 + 60r^2 - \frac{25}{4}\ , \quad |r| < 1 \tag{175}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -224r^5 + 260r^3 - 60r\ , \quad |r| < 1 \tag{176}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -672r^6 + 980r^4 - 360r^2 + 20\ , \quad |r| < 1 \tag{177}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -1920r^7 + 3360r^5 - 1700r^3 + 220r\ , \quad |r| < 1 \tag{178}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -5280r^8 + 10752r^6 - 6930r^4 + 1460r^2 - 50\ , \quad |r| < 1 \tag{179}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_7(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -14080r^9 + 32640r^7 - 25536r^5 + 7540r^3 - 620r\ , \quad |r| < 1 \tag{180}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{T_8(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -36608r^{10} + 95040r^8 - 87360r^6 + 33320r^4 - 4560r^2 + 104\ , \quad |r| < 1 \tag{181}$$

A.11 $I_4(U_n, 3, r),\ n = 0 \cdots 6$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{U_0(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -10r^2 + \frac{5}{2}\ , \quad |r| < 1 \tag{182}$$

$$\frac{1}{\pi} \not\int_{-1}^{1} \frac{U_1(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4} ds = -40r^3 + 20r\ , \quad |r| < 1 \tag{183}$$

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_2(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds = -140r^4+110r^2-10\ ,\quad |r|<1 \tag{184}$$

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_3(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds = -448r^5+480r^3-100r\ ,\quad |r|<1 \tag{185}$$

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_4(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds = -1344r^6+1820r^4-610r^2+30\ ,\quad |r|<1 \tag{186}$$

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_5(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds = -3840r^7+6272r^5-2920r^3+340r\ ,\quad |r|<1 \tag{187}$$

$$\frac{1}{\pi}\oint_{-1}^{1}\frac{U_6(s)(1-s^2)^{\frac{5}{2}}}{(s-r)^4}ds = -10560r^8+20160r^6-12040r^4+2310r^2-70\ ,\quad |r|<1 \tag{188}$$

A.12 Others

$$\int_{-1}^{1}(1-t^2)^{\frac{3}{2}}T_0(t)dt = \int_{-1}^{1}(1-t^2)^{\frac{3}{2}}dt = \frac{3}{8}\pi \tag{189}$$

$$\int_{-1}^{1}(1-t^2)^{\frac{3}{2}}T_2(t)dt = -\frac{\pi}{4} \tag{190}$$

$$\int_{-1}^{1}(1-t^2)^{\frac{3}{2}}T_4(t)dt = \frac{\pi}{16} \tag{191}$$

$$\int_{-1}^{1}(1-t^2)^{\frac{3}{2}}T_n(t)dt = 0\quad ,\quad \text{for all } n,\quad \text{and } n\neq 0,2,4\ . \tag{192}$$

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