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Self-Averaging Scaling Limits for Random Parabolic Waves

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Abstract

We consider several types of scaling limits for the Wigner-Moyal equation of the parabolic waves in random media, the limiting cases of which include the standard radiative transfer limit, the geometrical-optics limit and the white-noise limit. We show under fairly general assumptions on the random refractive index field that sufficient amount of medium diversity (thus excluding the white-noise limit) leads to statistical stability or self-averaging in the sense that the limiting law is deterministic and is governed by one of the 6 different types of transport (Boltzmann or Fokker-Planck) equations depending on the specific scaling involved. We discuss the connection to the statistical stability of time reversal procedure and the decoherence effect in quantum mechanics.

1. Introduction

The celebrated Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\Delta\Psi+\sigma V(t,\mathbf{x})\Psi=0,\quad \Psi(0,\mathbf{x})=\Psi_0(\mathbf{x})$$

describes the evolution of the wave function Ψ of a quantum spin-less particle in a potential σV where σ is the typical size of the variation.

A similar equation called the parabolic wave equation is also widely used to describe the propagation of the modulation of low intensity wave beam in turbulent or turbid media in the forward scattering approximation of the full wave equation [20]. In this connection the refractive index fluctuation plays

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the role of the potential in the equation. Nondimensionalized with respect to the propagation distances in the longitudinal and transverse directions, L_z and L_x, respectively, the parabolic wave equation for the modulation function Ψ reads

$$ik^{-1}L_z^{-1}\frac{\partial \Psi}{\partial z} + 2^{-1}k^{-2}L_x^{-2}\Delta\Psi + \sigma V(zL_z, \mathbf{x}L_x)\Psi = 0, \quad \Psi(0,\mathbf{x}) = \Psi_0(\mathbf{x}) \tag{1}$$

where k is the carrier wavenumber, Ψ the amplitude modulation and Δ the Laplacian operator in the transverse coordinates $\mathbf{x}$. Here we have assumed the random media has a constant background. In the sequel we will adopt the notation in eq. (1).

In this paper we study the scaling regimes where the wave beam experiences both longitudinal and transverse diversity of the random medium, represented by V, whose fluctuation is assumed to be weak. This gives rise to random spread of wave energy in the transverse directions.

Atmospheric turbulence is an example at hand. A widely used model is a Gaussian refractive index field V with the modified von Kármán spectral density [20]

$$\Phi(\xi, \mathbf{k}) \sim \left(L_0^{-2} + |\mathbf{k}| + \xi^2\right)^{-H-3/2} \exp\left[-\ell_0^2(|\mathbf{k}|^2 + \xi^2)\right], \quad (\xi, \mathbf{k}) \in \mathbb{R}^3, \quad H = 1/3 \tag{2}$$

with a slowly varying background mainly depending on the altitude. Here the positive constants L_0 and ℓ_0 are the outer and inner scales, respectively. Our method and results can easily be adapted to the case of slowing varying background. For the simplicity of presentation, however, we will focus on the case with constant background. One expects self-averaging of the wave beam when L_x, L_z are both much larger than L_0, which is roughly the correlation length. We will see that as far as the self-averaging effect is concerned the rapid decay in (2) at high wavenumbers $|\xi|^2 + |\mathbf{k}|^2 \gg \ell_0^{-2}$ can be significantly relaxed, cf. (10) below.

To fix the idea, let us choose the units of the longitudinal and transverse coordinates such that the correlation lengths of V equals $L_0 = O(1)$ in both directions and that

$$L_z \sim L_x \gg L_0. \tag{3}$$

There is no loss of generality in assuming the isotropy in the numerical values of the correlation lengths since their units may be different; analogously there is no loss of generality in the choice of the hyperbolic scaling (3), cf. Remark 2 below.

In addition to (3) we adjust the intensity σ of the medium fluctuations, depending on the actual length scales and anisotropy of the medium, in order to obtain a nontrivial limit. Below we digress to discuss the quadratic transformation of the wave field, called the Wigner distribution, and its connection to time reversal operation and quantum mechanics. The Wigner distribution will play an essential role in our analysis of the scaling limits.

2. Wigner distribution and time reversal

There has been a surge of interest in the radiative transfer limit in terms of the Wigner distribution (see below) because of its application to the spectacular phenomena related to time-reversal (or phase-conjugate) mirrors [4], [3], [8], [17].

The Wigner transform or distribution of the wave function Ψ is defined as

$$\begin{aligned} W(z,\mathbf{x},\mathbf{p}) &= \frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi(z,\mathbf{x}+\frac{\mathbf{y}}{2})\Psi^*(z,\mathbf{x}-\frac{\mathbf{y}}{2})d\mathbf{y} \\ &= \frac{1}{(2\pi)^d}\int\int e^{-i\mathbf{p}\cdot(\mathbf{y}_1-\mathbf{y}_2)}\delta(\mathbf{x}-\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\rho(\mathbf{y}_1,\mathbf{y}_2)d\mathbf{y}_1d\mathbf{y}_2 \end{aligned}$$

where $\rho(\mathbf{y}_1,\mathbf{y}_2)=\Psi(\mathbf{y}_1)\Psi^*(\mathbf{y}_2)$ is called the two-point function or the density matrix. As apparent from the definition the Wigner function contains all the information about ρ. The Wigner distribution have the simple properties

$$\int W(z,\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p}=\|\Psi\|_2^2,\quad \|W\|_2=(2\pi)^{-d/2}\|\Psi\|_2^2.$$

This is the case of pure state Wigner distribution. What is the more pertinent for us is the so called mixed state Wigner distribution.

Let us briefly review how a mixed state Wigner distribution arises in the time-reversal operation. Let $G_H(0,\mathbf{x},z,\mathbf{y})$ be the Green's function, with the point source located at $(z,\mathbf{y})$, for the reduced wave (Helmholtz) equation for which the Schrödinger equation is an approximation. By the self-adjointness of the Helmholtz equation, G_H satisfies the symmetry property

$$G_H(0,\mathbf{x},z,\mathbf{y})=G_H(z,\mathbf{y},0,\mathbf{x}).$$

The wave field Ψ_m received at the mirror is given by

$$\begin{aligned} \Psi_m(z,\mathbf{x}_m) &= \chi_A(\mathbf{x}_m)\int G_H(0,\mathbf{x}_m,z,\mathbf{x}_s)\Psi_0(\mathbf{x}_s)d\mathbf{x}_s \\ &= \chi_A(\mathbf{x}_m)\int G_H(z,\mathbf{x}_s,0,\mathbf{x}_m)\Psi_0(\mathbf{x}_s)d\mathbf{x}_s \end{aligned}$$

where χ_A is the aperture function of the phase-conjugating mirror A.

After phase conjugation and back-propagation we have at the source plane the wave field

$$\Psi^B(z,\mathbf{x};k)=\int G_H(z,\mathbf{x},0,\mathbf{x}_m)\overline{G_H(z,\mathbf{x}_s,0,\mathbf{x}_m)}\chi_A(\mathbf{x}_m)\overline{\Psi_0(\mathbf{x}_s)}d\mathbf{x}_md\mathbf{x}_s.$$

In the parabolic approximations the Green's function $G_H(z,\mathbf{x},0,\mathbf{y})$ is approximated by $e^{ikz}G_S(z,\mathbf{x},\mathbf{y})$ where $G_S(z,\mathbf{x},\mathbf{y})$ is the propagator of the

Schrödinger equation. Making the approximation in the above expression for the back-propagated field we obtain

$$\begin{aligned}\Psi^B(z,\mathbf{x};k) &= \int G_S(z,\mathbf{x},\mathbf{x}_m)\overline{G_S(z,\mathbf{x}_s,\mathbf{x}_m)\Psi_0\,(\mathbf{x}_s)}\chi_A(\mathbf{x}_m)d\mathbf{x}_m d\mathbf{x}_s \\ &= \int e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)}W(z,\frac{\mathbf{x}+\mathbf{x}_s}{2},\mathbf{p})\overline{\Psi_0\,(\mathbf{x}_s)}d\mathbf{p}d\mathbf{x}_s \qquad (4)\end{aligned}$$

where the Wigner distribution W is given by

$$\begin{aligned}&W(z,\mathbf{x},\mathbf{p}) \qquad (5)\\ &= \frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}G_S(z,\mathbf{x}+\mathbf{y}/2,\mathbf{x}_m)\overline{G_S(z,\mathbf{x}-\mathbf{y}/2,\mathbf{x}_m)}\chi_A(\mathbf{x}_m)d\mathbf{y}d\mathbf{x}_m.\end{aligned}$$

This is a mixed-state Wigner distribution. In general, the integral in (4) should be interpreted in the distributional sense.

The Wigner distribution in (5) has the initial condition W

$$W(0,\mathbf{x},\mathbf{p}) = \frac{\chi_A(\mathbf{x})}{(2\pi)^d} \qquad (6)$$

and can be treated as a generalized function on $\mathbb{R}^{2d}$. Indeed, for any $\Theta \in C_c^\infty(\mathbb{R}^d)$ we have

$$\langle \Psi^B, \Theta\rangle = \int\int W(z,\mathbf{r},\mathbf{p})\Theta(\mathbf{r},\mathbf{p})d\mathbf{r}d\mathbf{p} \qquad (7)$$

where the function Θ is defined as

$$\theta(\mathbf{r},\mathbf{p}) = 2^d\int \Theta(\mathbf{y})e^{i2\mathbf{p}\cdot(\mathbf{y}-\mathbf{r})/\gamma}\overline{\Psi_0(2\mathbf{r}-\mathbf{y})}d\mathbf{y}.$$

If for instance $\Psi_0 \in C_c^\infty(\mathbb{R}^d)$ then it is easy to see $\Theta(\mathbf{y},\mathbf{p})$ is compactly supported in $\mathbf{y}\in\mathbb{R}^d$ and decays rapidly (faster than any power) in $\mathbf{p}\in\mathbb{R}^d$. As a result we can always approximate to arbitrary accuracy the distributional initial data such as (6) by square-integrable initial data.

The fluctuations of the back-propagated wave field is thus determined by the fluctuations of the Wigner distribution. The statistical stability or self-averaging of the Wigner distribution in turn explains, modulo the scaling limit, the persistence and stability of the super-focusing of the time-reversed, back-propagated wave field observed experimentally and numerically.

Our main results show that under various scaling limits, sufficient amount of spatial-transverse diversity experienced by the propagating wave pulse results in self-averaging and deterministic limiting laws.

From the perspective of the quantum stochastic dynamics in a random environment, our results say that due to the spatio- temporal diversity experienced by the wave function of the quantum particle the quantum dynamics has in the scaling limit a classical probabilistic description which is independent of the particular realization of the environment. The transition from a unitary evolution to an irreversible process is of course the outcome of the phase-space coarse-graining by the test functions. The results presented below are a rigorous demonstration of decoherence, a mechanism believed to be responsible for the emergence of the classical world from the quantum one [13], [21].

3. Assumptions

We assume $V_z(\mathbf{x}) = V(z, \mathbf{x})$ is $\mathbf{x}$-homogeneous square-integrable process with the (partial) spectral measure $\widehat{V}(z, d\mathbf{q})$. which is an orthogonal random measure

$$\mathbb{E}[\widehat{V}(z, d\mathbf{p})\widehat{V}(z, d\mathbf{q})] = \delta(\mathbf{p} + \mathbf{q})\Phi_0(\mathbf{p})\ d\mathbf{p}\, d\mathbf{q}$$

and gives rise to the (partial) spectral representation of the refractive index field

$$V_z(\mathbf{x}) \equiv V(z, \mathbf{x}) = \int \exp{(i\mathbf{p} \cdot \mathbf{x})}\widehat{V}(z, d\mathbf{p}).$$

In case that $V(\mathbf{x}), \mathbf{x} \in \mathbb{R}^{d+1}$, is $\mathbf{x}$-homogeneous square-integrable random field with the full spectral density given by $\Phi(\xi, \mathbf{k})$ we have the following relation

$$\Phi_0(\mathbf{p}) = \int \Phi(w, \mathbf{p})\, dw.$$

We also have the following relation between the partial and full spectral measures

$$\hat{V}_z(d\mathbf{p}) = \int e^{izw}\hat{V}(dw, d\mathbf{p})$$

such that

$$\begin{aligned}\mathbb{E}[\hat{V}_z(d\mathbf{p})\hat{V}_s(d\mathbf{q})] &= \int e^{i(s-z)w}\Phi(w, \mathbf{p})\ dw\ \delta(\mathbf{p} + \mathbf{q})\ d\mathbf{p}\, d\mathbf{q} \\ &= \check{\Phi}(s - z, \mathbf{p})\delta(\mathbf{p} + \mathbf{q})\ d\mathbf{p}\, d\mathbf{q}\end{aligned}$$

where

$$\check{\Phi}(s, \mathbf{p}) = \int e^{isw}\Phi(w, \mathbf{p})\ dw.$$

Since $\Phi(\mathbf{k}) = \Phi(-\mathbf{k}), \forall \mathbf{k} \in \mathbb{R}^{d+1}$, we may assume in this case that

$$\Phi(w, \mathbf{q}) = \Phi(-w, \mathbf{p}) = \Phi(w, -\mathbf{p}) = \Phi(-w, -\mathbf{p}), \quad \forall w,\ \mathbf{p} \tag{8}$$

so that $\check{\Phi}(s, \mathbf{p})$ is real-valued and $\check{\Phi}(s, \mathbf{p}) = \check{\Phi}(-s, \mathbf{p})$. The property (8) is related to the detailed balance of the limiting scattering kernels.

First we assume

Assumption 1

$$\Phi(\xi, \mathbf{k}) \in C^\infty(\mathbb{R}^{d+1}) \tag{9}$$

and

$$\Phi(\xi, \mathbf{k}) \le K\left(1 + \ell_x^2|\mathbf{k}|^2 + \ell_z^2\xi^2\right)^{-\zeta/2}, \quad \zeta > d + 2 \tag{10}$$

for some positive constants K, ℓ_x, ℓ_z.

One can interpret ℓ_z and ℓ_x as the ultraviolet cutoff scales for the longitudinal and transverse coordinates, respectively. This is a slower decay at high wavenumbers $\ell_z^2\xi^2 + \ell_x^2|\mathbf{k}|^2 \gg 1$ than stipulated in (2). In further comparison, one sees that for $H > 1/2$ the spectrum (2) satisfies (10) even with $\ell_x = \ell_z = 0$. Such a Gaussian random field is only Hölder continuous with exponent less than H.

Let $\mathcal{F}_z$ and $\mathcal{F}_z^+$ be the sigma-algebras generated by $\{V_s : \forall s \le z\}$ and $\{V_s : \forall s \ge z\}$, respectively. Define the correlation coefficient

$$\rho(t) = \sup_{\substack{h\in\mathcal{F}_z\\ \mathbb{E}[h]=0,\mathbb{E}[h^2]=1}} \sup_{\substack{g\in\mathcal{F}^+_{z+t}\\ \mathbb{E}[g]=0,\mathbb{E}[g^2]=1}} \mathbb{E}[hg]. \tag{11}$$

Assumption 2 *The correlation coefficient $\rho(t)$ is integrable*

When V_z is a Gaussian process, the correlation coefficient $\rho(t)$ equals the *linear* correlation coefficient $r(t)$ which has the following useful expression

$$r(t) = \sup_{g_1,g_2} \int \check{\Phi}(t-\tau_1-\tau_2,\mathbf{k})g_1(\tau_1,\mathbf{k})g_2(\tau_2,\mathbf{k})d\mathbf{k}d\tau_1 d\tau_2 \tag{12}$$

where the supremum is taken over all $g_1, g_2 \in L^2(\mathbb{R}^{d+1})$ which are supported on $(-\infty, 0] \times \mathbb{R}^d$ and satisfy the constraint

$$\int \check{\Phi}(t-t',\mathbf{k})g_1(t,\mathbf{k})\bar{g}_1(t',\mathbf{k})dtdt'd\mathbf{k} = 1 \tag{13}$$

$$\int \check{\Phi}(t-t',\mathbf{k})g_2(t,\mathbf{k})\bar{g}_2(t',\mathbf{k})dtdt'd\mathbf{k} = 1. \tag{14}$$

There are various criteria for the decay rate of the linear correlation coefficients in the literature. For example, according to [12], Chapter VI, Theorem 6 a special class of spectral density functions give rise to exponentially decaying correlation coefficients.

Secondly, we assume a 6-th order sub-Gaussian property: Let

$$U_s^1(\mathbf{x}) = V_s(\mathbf{x}), \quad U_s^2(\mathbf{x}) = \mathbb{E}_z[V_s](\mathbf{x}), \quad s \ge z.$$

Assumption 3 *For any choices of $\sigma_j \in \{1,2\}, j = 1, 2, ..., N$ and a set of linear operators $\{T_j\}$, there exists a finite constant C*

$$\mathbb{E}\left[\prod_{j=1}^N T_j U_{s_j}^{\sigma_j}(\mathbf{x}_j)\right] = 0, \quad N = 3, 5$$

$$\left|\mathbb{E}\left[\prod_{j=1}^N T_j U_{s_j}^{\sigma_j}(\mathbf{x}_j)\right]\right| \le C \sum \left|\prod_{\widehat{mn}} \mathbb{E}\left[T_m U_{s_m}^{\sigma_m}(\mathbf{x}_m) T_n U_{s_n}^{\sigma_n}(\mathbf{x}_n)\right]\right|, \quad N = 4, 6$$

where the summation is over all possible pairings $\{\widehat{mn}\}$ among $\{1, 2, ..., N\}$.

Finally we assume

Assumption 4 *There exists a constant C such that for Theorem 1, 2, 3 (i), (iii) and 4 (i), (iii)*

$$\lim_{\varepsilon\to 0}\mathbb{E}[\sup_{z<z_0}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2]\leq \frac{C}{\varepsilon}\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2,\quad \forall\theta\in C^\infty_c(\mathbb{R}^{2d}),\quad \forall z_0<\infty;$$

and for Theorem 3 (ii) and 4 (ii)

$$\lim_{\varepsilon\to 0}\mathbb{E}[\sup_{z<z_0}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2]\leq \frac{C}{\varepsilon^\alpha}\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2,\quad \forall\theta\in C^\infty_c(\mathbb{R}^{2d}),\quad \forall z_0<\infty;$$

$$\lim_{\varepsilon\to 0}\mathbb{E}[\sup_{z<z_0}\|\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2]\leq \frac{C}{\varepsilon^{2\alpha}}\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2,\quad \forall\theta\in C^\infty_c(\mathbb{R}^{2d}),\quad \forall z_0<\infty;$$

where $\tilde{\mathcal{L}}^\varepsilon_z$ is defined, respectively, by (70), (94), (100) and (120) and $\alpha\in(0,1)$ as specified in the statements of the theorems.

Assumption 4 is readily satisfied for Gaussian random fields. This can be seen by first observing that $\tilde{\mathcal{L}}^\varepsilon_z\theta$ is a Gaussian process and $\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta$ is a χ^2-process and, secondly, by an application of Borell's inequality [1] that the supremum over $z<z_0$ inside the expectation can be over-estimated by a $\log(1/\varepsilon)$ factor for excursion on the scale of any power of $1/\varepsilon$:

$$\mathbb{E}[\sup_{z<z_0}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2]\leq C\log(\frac{1}{\varepsilon})\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2;\tag{15}$$

$$\mathbb{E}[\sup_{z<z_0}\|\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2]\leq C\log^2(\frac{1}{\varepsilon})\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2.\tag{16}$$

4. Main results

In the standard scaling, we set

$$L_z=L_x=\frac{1}{\varepsilon^2}\gg 1,\quad \sigma=\varepsilon.\tag{17}$$

To describe the small scale wave energy we consider the scaled version of the Wigner distribution

$$\begin{aligned}W^\varepsilon(z,\mathbf{x},\mathbf{p})&=\frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi(z,\mathbf{x}+\frac{\varepsilon^2\mathbf{y}}{2})\Psi^*(z,\mathbf{x}-\frac{\varepsilon^2\mathbf{y}}{2})d\mathbf{y}\\&=\frac{1}{(2\pi)^d}\int\int e^{-i\mathbf{p}\cdot(\mathbf{y}_1-\mathbf{y}_2)/\varepsilon^2}\delta(\mathbf{x}-\frac{\mathbf{y}_1+\mathbf{y}_2}{2})\rho(\mathbf{y}_1,\mathbf{y}_2)d\mathbf{y}_1d\mathbf{y}_2\end{aligned}\tag{18}$$

The Wigner distribution W^ε has a limit as certain measure, the Wigner measure, introduced in [16] (see also [11]. But as remarked in the introduction, we always consider uniformly L^2 initial condition induced by a mixed-state density matrix ρ.

The Wigner distribution satisfies the Wigner-Moyal equation

$$\frac{\partial W^\varepsilon_z}{\partial z}+\frac{\mathbf{p}}{k}\cdot\nabla W^\varepsilon_z+\frac{k}{\varepsilon}\mathcal{L}^\varepsilon_z W^\varepsilon_z=0\tag{19}$$

with $W_z^\varepsilon(\mathbf{x},\mathbf{p}) = W^\varepsilon(z,\mathbf{x},\mathbf{p})$. Here the integral operator $\mathcal{L}_z^\varepsilon$ is given by

$$\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \tag{20}$$
$$= i\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p}+\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^2},d\mathbf{q}),$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}, \alpha = 1$.

The more general case with $\alpha \in (0,1)$ can be derived from a somewhat different scaling (cf. the scaling leading to Theorem 2): We probe a highly anisotropic medium $V(z,\varepsilon^{2-2\alpha}\mathbf{x})$ with the strength

$$\sigma = \varepsilon^{2\alpha-1}$$

with a wave beam composed of waves of lengths comparable to that of the medium, so we replace k by $k\varepsilon^{2-2\alpha}$:

$$k \longrightarrow k\varepsilon^{2-2\alpha} \tag{21}$$

in the parabolic wave equation. We then use the following definition of the Wigner distribution to resolve the wave energy:

$$W^\varepsilon(z,\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^d}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi(z,\mathbf{x}+\frac{\varepsilon^{2\alpha}\mathbf{y}}{2})\Psi^*(z,\mathbf{x}-\frac{\varepsilon^{2\alpha}\mathbf{y}}{2})d\mathbf{y}. \tag{22}$$

The difference in scaling between (22) and (21) is, of course, due to the rescaling of coordinates (17).

Since the proof of convergence is the same for $\alpha \in (0,1]$ they are treated together. Eq. (23) and its variants studied in the sequel are understood in the weak sense and we consider their weak solutions with the test function space $C_c^\infty(\mathbb{R}^{2d})$: To find $W_z^\varepsilon \in L^2([0,\infty);L^2(\mathbb{R}^{2d}))$ such that $\|W_z^\varepsilon\|_2 \le \|W_0\|_2, \forall z > 0$, and

$$\langle W_z^\varepsilon,\theta\rangle - \langle W_0,\theta\rangle = k^{-1}\int_0^z \langle W_s^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle\, ds + \frac{k}{\varepsilon}\int_0^z \langle W_s^\varepsilon, \mathcal{L}_s^\varepsilon\theta\rangle\, ds. \tag{23}$$

We shall use the notation

$$\hat{V}_z^\varepsilon(d\mathbf{q}) = \hat{V}(\frac{z}{\varepsilon^2},d\mathbf{q}), \quad V_z^\varepsilon(\mathbf{x}) = V(\frac{z}{\varepsilon^2},\mathbf{x}).$$

Taking the partial inverse Fourier transform

$$\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \equiv \int e^{i\mathbf{p}\cdot\mathbf{y}}\theta(\mathbf{x},\mathbf{p})\, d\mathbf{p}$$

we see that $\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon\theta(\mathbf{x},\mathbf{y})$ acts in the following completely local manner

$$\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = -i\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \tag{24}$$

where

$$\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y}) \equiv V_z^\varepsilon(\tilde{\mathbf{x}}+\mathbf{y}/2) - V_z^\varepsilon(\tilde{\mathbf{x}}-\mathbf{y}/2) \tag{25}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$. The operator $\mathcal{L}_z^\varepsilon$ is skew-symmetric and real (i.e. mapping real-valued functions to real-valued functions).

Since our results do not depend on the transverse dimension d we hereafter take it to be any positive integer.

Remark 1. Since Eq. (23) is linear, the existence of weak solutions can be established straightforwardly by the weak-$\star$ compactness argument. Let us briefly comment on this. First, we introduce truncation $N < \infty$

$$V_N(z,\mathbf{x}) = V(z,\mathbf{x}), \quad |V(z,\mathbf{x})| < N$$

and zero otherwise. Clearly, for such bounded V_N the corresponding operator $\mathcal{L}_z^\varepsilon$ is a bounded self-adjoint operator on $L^2(\mathbb{R}^{2d})$. Hence the corresponding Wigner-Moyal equation preserves the L^2-norm of the initial data and produces a sequence of L^2-bounded weak solutions. Passing to the limit $N \to \infty$ we obtain a L^2-weak solution for the original Wigner-Moyal equation if V is locally square-integrable as is assumed here. However, due to the weak limiting procedure, there is no guarantee that the L^2-norm of the initial data is preserved in the limit.

We will not address the uniqueness of solution for the Wigner-Moyal equation (23) but we will show that as $\varepsilon \to 0$ any sequence of weak solutions to eq. (23) converges in a suitable sense to the unique solution of a deterministic transport equation.

We state our first result in the following theorem.

Theorem 1. *Let Assumptions 1, 2, 3 and 4 be satisfied. Then the weak solution W_z^ε of the Wigner-Moyal equation (23), (20) with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution W_z of the radiative transfer equation*

$$\frac{\partial W_z(\mathbf{x},\mathbf{p})}{\partial z} + \frac{\mathbf{p}}{k}\cdot\nabla W_z(\mathbf{x},\mathbf{p}) = k^2 \mathcal{L} W_z(\mathbf{x},\mathbf{p})$$

with the initial condition W_0 and one of the following scattering operator $\mathcal{L}$:

Case (i) $0 < \alpha < 1$,

$$\mathcal{L}W_z(\mathbf{x},\mathbf{p}) = 2\pi \int \Phi(0,\mathbf{q}-\mathbf{p})[W_z(\mathbf{x},\mathbf{q}) - W_z(\mathbf{x},\mathbf{p})]d\mathbf{q}; \tag{26}$$

Case (ii) $\alpha = 1$,

$$\mathcal{L}W_z(\mathbf{x},\mathbf{p}) = 2\pi \int \Phi(\frac{|\mathbf{q}|^2 - |\mathbf{p}|^2}{2k}, \mathbf{q}-\mathbf{p})[W_z(\mathbf{x},\mathbf{q}) - W_z(\mathbf{x},\mathbf{p})]d\mathbf{q}; \tag{27}$$

Case (iii) $\alpha > 1$,

$$\mathcal{L}W_z = 0. \tag{28}$$

The case of $\alpha = 0$ corresponds to the so-called white-noise scaling whose limit is a Markovian process [6]. Eq. (27) has recently been obtained in [3] for strongly mixing z-Markovian refractive index fields with a bounded generator.

In order to obtain a nontrivial scattering kernel for $\alpha > 1$ we need to boost up the intensity of V (cf. Theorem 3).

Next we consider a second type of scaling limits which starts with the highly anisotropic medium $V(z, \varepsilon^{2-2\alpha}\mathbf{x})$. We then set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^{2\alpha-1}, \quad 0 < \alpha < 1 \tag{29}$$

under which the parabolic wave equation becomes

$$ik^{-1}\frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1}k^{-2}\varepsilon^2 \Delta\Psi^\varepsilon + \varepsilon^{2\alpha-3}V(z\varepsilon^{-2}, \mathbf{x}\varepsilon^{-2\alpha})\Psi^\varepsilon = 0, \tag{30}$$
$$\Psi^\varepsilon(0,\mathbf{x}) = \Psi_0(\mathbf{x}).$$

The radiative transfer scaling (17) is the limiting case $\alpha = 1$. The time-evolution of the Wigner function (18) is governed by the Wigner-Moyal equation (23) with the following operator $\mathcal{L}_z^\varepsilon$

$$\begin{aligned} &\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) \\ &= i\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\varepsilon^{2\alpha-2}\left[W_z^\varepsilon(\mathbf{x}, \mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x}, \mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\right]\widehat{V}_z^\varepsilon(d\mathbf{q}), \end{aligned} \tag{31}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$. The partial Fourier transform of $\mathcal{L}_z^\varepsilon\theta$ is now given by (24) with the following $\delta_\varepsilon V_z$

$$\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}}, \mathbf{y}) = \varepsilon^{2\alpha-2}\left[V_z^\varepsilon(\tilde{\mathbf{x}} + \mathbf{y}\varepsilon^{2-2\alpha}/2) - V_z^\varepsilon(\tilde{\mathbf{x}} - \mathbf{y}\varepsilon^{2-2\alpha}/2)\right]. \tag{32}$$

We now state the result for the scaling limit (29), (30).

Theorem 2. *Let $0 < \alpha < 1$. Let Assumptions 1, 2, 3 and 4 be satisfied. Then the weak solution W_z^ε of the Wigner-Moyal equation (23), (31) with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution W_z of the following Fokker-Planck equations with the initial condition W_0:*

$$\frac{\partial W_z}{\partial z} + \frac{\mathbf{p}}{k}\cdot\nabla W_z = k^2\nabla_{\mathbf{p}}\cdot\mathbf{D}\nabla_{\mathbf{p}}W_z, \tag{33}$$

with one of the following diffusion tensors $\mathbf{D}$.

– $\alpha \in (0,1)$*:*

$$\mathbf{D} = \pi\int \Phi(0,\mathbf{q})\mathbf{q}\otimes\mathbf{q}d\mathbf{q}; \tag{34}$$

– $\alpha > 1$*:*

$$\mathbf{D} = 0.$$

For $\alpha = 1$ the limit is the same as that in Theorem 1 Case (ii). $\alpha = 0$ gives rise to the white-noise limit for the Liouville equation. The Fokker-Planck equation (33) can be obtained from (26) under the geometrical optics limit of the latter.

Let us consider yet another type of scaling limit parametrized by β. We first assume a highly anisotropic medium $V(\varepsilon^{2-2\beta}z, \mathbf{x})$ and set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon \tag{35}$$

i.e. the standard radiative transfer scaling. The Schrödinger equation then becomes

$$ik^{-1}\frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1}k^{-2}\varepsilon^2 \Delta\Psi^\varepsilon + \varepsilon^{-1}V(z\varepsilon^{-2\beta}, \mathbf{x}\varepsilon^{-2})\Psi^\varepsilon = 0, \tag{36}$$
$$\Psi^\varepsilon(0,\mathbf{x}) = \Psi_0(\mathbf{x}),$$

the corresponding Wigner-Moyal equation is

$$\frac{\partial W_z^\varepsilon}{\partial z} + \frac{\mathbf{p}}{k}\cdot\nabla W_z^\varepsilon + \frac{k}{\varepsilon^\beta}\mathcal{L}_z^\varepsilon W_z^\varepsilon = 0 \tag{37}$$

with

$$\begin{aligned}&\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \\ &= i\varepsilon^{\beta-1}\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p}+\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^{2\beta}}, d\mathbf{q}),\end{aligned} \tag{38}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2}$.

Eq. (38) is a borderline case of the following family of scaling limits. Let us consider probing an anisotropic medium

$$V(\varepsilon^{2-2\beta}z, \varepsilon^{2-2\alpha}\mathbf{x}), \quad \alpha,\beta > 0,$$

with a wave beam composed of waves of lengths comparable to that of the medium, so we switch to (21) and (22) for the formulation of scaling limits.

Three situations arise: Case (i) $\alpha < \beta$, Case (ii) $\alpha > \beta$ and Case (iii) $\alpha = \beta$.

In the first case $\alpha < \beta$ we set the strength of the medium fluctuation to be

$$\sigma = \varepsilon^{2\alpha-\beta}.$$

The resulting equation is (37) with

$$\begin{aligned}&\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \\ &= i\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\left[W_z^\varepsilon(\mathbf{x},\mathbf{p}+\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^{2\beta}}, d\mathbf{q}),\end{aligned} \tag{39}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$. In the second case $\alpha > \beta$ we set the strength of the medium fluctuation to be

$$\sigma = \varepsilon^{\alpha}.$$

The resulting equation is (37) with

$$\begin{aligned}&\mathcal{L}^\varepsilon_z W^\varepsilon_z(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \qquad (40)\\ &= i\varepsilon^{\beta-\alpha}\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\left[W^\varepsilon_z(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-W^\varepsilon_z(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^{2\beta}},d\mathbf{q}),\end{aligned}$$

with $\tilde{\mathbf{x}}=\mathbf{x}\varepsilon^{-2\alpha}$.

In the third case $\alpha=\beta$ the strength of the medium fluctuation is

$$\sigma=\varepsilon^\alpha.$$

The resulting equation is eq. (37) with $\mathcal{L}^\varepsilon_z$ given by either (39) or (40).

We have the following theorem.

Theorem 3. *Let $\alpha,\beta>0$. Let Assumptions 1, 2, 3 and 4 be satisfied. Then the weak solution W^ε_z of the Wigner-Moyal equation (37), with (39) for $\alpha\le\beta$ or with (40) for $\alpha>\beta$, and the initial condition $W_0\in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution W_z of the transport equation with the initial condition W_0:*

$$\frac{\partial W_z}{\partial z}+\frac{\mathbf{p}}{k}\cdot\nabla W_z=k^2\mathcal{L}W_z, \qquad (41)$$

with one of the following the scattering operators $\mathcal{L}$.

Case (i) $\alpha<\beta$

$$\mathcal{L}W_z(\mathbf{x},\mathbf{p})=2\pi\int\Phi(0,\mathbf{q}-\mathbf{p})\left[W_z(\mathbf{x},\mathbf{q})-W_z(\mathbf{x},\mathbf{p})\right]d\mathbf{q}. \qquad (42)$$

Case (ii) $1<\alpha/\beta<4/3, d\ge 3$

$$\begin{aligned}&\mathcal{L}W_z(\mathbf{x},\mathbf{p}) \qquad (43)\\ &=2\pi\int\delta(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k})\left[\int\Phi(w,\mathbf{q}-\mathbf{p})dw\right]\left[W_z(\mathbf{x},\mathbf{q})-W_z(\mathbf{x},\mathbf{p})\right]d\mathbf{q}.\end{aligned}$$

Case (iii) $\alpha=\beta$

$$\mathcal{L}W_z(\mathbf{x},\mathbf{p})=2\pi\int\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k},\mathbf{q}-\mathbf{p})\left[W_z(\mathbf{x},\mathbf{q})-W_z(\mathbf{x},\mathbf{p})\right]d\mathbf{q} \qquad (44)$$

Theorem 3 (i) probably holds for $d=2$ and $\alpha/\beta>4/3$ but we do not pursue it here in order to keep the argument as simple as possible.

Earlier [19], [5] have established the convergence of the mean field $\mathbb{E}W^\varepsilon_z$ for z-independent Gaussian media and $d\ge 3$. Their transport equation can be viewed as a limiting case of (41) in which $\Phi(\xi,\mathbf{k})$ is a δ-function concentrated at $\xi=0$. See also [18] for mean-field results for z-finitely dependent potentials.

Unlike the transport equations (27), (26), the scattering kernel (43) is elastic in the sense that it preserves the kinetic energy of the scattered particle so that the incoming and outgoing momenta $\mathbf{q}, \mathbf{p}$ have the same magnitude.

Finally let us consider two other types of scaling limit starting with the slowly-varying, anisotropic refractive index field $V(\varepsilon^{2-2\beta}z, \varepsilon^{2-2\alpha}\mathbf{x}), \alpha, \beta \in (0,1)$. In the first case

$$\beta > \alpha, \quad 0 < \alpha < 1 \tag{45}$$

we set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^{2\alpha-\beta} \tag{46}$$

under which we have the following parabolic wave equation

$$ik^{-1}\frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1}k^{-2}\varepsilon^2 \Delta\Psi^\varepsilon + \varepsilon^{2\alpha-\beta-2}V(z\varepsilon^{-2\beta}, \mathbf{x}\varepsilon^{-2\alpha})\Psi^\varepsilon = 0, \qquad \Psi^\varepsilon(0,\mathbf{x}) = \Psi_0(\mathbf{x}), \tag{47}$$

and the corresponding Wigner-Moyal equation (37) with

$$\begin{aligned}&\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) \\ &= i\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\varepsilon^{2\alpha-2}\left[W_z^\varepsilon(\mathbf{x}, \mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x}, \mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\right]\widehat{V}(\frac{z}{\varepsilon^{2\beta}}, d\mathbf{q}),\end{aligned} \tag{48}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$.

In the second case

$$\alpha > \beta, \quad 0 < \alpha < 1, \tag{49}$$

we set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^{\alpha}. \tag{50}$$

After rescaling the parabolic wave equation reads as follows

$$ik^{-1}\frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1}k^{-2}\varepsilon^2 \Delta\Psi^\varepsilon + \varepsilon^{\alpha-2}V(z\varepsilon^{-2\beta}, \mathbf{x}\varepsilon^{-2\alpha})\Psi^\varepsilon = 0, \qquad \Psi^\varepsilon(0,\mathbf{x}) = \Psi_0(\mathbf{x}), \tag{51}$$

and the corresponding Wigner-Moyal equation takes the form of (37) with

$$\begin{aligned}\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) = i\varepsilon^{\beta-\alpha}\int \widehat{V}(\frac{z}{\varepsilon^{2\beta}}, d\mathbf{q})e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\varepsilon^{2\alpha-2} \\ \times\left[W_z^\varepsilon(\mathbf{x}, \mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x}, \mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\right]\end{aligned} \tag{52}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$.

In the third case,

$$\alpha = \beta, \alpha \in (0,1) \tag{53}$$

we set

$$\sigma = \varepsilon^{\alpha}.$$

The resulting equation is (37) with either (48) or (52).

Theorem 4. *Let $\alpha, \beta \in (0,1)$. Let Assumptions 1, 2, 3 and 4 be satisfied. Then the weak solution W_z^ε of the Wigner-Moyal equation (37), with (48) for $\alpha \leq \beta$ or with (52) otherwise, and the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution W_z of the Fokker-Planck equation (33) with the following diffusion tensors:*

Case (i)-(45), (46)

$$\mathbf{D} = \pi \int \Phi(0, \mathbf{q})\mathbf{q} \otimes \mathbf{q} d\mathbf{q}. \tag{54}$$

Case (ii)-(49), (50) $d \geq 3$, $1 < \alpha/\beta < 4/3$,

$$\mathbf{D}(\mathbf{p}) = \pi k |\mathbf{p}|^{-1} \int \left[\int \Phi(w, \mathbf{p}_\perp) dw \right] \mathbf{p}_\perp \otimes \mathbf{p}_\perp d\mathbf{p}_\perp \tag{55}$$

where $\mathbf{p}_\perp \in \mathbb{R}^{d-1}, \mathbf{p}_\perp \cdot \mathbf{p} = 0$.
Case (iii) $\alpha = \beta$

$$\mathbf{D}(\mathbf{p}) = \pi \int \Phi(k^{-1}\mathbf{p} \cdot \mathbf{q}, \mathbf{q})\mathbf{q} \otimes \mathbf{q}\, d\mathbf{q} \tag{56}$$

The Fokker-Planck equation with (54), (55) and (56) are the geometrical optics limit of the transport equations (43) and (27), respectively. The limiting case of $\alpha = 0$ gives rise to the white-noise model of the Liouville equation [6]. We believe that the result for Case (ii) can be extended to $d = 2$ and $\beta/\alpha \in (0,1)$.

Remark 2. Taken together, our results have roughly covered all the super-parabolic scaling

$$L_x \gg \sqrt{L_z}.$$

To see this, let us set

$$L_z \sim L_x^\gamma = \varepsilon^{-2\gamma}, \quad 0 < \gamma < 2$$

and define the following Wigner transform

$$\begin{aligned} W^\varepsilon(z, \mathbf{x}, \mathbf{p}) &= \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p}\cdot\mathbf{y}} \Psi(z, \mathbf{x} + \frac{\tilde{\varepsilon}^2 \mathbf{y}}{2}) \Psi^*(z, \mathbf{x} - \frac{\tilde{\varepsilon}^2 \mathbf{y}}{2}) d\mathbf{y} \\ &= \frac{1}{(2\pi)^d} \int \int e^{-i\mathbf{p}\cdot(\mathbf{y}_1 - \mathbf{y}_2)/\tilde{\varepsilon}^2} \delta(\mathbf{x} - \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}) \rho(\mathbf{y}_1, \mathbf{y}_2) d\mathbf{y}_1 d\mathbf{y}_2 \end{aligned}$$

with the new parameter

$$\tilde{\varepsilon} = \varepsilon^{2-\gamma}$$

and analyze analogously the preceding scaling limits as parametrized by $\tilde{\varepsilon}$. For an alternative treatment of scaling limits resulting in a transport equation, see [7].

Our approach is to use the conditional shift [14] to formulate the corresponding martingale problem parametrized by ε and adapt the perturbed test function technique to the probabilistic setting to establish the convergence of the martingales. It then turns out that after subtracting the drift and the Stratonovich correction term the limiting martingale has null quadratic variation (see Proposition 6) implying that the limit is deterministic. The perturbed test functions constructed here (see e.g. (73), (83) and (84)) are related to those in [2], [3] but our analysis is carried out in a more general framework as formulated in [6] and provides a unified treatment of a range of scaling limits from the radiative transfer to the geometrical optics limit and the white-noise limit.

5. Proof of Theorem 1

5.1. Martingale formulation

We consider the weak formulation of the Wigner-Moyal equation:

$$[\langle W_z^\varepsilon, \theta\rangle - \langle W_0, \theta\rangle] = k^{-1}\int_0^z \langle W_s^\varepsilon, \mathbf{p}\cdot\nabla\theta\rangle\, ds + \frac{k}{\varepsilon}\int_0^z \langle W_s^\varepsilon, \mathcal{L}_z^\varepsilon\theta\rangle\, ds \tag{57}$$

for any test function $\theta \in C_c^\infty(\mathbb{R}^{2d})$. which is a dense subspace in $L^2(\mathbb{R}^{2d})$. The tightness result (see below) implies that for L^2 initial data the limiting measure $\mathbb{P}$ is supported in $L^2([0, z_0]; L^2(\mathbb{R}^{2d}))$.

For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $V_z^\varepsilon \equiv V(z/\varepsilon^2, \cdot)$. Let $\mathcal{F}_z^\varepsilon$ be the σ-algebras generated by $\{V_s^\varepsilon,\ s \le z\}$ and $\mathbb{E}_z^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_z^\varepsilon$. Let $\mathcal{M}^\varepsilon$ be the space of measurable function adapted to $\{\mathcal{F}_z^\varepsilon, \forall t\}$ such that $\sup_{z<z_0}\mathbb{E}|f(z)| < \infty$. We say $f(\cdot) \in \mathcal{D}(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f = g$ if $f, g \in \mathcal{M}^\varepsilon$ and for $f^\delta(z) \equiv \delta^{-1}[\mathbb{E}_z^\varepsilon f(z+\delta) - f(z)]$ we have

$$\sup_{z,\delta} \mathbb{E}|f^\delta(z)| < \infty$$
$$\lim_{\delta\to 0} \mathbb{E}|f^\delta(z) - g(z)| = 0, \quad \forall z.$$

Consider a special class of admissible functions $f(z) = \phi(\langle W_z^\varepsilon, \theta\rangle), f'(z) = \phi'(\langle W_z^\varepsilon, \theta\rangle), \forall\phi \in C^\infty(\mathbb{R})$ we have the following expression from (57) and the chain rule

$$\mathcal{A}^\varepsilon f(z) = f'(z)\left[\frac{1}{k}\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla\theta\rangle + \frac{k}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\rangle\right]. \tag{58}$$

In case of the test function θ that is also a functional of the media we have

$$\mathcal{A}^\varepsilon f(z) = f'(z)\left[\frac{1}{k}\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla\theta\rangle + \frac{k}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\rangle + \langle W_z^\varepsilon, \mathcal{A}^\varepsilon\theta\rangle\right] \tag{59}$$

and when θ depends explicitly on the fast spatial variable

$$\tilde{\mathbf{x}} = \mathbf{x}/\varepsilon^{2\alpha}$$

the gradient ∇ is conveniently decomposed into the gradient w.r.t. the slow variable $\nabla_{\mathbf{x}}$ and that w.r.t. the fast variable $\nabla_{\tilde{\mathbf{x}}}$

$$\nabla = \nabla_{\mathbf{x}} + \varepsilon^{-2\alpha}\nabla_{\tilde{\mathbf{x}}}.$$

A main property of $\mathcal{A}^\varepsilon$ is that

$$f(z) - \int_0^z \mathcal{A}^\varepsilon f(s)ds \quad \text{is a } \mathcal{F}_z^\varepsilon\text{-martingale}, \quad \forall f \in \mathcal{D}(\mathcal{A}^\varepsilon). \tag{60}$$

Also,

$$\mathbb{E}_s^\varepsilon f(z) - f(s) = \int_s^z \mathbb{E}_s^\varepsilon \mathcal{A}^\varepsilon f(\tau)d\tau \quad \forall s < z \quad \text{a.s.} \tag{61}$$

(see [14]). We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $V_z(\cdot) = V(z,\cdot)$.

5.2. Tightness

In the sequel we will adopt the following notation

$$f(z) \equiv \phi(\langle W_z^\varepsilon, \theta\rangle), f'(z) \equiv \phi'(\langle W_z^\varepsilon, \theta\rangle), f''(z) \equiv \phi''(\langle W_z^\varepsilon, \theta\rangle), \forall \phi \in C^\infty(\mathbb{R}).$$

Namely, the prime stands for the differentiation w.r.t. the original argument (not t) of f, f' etc.

Let $D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ be the L^2-valued right continuous processes with left limits endowed with the Skorohod topology. A family of processes $\{W^\varepsilon, 0 < \varepsilon < 1\} \subset D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ is tight if and only if the family of processes $\{\langle W^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\} \subset D([0,\infty); L^2_w(\mathbb{R}^{2d}))$ is tight for all $\theta \in C_c^\infty$ [10]. We use the tightness criterion of [15] (Chap. 3, Theorem 4), namely, we will prove: Firstly,

$$\lim_{N\to\infty} \limsup_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |\langle W_z^\varepsilon, \theta\rangle| \geq N\} = 0, \quad \forall z_0 < \infty. \tag{62}$$

Secondly, for each $\phi \in C^\infty(\mathbb{R})$ there is a sequence $f^\varepsilon(z) \in \mathcal{D}(\mathcal{A}^\varepsilon)$ such that for each $z_0 < \infty$ $\{\mathcal{A}^\varepsilon f^\varepsilon(z), 0 < \varepsilon < 1, 0 < z < z_0\}$ is uniformly integrable and

$$\lim_{\varepsilon\to 0} \mathbb{P}\{\sup_{z<z_0} |f^\varepsilon(z) - \phi(\langle W_z^\varepsilon, \theta\rangle)| \geq \delta\} = 0, \quad \forall \delta > 0. \tag{63}$$

Then it follows that the laws of $\{\langle W^\varepsilon, \theta\rangle, 0 < \varepsilon < 1\}$ are tight in the space of $D([0,\infty); \mathbb{R})$.

First, condition (62) is satisfied because the L^2-norm is uniformly bounded.

Let

$$\tilde{\mathcal{L}}^\varepsilon_z\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \equiv i\varepsilon^{-2}\int_z^\infty \int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)] \quad (64)$$
$$\times e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}^\varepsilon_z\hat{V}^\varepsilon_s(d\mathbf{q})ds.$$

Note that the operator $\tilde{\mathcal{L}}^\varepsilon_z$ maps a real-valued function θ to a real-valued z-stationary random function $\tilde{\mathcal{L}}^\varepsilon_z\theta$.

We have the following estimate.

Lemma 1.

$$\mathbb{E}\left[\tilde{\mathcal{L}}^\varepsilon_z\theta\right]^2(\mathbf{x},\mathbf{p}) \quad (65)$$
$$\leq \left[\int_0^\infty \rho(s)ds\right]^2\int\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.$$

Proof. Consider the following trial functions in the definition of the maximal correlation coefficient

$$\begin{aligned}
h &= h_s(\mathbf{x},\mathbf{p})\\
&= i\int e^{i\mathbf{q}\cdot\mathbf{x}\varepsilon^{-2\alpha}}[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}^\varepsilon_z\hat{V}^\varepsilon_s(d\mathbf{q})\\
g &= g_t(\mathbf{x},\mathbf{p})\\
&= i\int e^{i\mathbf{q}\cdot\mathbf{x}\varepsilon^{-2\alpha}}[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]e^{ik^{-1}(t-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\hat{V}^\varepsilon_t(d\mathbf{q})
\end{aligned}$$

It is easy to see that

$$h_s \in L^2(P,\Omega,\mathcal{F}_{\varepsilon^{-2}z})$$
$$g_t \in \in L^2(P,\Omega,\mathcal{F}^+_{\varepsilon^{-2}t})$$

and their second moments are uniformly bounded in $\mathbf{x},\mathbf{p},\varepsilon$ since

$$\mathbb{E}[h_s^2](\mathbf{x},\mathbf{p}) \leq \mathbb{E}[g_s^2](\mathbf{x},\mathbf{p}) \quad (66)$$
$$\mathbb{E}[g_s^2](\mathbf{x},\mathbf{p}) = \int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q} \quad (67)$$

which is uniformly bounded for any integrable spectral density Φ.

From the definition (11) we have

$$|\mathbb{E}[h_s(\mathbf{x},\mathbf{p})h_t(\mathbf{y},\mathbf{q})]|$$
$$\leq \rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}\left[h_s^2(\mathbf{x},\mathbf{p})\right]\mathbb{E}^{1/2}\left[g_t^2(\mathbf{y},\mathbf{q})\right].$$

Hence by setting $s=t,\mathbf{x}=\mathbf{y},\mathbf{p}=\mathbf{q}$ first and the Cauchy-Schwartz inequality we have

$$\mathbb{E}\left[h_s^2(\mathbf{x},\mathbf{p})\right] \leq \rho^2(\varepsilon^{-2}(s-z))\mathbb{E}[g_t^2(\mathbf{x},\mathbf{p})]$$

and

$$
\begin{aligned}
&|\mathbb{E}[h_s(\mathbf{x},\mathbf{p})h_t(\mathbf{y},\mathbf{q})]| \\
&\leq \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(s-z))\mathbb{E}^{1/2}[g_t^2(\mathbf{x},\mathbf{p})]\mathbb{E}^{1/2}[g_t^2(\mathbf{y},\mathbf{q})],
\end{aligned}
$$

$\forall s,t \geq z, \forall \mathbf{x},\mathbf{y}$. Hence

$$
\varepsilon^{-4}\int_z^\infty\int_z^\infty \mathbb{E}[h_s(\mathbf{x},\mathbf{p})g_t(\mathbf{x},\mathbf{p})]dsdt \leq \mathbb{E}[g_t^2](\mathbf{x},\mathbf{p})\left[\int_0^\infty \rho(s)ds\right]^2
$$

which together with (67) yields (65). ■

Corollary 1

$$
\begin{aligned}
&\mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta\right]^2(\mathbf{x},\mathbf{p}) &(68)\\
&\leq \left[\int_0^\infty \rho(s)ds\right]^2\int\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.
\end{aligned}
$$

Inequality (68) can be obtained from the expression

$$
\begin{aligned}
&\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\
&\equiv i\varepsilon^{-2}\int_z^\infty\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\mathbf{p}\cdot\nabla_{\mathbf{x}}[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]\\
&\qquad\times e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\mathbb{E}_z^\varepsilon\hat{V}_s^\varepsilon(d\mathbf{q})ds
\end{aligned}
$$

as in Lemma 1.

The main property of $\tilde{\mathcal{L}}_z^\varepsilon\theta$ is that it solves the corrector equation

$$
\left[\varepsilon^{-2\alpha}\frac{\mathbf{p}}{k}\cdot\nabla_{\tilde{\mathbf{x}}}+\mathcal{A}^\varepsilon\right]\tilde{\mathcal{L}}_z^\varepsilon\theta=\varepsilon^{-2}\mathcal{L}_z^\varepsilon\theta. \tag{69}
$$

Eq. (69) can also be solved by using (24), yielding the solution

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) &(70)\\
&=\varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z^\varepsilon\left[\delta_\varepsilon V_s^\varepsilon\right]\mathcal{F}_2^{-1}\theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds
\end{aligned}
$$

where

$$
\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})=V_z^\varepsilon(\tilde{\mathbf{x}}+\mathbf{y}/2)-V_z^\varepsilon(\tilde{\mathbf{x}}-\mathbf{y}/2).
$$

Recall that $\nabla_{\tilde{\mathbf{x}}}$ and $\nabla_{\mathbf{x}}$ are the gradients w.r.t. the fast variable $\tilde{\mathbf{x}}$ and the slow variable $\mathbf{x}$, respectively.

We will need to estimate the iteration of $\mathcal{L}_z^\varepsilon$ and $\tilde{\mathcal{L}}_z^\varepsilon$:

$$
\begin{aligned}
&\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\
&=-\varepsilon^{-2}\int_z^\infty\int\hat{V}_z^\varepsilon(d\mathbf{q})\mathbb{E}_z^\varepsilon[\hat{V}_s^\varepsilon(d\mathbf{q}')]e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}}e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\\
&\quad\Big\{[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}'/2+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}+\mathbf{q}'/2-\mathbf{q}/2)]e^{ik^{-1}(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\\
&\quad-[\theta(\mathbf{x},\mathbf{p}-\mathbf{q}'/2+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}'/2-\mathbf{q}/2)]e^{-ik^{-1}(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\Big\}\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\
&= -\varepsilon^{-4}\int_z^\infty\int_z^\infty\int \mathbb{E}_z^\varepsilon[\hat{V}_s^\varepsilon(d\mathbf{q})]\mathbb{E}_z^\varepsilon[\hat{V}_t^\varepsilon(d\mathbf{q}')]\\
&\quad\times e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}e^{i\mathbf{q}'\cdot\tilde{\mathbf{x}}}e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}e^{ik^{-1}(t-z)\mathbf{p}\cdot\mathbf{q}'/\varepsilon^{2\alpha}}\\
&\quad\times\Big\{[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}'/2+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}+\mathbf{q}'/2-\mathbf{q}/2)]e^{ik^{-1}(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\\
&\quad-[\theta(\mathbf{x},\mathbf{p}-\mathbf{q}'/2+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}'/2-\mathbf{q}/2)]e^{-ik^{-1}(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}\Big\}\,dsdt.
\end{aligned}
$$

Their second moments can be estimated as in Lemma 1 by using the 6-th order sub-Gaussian property (Assumption 3). In order to carry out the same argument we need to approximate the terms of non-product form such as $\theta(\mathbf{x},\mathbf{p}+\mathbf{q}'/2+\mathbf{q}/2)e^{ik^{-1}(s-z)\mathbf{q}'\cdot\mathbf{q}/(2\varepsilon^{2\alpha})}$ by sum of product of functions of variables that are statistically coupled in the pairing.

Since we do not need the pointwise estimate such as stated in Lemma 1 we shall demonstrate a simpler approach based on the inverse Fourier transform:

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) \qquad (71)\\
&=\varepsilon^{-2}\int_z^\infty \delta_\varepsilon V_z^\varepsilon e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) \qquad (72)\\
&=-\varepsilon^{-4}\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\quad\left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})dsdt.
\end{aligned}
$$

Lemma 2.

$$
\begin{aligned}
\mathbb{E}\|\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2 &\le 8C\left(\int_0^\infty\rho(s)ds\right)^2\mathbb{E}[V_z]^2\\
&\quad\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2 &\le 8C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z]^2\\
&\quad\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
\end{aligned}
$$

for some constant C independent of ε.

Proof. Let us consider $\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta$. The calculation for $\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta$ is analogous and simpler.

By the Parseval theorem and the unitarity of $\exp(i\tau\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}), \tau\in\mathbb{R}$,

$$
\begin{aligned}
&\mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2\\
&=\varepsilon^{-8}\int\mathbb{E}\Big\{\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\qquad\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})dsdt\\
&\quad\times\int_z^\infty e^{i\varepsilon^{-2\alpha}k^{-1}(t'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\qquad\Big\{\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]e^{i\varepsilon^{-2\alpha}k^{-1}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds'dt'\Big\}d\mathbf{x}d\mathbf{y}\\
&=\varepsilon^{-8}\int\mathbb{E}\Big\{\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-t')/2\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\qquad\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})dsdt\\
&\quad\times\int_z^\infty e^{i\varepsilon^{-2\alpha}k^{-1}(t-t')/2\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\quad\Big\{\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]e^{i\varepsilon^{-2\alpha}k^{-1}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds'dt'\Big\}d\mathbf{x}d\mathbf{y}\\
&=\varepsilon^{-8}\int\mathbb{E}\Big\{\int_z^\infty\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})dsdt\\
&\quad\times\int_z^\infty\Big\{\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]e^{i\varepsilon^{-2\alpha}k^{-1}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds'dt'\Big\}d\mathbf{x}d\mathbf{y}\\
&\leq C\varepsilon^{-8}\int\int_z^\infty\left|\mathbb{E}\Big\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\right|dsdt\\
&\quad\times\int_z^\infty\left|\mathbb{E}\Big\{\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]e^{i\varepsilon^{-2\alpha}k^{-1}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\Big\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\right|ds'dt'd\mathbf{x}d\mathbf{y}\\
&\quad+C\varepsilon^{-8}\int\int_z^\infty\left|\mathbb{E}[\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]\mathbb{E}_z[\delta_\varepsilon V_{t'}^\varepsilon]]\right|\\
&\qquad\Big|\mathbb{E}\Big\{e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\\
&\qquad\times e^{i\varepsilon^{-2\alpha}k^{-1}(s'-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_{s'}^\varepsilon]\mathcal{F}_2^{-1}\theta\right](\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\Big\}\Big|dsdtds'dt'd\mathbf{x}d\mathbf{y}.
\end{aligned}
$$

The last inequality follows from the sub-Gaussian assumption. Note that in the $\mathbf{x}$ integrals above the fast variable $\tilde{\mathbf{x}}$ is integrated and is not treated as independent of $\mathbf{x}$.

Let

$$g(t)=\delta_\varepsilon V_t^\varepsilon$$

and

$$h(s)=e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\delta_\varepsilon V_s^\varepsilon\mathcal{F}_2^{-1}\theta\right].$$

The same argument as that for Lemma 1 yields that for $t, t', s, s' \geq z$

$$\begin{aligned}
|\mathbb{E}[\mathbb{E}_z[g(t)]\mathbb{E}_z[h(s)]]| &\leq \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2] \\
&\leq \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(s-z))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[h^2(s)]; \\
|\mathbb{E}[\mathbb{E}_z[g(t)]\mathbb{E}_z[g(t')]]| &\leq \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[g(t')]^2] \\
&\leq \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(t'-z))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[g^2(t')]; \\
|\mathbb{E}[\mathbb{E}_z[h(s)]\mathbb{E}_z[h(s')]]| &\leq \mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s')]^2] \\
&\leq \rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(s'-z))\mathbb{E}^{1/2}[h^2(s)]\mathbb{E}^{1/2}[h^2(s')].
\end{aligned}$$

Combining the above estimates we get

$$\begin{aligned}
&\mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 \\
&\leq 2C\left(\int_0^\infty \rho(s)ds\right)^4 \int \mathbb{E}[g(z)]^2\,\mathbb{E}[h(z)]^2 d\mathbf{x}d\mathbf{y} \\
&\leq 2C\left(\int_0^\infty \rho(s)ds\right)^4 \int \mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2 \mathbb{E}\left[e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\delta_\varepsilon V_s^\varepsilon \mathcal{F}_2^{-1}\theta\right]\right]^2 d\mathbf{x}d\mathbf{y} \\
&\leq 8C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z^\varepsilon]^2 \int \mathbb{E}\left[e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\delta_\varepsilon V_s^\varepsilon \mathcal{F}_2^{-1}\theta\right]\right]^2 d\mathbf{x}d\mathbf{y} \\
&\leq 8C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z^\varepsilon]^2 \int \mathbb{E}\left\{e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]\right. \\
&\qquad\qquad \left.\times e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}}\hat{V}_s^\varepsilon(d\mathbf{q})\right\}^2 d\mathbf{x}d\mathbf{p} \\
&\leq 8C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z^\varepsilon]^2 \\
&\quad\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2 \Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
\end{aligned}$$

■

Eq. (72) is convenient for estimating the second moment of $\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta$ and $\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta$ which by (72) and (24) have the following expressions

$$\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) \\
&= i\varepsilon^{-2}\nabla_{\mathbf{y}}\cdot\nabla_{\mathbf{x}}\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}} \\
&\qquad \left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt \\
&= i\varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}} \\
&\qquad \left\{\mathbb{E}_z[\delta_\varepsilon V_t^\varepsilon]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\nabla_{\mathbf{y}}\delta_\varepsilon V_s^\varepsilon]\cdot\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt \\
&\quad + i\varepsilon^{-2}\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}} \\
&\qquad \left\{\mathbb{E}_z[\nabla_{\mathbf{y}}\delta_\varepsilon V_t^\varepsilon]\cdot e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\nabla_{\mathbf{x}}\theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{F}_2^{-1}\left\{\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\\
&= i\varepsilon^{-4}\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})\int_z^\infty e^{-i\varepsilon^{-2\alpha}k^{-1}(t-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\\
&\qquad\left\{\mathbb{E}_z\left[\delta_\varepsilon V_t^\varepsilon\right]e^{-i\varepsilon^{-2\alpha}k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}\left[\mathbb{E}_z\left[\delta_\varepsilon V_s^\varepsilon\right]\mathcal{F}_2^{-1}\theta\right]\right\}(\mathbf{x},\mathbf{y})dsdt.
\end{aligned}
$$

The same calculation as in Lemma 2 yields the following estimates:

Corollary 2

$$
\begin{aligned}
&\mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2\\
&\le 32C\left(\int_0^\infty\rho(s)ds\right)^4\\
&\quad\times\left\{\mathbb{E}[\nabla_{\mathbf{y}}V_z^\varepsilon]^2\int[\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\right.\\
&\quad\left.+\ \mathbb{E}[V_z^\varepsilon]^2\int[\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2|\mathbf{p}|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\right\};
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\|\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2 &\le 32C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z^\varepsilon]^4\\
&\qquad\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
\end{aligned}
$$

for some constant C independent of ε.

Let

$$f_1^\varepsilon(z)=k\varepsilon f'(z)\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle \tag{73}$$

be the 1-st perturbation of $f(z)$.

Proposition 1

$$\lim_{\varepsilon\to0}\sup_{z<z_0}\mathbb{E}|f_1^\varepsilon(z)|=0,\quad \lim_{\varepsilon\to0}\sup_{z<z_0}|f_1^\varepsilon(z)|=0\quad \textit{in probability}$$

.

Proof. We have

$$\mathbb{E}[|f_1^\varepsilon(z)|]\le\varepsilon\|f'\|_\infty\|W_0\|_2\mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2 \tag{74}$$

and

$$\sup_{z<z_0}|f_1^\varepsilon(z)|\le\varepsilon\|f'\|_\infty\|W_0\|_2\sup_{z<z_0}\|\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2. \tag{75}$$

The right side of (74) is $O(\varepsilon)$ while the right side of (75) is $o(1)$ in probability by Chebyshev's inequality and Assumption 4.

Proposition 1 now follows from (74) and (75). ■

Set $f^\varepsilon(z) = f(z) + f_1^\varepsilon(z)$. A straightforward calculation yields

$$\begin{aligned}\mathcal{A}^\varepsilon f_1^\varepsilon &= \varepsilon f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle + \varepsilon f''(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}^\varepsilon\theta\right\rangle\\ &\quad + k^2 f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle + k^2 f''(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\\ &\quad -\frac{k}{\varepsilon}f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\right\rangle\end{aligned}$$

and, hence

$$\begin{aligned}\mathcal{A}^\varepsilon f^\varepsilon(z) &= \frac{1}{k}f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\\ &\quad + k^2 f'(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle + k^2 f''(z)\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\\ &\quad + \varepsilon\left[f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle + f''(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\right]\\ &= A_0^\varepsilon(z) + A_1^\varepsilon(z) + A_2^\varepsilon(z) + R_1^\varepsilon(z) \end{aligned} \tag{76}$$

where $A_1^\varepsilon(z)$ and $A_2^\varepsilon(z)$ are the $O(1)$ statistical coupling terms.

Proposition 2

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_1^\varepsilon(z)|^2 = 0$$

.

Proof.

$$|R_1^\varepsilon| \le \varepsilon\left[\|f''\|_\infty\|W_0\|_2^2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\|_2\|\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2 + \|f'\|_\infty\|W_z^\varepsilon\|_2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^\varepsilon\theta)\|_2\right].$$

Clearly

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R_1^\varepsilon(z)|^2 = 0.$$

by Lemma 1 and Corollary 1. ■

For the tightness criterion stated in the beginnings of the section, it remains to show

Proposition 3 $\{\mathcal{A}^\varepsilon f^\varepsilon\}$ *are uniformly integrable.*

Proof. We show that $\{A_i^\varepsilon\}, i = 0,1,2,3$ are uniformly integrable.

For this we have the following estimates:

$$\begin{aligned}|A_0^\varepsilon(z)| &\le \frac{1}{k}\|f'\|_\infty\|W_0\|_2\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\|_2\\ |A_1^\varepsilon(z)| &\le k^2\|f'\|_\infty\|W_0\|_2\|\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2\\ |A_2^\varepsilon(z)| &\le k^2\|f''\|_\infty\|W_0\|_2^2\|\mathcal{L}_z^\varepsilon\theta\|_2\|\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2.\end{aligned}$$

The second moments of the right hand side of the above expressions are uniformly bounded as $\varepsilon \to 0$ by Lemmas 1 and 2 and hence $A_0^\varepsilon(z), A_1^\varepsilon(z), A_2^\varepsilon(z)$ are uniformly integrable. By Proposition 2, R_1^ε is uniformly integrable. ■

5.3. Identification of the limit

Our strategy is to show directly that in passing to the weak limit the limiting process solves the martingale problem with zero quadratic variation. The uniqueness of the limiting deterministic problem then identifies the limit.

For this purpose, we introduce the next perturbations $f_2^\varepsilon, f_3^\varepsilon$. Let

$$A_2^{(1)}(\psi) \equiv \int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q} \tag{77}$$

$$A_1^{(1)}(\psi) \equiv \int \mathcal{Q}_1'\theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})\;d\mathbf{x}d\mathbf{p}, \quad \forall\psi\in L^2(\mathbb{R}^{2d}) \tag{78}$$

where

$$\mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) = \mathbb{E}\left[\mathcal{L}_z^\varepsilon\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{y},\mathbf{q})\right] \tag{79}$$

and

$$\mathcal{Q}_1'\theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\mathbf{p})\right].$$

Clearly,

$$A_2^{(1)}(\psi) = \mathbb{E}\left[\langle\psi,\mathcal{L}_z^\varepsilon\theta\rangle\left\langle\psi,\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\right]. \tag{80}$$

Let

$$\mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q}) \equiv \mathbb{E}\left[\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{y},\mathbf{q})\right]$$

and

$$\mathcal{Q}_2'\theta(\mathbf{x},\mathbf{p}) = \mathbb{E}\left[\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\mathbf{p})\right].$$

Let

$$A_2^{(2)}(\psi) \equiv \int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\;d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q} \tag{81}$$

$$A_1^{(2)}(\psi) \equiv \int \mathcal{Q}_2'\theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})\;d\mathbf{x}\,d\mathbf{p} \tag{82}$$

Define

$$f_2^\varepsilon(z) = \frac{\varepsilon^2k^2}{2}f''(z)\left[\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right] \tag{83}$$

$$f_3^\varepsilon(z) = \frac{\varepsilon^2k^2}{2}f'(z)\left[\left\langle W_z^\varepsilon,\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle - A_1^{(2)}(W_z^\varepsilon)\right]. \tag{84}$$

Proposition 4

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_2^\varepsilon(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|f_3^\varepsilon(z)| = 0.$$

Proof. We have the bounds

$$\sup_{z<z_0} \mathbb{E}|f_2^\varepsilon(z)| \leq \sup_{z<z_0} \varepsilon^2 k^2 \|f''\|_\infty \left[\|W_0\|_2^2 \mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 + \mathbb{E}[A_2^{(2)}(W_z^\varepsilon)]\right]$$
$$\sup_{z<z_0} \mathbb{E}|f_3^\varepsilon(z)| \leq \sup_{z<z_0} \varepsilon^2 k^2 \|f'\|_\infty \left[\|W_0\|_2 \mathbb{E}\|\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2 + \mathbb{E}[A_1^{(2)}(W_z^\varepsilon)]\right].$$

The right sides of both tend to zero as $\varepsilon \to 0$ by Lemma 1 and 2. ■

We have

$$\mathcal{A}^\varepsilon f_2^\varepsilon(z) = k^2 f''(z) \left[-\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \theta\rangle \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta\right\rangle + A_2^{(1)}(W_z^\varepsilon)\right] + R_2^\varepsilon(z)$$
$$\mathcal{A}^\varepsilon f_3^\varepsilon(z) = k^2 f'(z) \left[-\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon(\tilde{\mathcal{L}}_z^\varepsilon \theta)\right\rangle + A_1^{(1)}(W_z^\varepsilon)\right] + R_3^\varepsilon(z)$$

with

$$\begin{aligned} R_2^\varepsilon(z) = {}& \varepsilon^2 k^2 \frac{f'''(z)}{2} \left[\frac{1}{k}\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle + \frac{k}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\rangle\right] \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right] \\ & + \varepsilon^2 k^2 f''(z) \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta\right\rangle \left[\frac{1}{k}\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^\varepsilon\theta)\right\rangle + \frac{k}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\right] \\ & - \varepsilon^2 k^2 f''(z) \left[\frac{1}{k}\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\theta^{(2)} W_z^\varepsilon)\right\rangle + \frac{k}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon\right\rangle\right] \end{aligned} \tag{85}$$

where $G_\theta^{(2)}$ denotes the operator

$$G_\theta^{(2)}\psi \equiv \int \mathcal{Q}_2(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\, d\mathbf{y} d\mathbf{q}.$$

Similarly

$$\begin{aligned} & R_3^\varepsilon(z) \\ & = \varepsilon^2 k^2 f'(z) \left[\frac{1}{k}\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta)\right\rangle + \frac{k}{\varepsilon}\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\right] \\ & \quad + \varepsilon^2 \frac{k^2}{2} f''(z) \left[\frac{1}{k}\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\rangle + \frac{k}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\theta\rangle\right] \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle - A_1^{(2)}(W_z^\varepsilon)\right] \\ & \quad - \varepsilon^2 k^2 f'(z) \left[\frac{1}{k}\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(\mathcal{Q}_2'\theta)\rangle + \frac{k}{\varepsilon}\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon\mathcal{Q}_2'\theta\rangle\right]. \end{aligned} \tag{86}$$

Proposition 5

$$\lim_{\varepsilon\to 0}\sup_{z<z_0} \mathbb{E}|R_2^\varepsilon(z)| = 0, \quad \lim_{\varepsilon\to 0}\sup_{z<z_0} \mathbb{E}|R_3^\varepsilon(z)| = 0.$$

Proof. Part of the argument is analogous to that given for Proposition 4. The additional estimates that we need to consider are the following.

In R_2^ε: First

$$
\begin{aligned}
&\sup_{z<z_0} \varepsilon^2 \mathbb{E}\left|\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}(G_\theta^{(2)} W_z^\varepsilon)\right\rangle\right| \\
&= \varepsilon^2 \int \mathbb{E}[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})]\, \mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\mathbf{p})\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{y},\mathbf{q})\right] d\mathbf{x}d\mathbf{y}d\mathbf{p}d\mathbf{q} \\
&\le \varepsilon^2 \int \mathbb{E}[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})]\, \mathbb{E}^{1/2}[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta]^2(\mathbf{x},\mathbf{p})\mathbb{E}^{1/2}[\tilde{\mathcal{L}}_z^\varepsilon\theta]^2(\mathbf{y},\mathbf{q})d\mathbf{x}d\mathbf{y}d\mathbf{p}d\mathbf{q}
\end{aligned}
$$

which is $O(\varepsilon^2)$ by using Lemma 1, Corollary 1 and the fact $\mathbb{E}[W_z^\varepsilon(\mathbf{x},\mathbf{p})W_z^\varepsilon(\mathbf{y},\mathbf{q})] \in L^2(\mathbb{R}^{4d})$ in conjunction with the same argument as in proof of Lemma 1; Secondly,

$$
\begin{aligned}
&\sup_{z<z_0} \varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon\right\rangle\right| \\
&= \sup_{z<z_0} \varepsilon\|W_0\|_2 \mathbb{E}\|\mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon\|_2 \\
&= \sup_{z<z_0} \varepsilon\|W_0\|_2 \mathbb{E}\|\mathcal{L}_z^\varepsilon \mathbb{E}\left[\tilde{\mathcal{L}}_z^\varepsilon\theta\otimes\tilde{\mathcal{L}}_z^\varepsilon\theta\right] W_z^\varepsilon\|_2 \\
&= \sup_{z<z_0} \varepsilon\|W_0\|_2 \mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon \mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\right] \mathcal{F}_2^{-1}W_z^\varepsilon\|_2.
\end{aligned}
$$

Let

$$
h_s = e^{-ik^{-1}\varepsilon^{-2\alpha}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}}[\delta_\varepsilon V_z^\varepsilon \mathcal{F}_2^{-1}\theta].
$$

We then have

$$
\begin{aligned}
&\mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon \mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\right] \mathcal{F}_2^{-1}W_z^\varepsilon\|_2 \\
&= \mathbb{E}\left\{\int\left|\varepsilon^{-4}\int\int_z^\infty \delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})\mathbb{E}\left[\mathbb{E}_z[h_s(\mathbf{x},\mathbf{y})]\mathbb{E}_z[h_t(d\mathbf{x}',d\mathbf{y}')]\right]\right.\right. \\
&\qquad \left.\left.\times\mathcal{F}_2^{-1}W_z^\varepsilon(\mathbf{x}',\mathbf{y}')d\mathbf{x}'d\mathbf{y}'dsdt\right|^2 d\mathbf{x}d\mathbf{y}\right\}^{1/2} \\
&\le \mathbb{E}^{1/2}\left\{\int\left|\varepsilon^{-4}\int_z^\infty |\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})|\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}[h_s(\mathbf{x},\mathbf{y})]^2\right.\right. \\
&\qquad \left.\left.\int\mathbb{E}^{1/2}[h_t(d\mathbf{x}',d\mathbf{y}')]^2|\mathcal{F}_2^{-1}W_z^\varepsilon(\mathbf{x}',\mathbf{y}')|d\mathbf{x}'d\mathbf{y}'dsdt\right|^2 d\mathbf{x}d\mathbf{y}\right\}^2 \\
&\le \mathbb{E}^{1/2}\left\{\int\left|\varepsilon^{-4}\int_z^\infty |\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})|\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}[h_s(\mathbf{x},\mathbf{y})]^2\right.\right. \\
&\qquad \left.\left.\left(\int\mathbb{E}[h_t(d\mathbf{x}',d\mathbf{y}')]^2 d\mathbf{x}'d\mathbf{y}'\right)\left(\int|W_z^\varepsilon(\mathbf{x}',\mathbf{p}')|^2 d\mathbf{x}'d\mathbf{p}'\right)dsdt\right|^2 d\mathbf{x}d\mathbf{y}\right\}.
\end{aligned}
$$

Recall that $\|W_z^\varepsilon\|_2 \le \|W_0\|_2$ and

$$
\int\mathbb{E}[h_t(d\mathbf{x}',d\mathbf{y}')]^2 d\mathbf{x}'d\mathbf{y}' = \int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}d\mathbf{x}d\mathbf{p} < \infty
$$

so that

$$
\begin{aligned}
&\mathbb{E}\|\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon\mathbb{E}\left[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\otimes\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta\right]\mathcal{F}_2^{-1}W_z^\varepsilon\|_2\\
&\leq\|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^2\mathbb{E}^{1/2}\left\{\int\left|\varepsilon^{-4}\int_z^\infty\right.\right.\\
&\quad\left.\left.|\delta_\varepsilon V_z^\varepsilon(\mathbf{x},\mathbf{y})|\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\mathbb{E}^{1/2}[h_s(\mathbf{x},\mathbf{y})]^2dsdt\right|^2d\mathbf{x}d\mathbf{y}\right\}\\
&\leq\|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2\right)\varepsilon^{-8}\int_z^\infty\rho(\varepsilon^{-2}(s-z))\rho(\varepsilon^{-2}(t-z))\\
&\qquad\qquad\times\rho(\varepsilon^{-2}(s'-z))\rho(\varepsilon^{-2}(t'-z))\mathbb{E}^{1/2}\|h_s\|_2^2\mathbb{E}^{1/2}\|h_{s'}\|_2^2dsdtds'dt'\\
&\leq\|W_0\|_2\mathbb{E}^{3/2}\|h_s\|_2^2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}[\delta_\varepsilon V_z^\varepsilon]^2\right)\left|\int_0^\infty\rho(s)ds\right|^2<\infty.
\end{aligned}
$$

Recall from (67) that

$$
\mathbb{E}\|h_s\|_2^2=\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}d\mathbf{x}d\mathbf{p}<\infty.
$$

Hence

$$
\sup_{z<z_0}\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathcal{L}_z^\varepsilon G_\theta^{(2)}W_z^\varepsilon\right\rangle\right|=O(\varepsilon).
$$

In R_3^ε:

$$
\sup_{z<z_0}\varepsilon\mathbb{E}\left|\left\langle W_z^\varepsilon,\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\right\rangle\right|\leq\varepsilon\|W_0\|_2\sup_{z<z_0}\mathbb{E}\|\mathcal{L}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2
$$

which is $O(\varepsilon)$ by Corollary 2.

The two other terms in R_3^ε

$$
\varepsilon^2\mathbb{E}|\langle W_z^\varepsilon,\mathbf{p}\cdot\nabla_\mathbf{x}(\mathcal{Q}_2'\theta)\rangle|\leq\varepsilon^2\|W_0\|_2\|\mathbf{p}\cdot\nabla_\mathbf{x}\mathbb{E}[\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta]\|_2
$$

which is $O(\varepsilon^2)$ by Corollary 2;

$$
\begin{aligned}
\varepsilon\mathbb{E}|\langle W_z^\varepsilon,\mathcal{L}_z^\varepsilon\mathcal{Q}_2'\theta\rangle|&\leq\varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{L}_z^\varepsilon\mathbb{E}[\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta]\|_2\\
&\leq\varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{F}_2^{-1}[\mathcal{L}_z^\varepsilon]\mathbb{E}[\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta]\|_2\\
&\leq\varepsilon\|W_0\|_2\left(\sup_{\mathbf{x},\mathbf{y}}\mathbb{E}^{1/2}|\delta_\varepsilon V_z^\varepsilon|^2\right)\mathbb{E}^{1/2}\|\tilde{\mathcal{L}}_z^\varepsilon\tilde{\mathcal{L}}_z^\varepsilon\theta\|_2^2
\end{aligned}
$$

which is $O(\varepsilon)$ by Lemma 2.

■

Consider the test function $f^\varepsilon(z)=f(z)+f_1^\varepsilon(z)+f_2^\varepsilon(z)+f_3^\varepsilon(z)$. We have

$$
\begin{aligned}
\mathcal{A}^\varepsilon f^\varepsilon(z)=&\frac{1}{k}f'(z)\langle W_z^\varepsilon,\mathbf{p}\cdot\nabla_\mathbf{x}\theta\rangle+k^2f''(z)A_2^{(1)}(W_z^\varepsilon)+k^2f'A_1^{(1)}(W_z^\varepsilon)\\
&+R_1^\varepsilon(z)+R_2^\varepsilon(z)+R_3^\varepsilon(z).
\end{aligned}\tag{87}
$$

Set

$$R^\varepsilon(z) = R_1^\varepsilon(z) + R_2^\varepsilon(z) + R_3^\varepsilon(z). \tag{88}$$

It follows from Propositions 3 and 5 that

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\mathbb{E}|R^\varepsilon(z)| = 0.$$

Proposition 6

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\sup_{\|\psi\|_2=1} A_2^{(1)}(\psi) = 0.$$

Proof. We have

$$A_2^{(1)}(\psi) = \frac{1}{2}\int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1^s(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\, d\mathbf{x}d\mathbf{p}\, d\mathbf{y}d\mathbf{q}$$

with the symmetrized kernel

$$\begin{aligned}
&\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{y},\mathbf{q},\mathbf{x},\mathbf{p}) + \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \int_{-\infty}^{\infty} ds\int d\mathbf{p}'\check{\Phi}(s,\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha}}\\
&\qquad\times[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}'/2)]\,[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}'/2)]\\
&= 2\pi\int e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}\,[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}'/2)]\\
&\qquad\times[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}'/2)]\,\Phi(k^{-1}\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha},\mathbf{p}')d\mathbf{p}'.
\end{aligned}$$

which, because of Assumption 1, tends to zero uniformly outside any neighborhood of $\mathbf{x}=\mathbf{y}$ while stays uniformly bounded everywhere. Therefore the L^2-norm of $\mathcal{Q}_1^s$ tends to zero and the proposition follows. ■

Similar calculation leads to the following expression: For any real-valued, L^2-weakly convergent sequence $\psi^\varepsilon\to\psi$, we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi)\\
&= \lim_{\varepsilon\to 0}\int_0^{\infty} ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\\
&\qquad\times\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{2-2\alpha}/2}\,[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})]\\
&\qquad\quad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{2-2\alpha}/2}\,[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})]\Big].
\end{aligned}$$

Note that the integrand is invariant under the change of variables:

$$s\to -s,\quad \mathbf{q}\to-\mathbf{q}.$$

Thus we can write

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi)\\
&=\lim_{\varepsilon\to 0}\frac{1}{2}\int_{-\infty}^{\infty} ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\\
&\qquad\qquad\times\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{2-2\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad\qquad\quad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{2-2\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\Big]\\
&=\lim_{\varepsilon\to 0}\pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\\
&\quad\times\big\{\Phi(\varepsilon^{2-2\alpha}k^{-1}(\mathbf{p}+\mathbf{q}/2)\cdot\mathbf{q},\mathbf{q})\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\quad\ -\Phi(\varepsilon^{2-2\alpha}k^{-1}(\mathbf{p}-\mathbf{q}/2)\cdot\mathbf{q},\mathbf{q})\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\big\}\\
&=\lim_{\varepsilon\to 0}\pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\\
&\quad\times\Big\{\Phi(\varepsilon^{2-2\alpha}\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k},\mathbf{q}-\mathbf{p})\left[\theta(\mathbf{x},\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\quad\ -\Phi(\varepsilon^{2-2\alpha}\frac{|\mathbf{p}|^2-|\mathbf{q}|^2}{2k},\mathbf{p}-\mathbf{q})\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{q})\right]\Big\}\\
&=\lim_{\varepsilon\to 0}2\pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\\
&\quad\times\Phi(\varepsilon^{2-2\alpha}\frac{|\mathbf{p}|^2-|\mathbf{q}|^2}{2k},\mathbf{p}-\mathbf{q})\left[\theta(\mathbf{x},\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]
\end{aligned}
$$

Clearly, we have that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int d\mathbf{q}[\theta(\mathbf{x},\mathbf{q})-\theta(\mathbf{x},\mathbf{p})]\Phi(\varepsilon^{2-2\alpha}\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k},\mathbf{q}-\mathbf{p})\\
&=\int d\mathbf{q}[\theta(\mathbf{x},\mathbf{q})-\theta(\mathbf{x},\mathbf{p})]\times\begin{cases}\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k},\mathbf{q}-\mathbf{p}), & \alpha=1\\ \Phi(0,\mathbf{q}-\mathbf{p}), & 0<\alpha<1\\ 0, & \alpha>1\end{cases}
\end{aligned}
$$

in $L^2(\mathbb{E}^{2d})$. Therefore,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&=\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\psi(\mathbf{x},\mathbf{p})[\theta(\mathbf{x},\mathbf{q})-\theta(\mathbf{x},\mathbf{p})]\times\begin{cases}\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k},\mathbf{q}-\mathbf{p}), & \alpha=1\\ \Phi(0,\mathbf{q}-\mathbf{p}), & 0<\alpha<1\\ 0, & \alpha>1\end{cases}\\
&\equiv\bar{A}_1(\psi)
\end{aligned}
$$

Recall that

$$
\begin{aligned}
M_z^\varepsilon(\theta) &= f^\varepsilon(z) - \int_0^z \mathcal{A}^\varepsilon f^\varepsilon(s)\,ds \qquad (89)\\
&= f(z) + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z) - \int_0^z \frac{1}{k} f'(z)\left\langle W_z^\varepsilon, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle\,ds\\
&\quad - \int_0^z k^2\left[f''(s)A_2^{(1)}(W_s^\varepsilon) + f'(s)A_1^{(1)}(W_s^\varepsilon)\right]\,ds - \int_0^z R^\varepsilon(s)\,ds
\end{aligned}
$$

is a martingale. The martingale property implies that for any finite sequence $0 < z_1 < z_2 < z_3 < ... < z_n \le z$, C^2-function f and bounded continuous function h with compact support, we have

$$
\begin{aligned}
\mathbb{E}\left\{ h\left(\left\langle W_{z_1}^\varepsilon, \theta\right\rangle, \left\langle W_{z_2}^\varepsilon, \theta\right\rangle, ..., \left\langle W_{z_n}^\varepsilon, \theta\right\rangle\right)\left[M_{z+s}^\varepsilon(\theta) - M_z^\varepsilon(\theta)\right]\right\} = 0, \quad (90)\\
\forall s > 0, \quad z_1 \le z_2 \le \cdots \le z_n \le z.
\end{aligned}
$$

Let

$$
\bar{A}f(z) \equiv f'(s)\left[\frac{1}{k}\left\langle W_z, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + k^2\bar{A}_1(W_z)\right].
$$

In view of the results of Propositions 1,2, 3, 4, 5 we see that $f^\varepsilon(z)$ and $\mathcal{A}^\varepsilon f^\varepsilon(z)$ in (89) can be replaced by $f(z)$ and $\bar{A}f(z)$, respectively, modulo an error that vanishes as $\varepsilon \to 0$. With this and the tightness of $\{W_z^\varepsilon\}$ we can pass to the limit $\varepsilon \to 0$ in (90). We see that the limiting process satisfies the martingale property that

$$
\mathbb{E}\{h\left(\langle W_{z_1}, \theta\rangle, \langle W_{z_2}, \theta\rangle, ..., \langle W_{z_n}, \theta\rangle\right)[M_{z+s}(\theta) - M_z(\theta)]\} = 0, \quad \forall s > 0.
$$

where

$$
M_z(\theta) = f(z) - \int_0^z \bar{A}f(s)\,ds. \qquad (91)
$$

Then it follows that

$$
\mathbb{E}[M_{z+s}(\theta) - M_z(\theta)|W_u, u \le z] = 0, \quad \forall z, s > 0
$$

which proves that $M_z(\theta)$ is a martingale given by

$$
M_z(\theta) = f(z) - \int_0^z \left\{f'(s)\left[\frac{1}{k}\left\langle W_s, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + k^2\bar{A}_1(W_s)\right]\right\}ds. \qquad (92)
$$

Choose $f(r) = r$ and r^2 in (92) we see that

$$
M_z^{(1)}(\theta) = \langle W_z, \theta\rangle - \int_0^z \left[\frac{1}{k}\left\langle W_s, \mathbf{p}\cdot\nabla_{\mathbf{x}}\theta\right\rangle + k^2\bar{A}_1(W_s)\right]\,ds
$$

is a martingale with the null quadratic variation

$$
\left[M^{(1)}(\theta), M^{(1)}(\theta)\right]_z = 0.
$$

Thus

$$f(z) - \int_0^z \left\{ f'(s) \left[\frac{1}{k} \langle W_s, \mathbf{p} \cdot \nabla_{\mathbf{x}} \theta \rangle + k^2 \bar{A}_1(W_s) \right] \right\} ds = f(0), \quad \forall z > 0.$$

Since $\langle W_z^\varepsilon, \theta \rangle$ is uniformly bounded

$$|\langle W_z^\varepsilon, \theta \rangle| \leq \|W_0\|_2 \|\theta\|_2$$

we have the convergence of the second moment

$$\lim_{\varepsilon \to 0} \mathbb{E} \left\{ \langle W_z^\varepsilon, \theta \rangle^2 \right\} = \langle W_z, \theta \rangle^2$$

and hence the convergence in probability.

6. Proof of Theorem 2

6.1. Tightness

Instead of (64) we use the following the corrector

$$\begin{aligned} \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) = \frac{i}{\varepsilon^2} \int_z^\infty \int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}} e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\varepsilon^{2\alpha}} \\ \times \varepsilon^{2\alpha-2} \left[\theta(\mathbf{x}, \mathbf{p} + \varepsilon^{2-2\alpha}\mathbf{q}/2) - \theta(\mathbf{x}, \mathbf{p} - \varepsilon^{2-2\alpha}\mathbf{q}/2) \right] \mathbb{E}_z^\varepsilon \hat{V}_s^\varepsilon(d\mathbf{q}) \end{aligned} \tag{93}$$

which satisfies the corrector equation (69). Its inverse Fourier transform is given by

$$\begin{aligned} &\mathcal{F}_2^{-1} \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}) \\ &= \varepsilon^{-2} \int_z^\infty e^{-i\varepsilon^{-2\alpha} k^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}} \left[\mathbb{E}_z \left[\delta_\varepsilon V_s^\varepsilon \right] \mathcal{F}_2^{-1} \theta \right] (\mathbf{x}, \mathbf{y}) ds. \end{aligned} \tag{94}$$

Instead of Lemma 1, Corollary 1, Lemma 2 and Corollary 2 we have

Lemma 3.

$$\limsup_{\varepsilon \to 0} \mathbb{E} \left[\tilde{\mathcal{L}}_z^\varepsilon \theta \right]^2 (\mathbf{x}, \mathbf{p}) \leq \left[\int_0^\infty \rho(s) ds \right]^2 \int \left[\mathbf{q} \cdot \nabla_{\mathbf{p}} \theta(\mathbf{x}, \mathbf{p}) \right]^2 \Phi(\xi, \mathbf{q}) d\xi d\mathbf{q}.$$

Corollary 3

$$\begin{aligned} &\limsup_{\varepsilon \to 0} \mathbb{E} \left[\mathbf{p} \cdot \nabla_{\mathbf{x}} \tilde{\mathcal{L}}_z^\varepsilon \theta \right]^2 (\mathbf{x}, \mathbf{p}) \\ &\leq \left[\int_0^\infty \rho(s) ds \right]^2 \int \left[\mathbf{p} \cdot \nabla_{\mathbf{x}} [\mathbf{q} \cdot \nabla_{\mathbf{p}} \theta(\mathbf{x}, \mathbf{p})] \right]^2 \Phi(\xi, \mathbf{q}) d\xi d\mathbf{q}. \end{aligned}$$

Lemma 4.

$$\limsup_{\varepsilon\to 0}\mathbb{E}\|\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2$$
$$\leq 8C\left(\int_0^\infty\rho(s)ds\right)^2\mathbb{E}[V_z]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}$$

$$\limsup_{\varepsilon\to 0}\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon\tilde{\mathcal{L}}^\varepsilon\theta\|_2^2$$
$$\leq 8C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}$$

for some constant C.

Corollary 4

$$\limsup_{\varepsilon\to 0}\mathbb{E}\|\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2$$
$$\leq 32C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z]^4\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p};$$

$$\limsup_{\varepsilon\to 0}\mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2$$
$$\leq 32C\left(\int_0^\infty\rho(s)ds\right)^4\Big\{\mathbb{E}[\nabla_{\mathbf{y}}V_z]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}$$
$$+\mathbb{E}[V_z]^2\int[\nabla_{\mathbf{x}}\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2|\mathbf{p}|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\Big\}$$

for some constant C.

The rest of the argument for tightness proceeds without changes.

6.2. Identification of the limit

With straightforward modification on the estimates of the remainder terms R_2^ε and R_3^ε the same argument for passing to the limit $\varepsilon\to 0$ as before carries over here.

In particular, Proposition 6 can be proved as follows.

Proposition 7

$$\lim_{\varepsilon\to 0}\sup_{\|\psi\|_2=1}A_2^{(1)}(\psi)=0.$$

Proof. As in (89) we have

$$A_2^{(1)}(\psi) = \frac{1}{2}\int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q}$$

with the symmetrized kernel

$$\begin{aligned}
&\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{y},\mathbf{q},\mathbf{x},\mathbf{p}) + \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \int_0^\infty \int \check{\Phi}(s,\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha}}\\
&\quad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{p}'/2) - \theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\\
&\quad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{y},\mathbf{q}+\varepsilon^{2-2\alpha}\mathbf{p}'/2) - \theta(\mathbf{y},\mathbf{q}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\; d\mathbf{p}'\; ds\\
&= \lim_{\varepsilon\to 0}\pi\int \Phi(k^{-1}\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2-2\alpha},\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}\\
&\quad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{p}'/2) - \theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\\
&\quad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{y},\mathbf{q}+\varepsilon^{2-2\alpha}\mathbf{p}'/2) - \theta(\mathbf{y},\mathbf{q}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\; d\mathbf{p}'.
\end{aligned}$$

Note that

$$\varepsilon^{2\alpha-2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{p}'/2) - \theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]$$

converges to $\mathbf{p}'\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})$ in $C_c^\infty(\mathbb{R}^{2d})$. Thus the L^2-norm of $\mathcal{Q}_1^s$ tends to zero for the same reason as given in the proof of Proposition 6. ■

To identify the limit we have the following straightforward calculation: For any real-valued, L^2-weakly convergent sequence $\psi^\varepsilon\to\psi$

$$\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&= \lim_{\varepsilon\to 0}\int_0^\infty ds\int dw d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\Phi(w,\mathbf{q})e^{isw}e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\varepsilon^{4\alpha-4}\\
&\qquad\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{4-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}) - \theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{4-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}) - \theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\Big]\\
&= \lim_{\varepsilon\to 0}\int_0^\infty ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2-2\alpha}}\varepsilon^{4\alpha-4}\\
&\qquad\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{4-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}) - \theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{4-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}) - \theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\Big]\\
&= \pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi(\mathbf{x},\mathbf{p})\Phi(0,\mathbf{q})(\mathbf{q}\cdot\nabla_{\mathbf{p}})^2\theta(\mathbf{x},\mathbf{p})\\
&= \bar{A}_1(\psi)
\end{aligned}$$

for $\alpha\in(0,1)$. For $\alpha=1$ we have the same result as in Theorem 1 Case (ii); for $\alpha>1$, the limit is identically zero.

7. Proof of Theorem 3

The proof of the result for Case (i) and (iii) is identical to that for Theorem 1, Case (i) and (iii), respectively. So in the sequel we focus on the second case $\alpha > \beta$.

Introducing a new parameter

$$\tilde{\varepsilon} = \varepsilon^{\beta}$$

we can rewrite the equation as

$$\frac{\partial W_z^\varepsilon}{\partial z} + \frac{\mathbf{p}}{k} \cdot \nabla W_z^\varepsilon + \frac{k}{\tilde{\varepsilon}} \mathcal{L}_z^\varepsilon W_z^\varepsilon = 0 \tag{95}$$

with

$$\begin{aligned} &\mathcal{L}_z^\varepsilon W_z^\varepsilon(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) \\ &= i\tilde{\varepsilon}^{1-\alpha/\beta} \int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}} \left[W_z^\varepsilon(\mathbf{x}, \mathbf{p}+\mathbf{q}/2) - W_z^\varepsilon(\mathbf{x}, \mathbf{p}-\mathbf{q}/2)\right] \widehat{V}_z^\varepsilon(d\mathbf{q}), \beta < 1 \end{aligned} \tag{96}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\tilde{\varepsilon}^{-2\alpha/\beta}$ and

$$\widehat{V}_z^\varepsilon(d\mathbf{q}) = \widehat{V}(\frac{z}{\tilde{\varepsilon}^2}, d\mathbf{q}), \quad V_z^\varepsilon(\mathbf{x}) = V(\frac{z}{\tilde{\varepsilon}^2}, \mathbf{x}). \tag{97}$$

Note again that

$$\mathcal{F}_2^{-1} \mathcal{L}_z^\varepsilon \theta = -i\tilde{\varepsilon}^{1-\alpha/\beta} \delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}}, \mathbf{y}) \mathcal{F}_2^{-1}\theta \tag{98}$$

with

$$\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}}, \mathbf{y}) = V_z^\varepsilon(\tilde{\mathbf{x}} + \mathbf{y}/2) - V_z^\varepsilon(\tilde{\mathbf{x}} - \mathbf{y}/2). \tag{99}$$

We will work with (96) and (97) and construct the perturbed test function in the power of $\tilde{\varepsilon}$.

First we note that

$$\begin{aligned} &\mathbb{E}\left[\mathcal{L}_z^\varepsilon \theta\right]^2(\mathbf{x}, \mathbf{p}) \\ &= \tilde{\varepsilon}^{2-2\alpha/\beta} \int [\theta(\mathbf{x}, \mathbf{p}+\mathbf{q}/2) - \theta(\mathbf{x}, \mathbf{p}-\mathbf{q}/2)]^2 \Phi(\xi, \mathbf{q}) d\xi d\mathbf{q}. \end{aligned}$$

Instead of (64) we define

$$\begin{aligned} \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}) = \frac{i}{\tilde{\varepsilon}^{1+\alpha/\beta}} \int_z^\infty \int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}} e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\tilde{\varepsilon}^{2\alpha/\beta}} \\ \times [\theta(\mathbf{x}, \mathbf{p}+\mathbf{q}/2) - \theta(\mathbf{x}, \mathbf{p}-\mathbf{q}/2)] \mathbb{E}_z^\varepsilon \hat{V}_s^\varepsilon(d\mathbf{q}) \end{aligned}$$

with $\tilde{\mathbf{x}} = \mathbf{x}\tilde{\varepsilon}^{-2\alpha/\beta}$ which becomes after the partial inverse Fourier transform

$$\begin{aligned} &\mathcal{F}_2^{-1} \tilde{\mathcal{L}}_z^\varepsilon \theta(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}) \\ &= \frac{i}{\tilde{\varepsilon}^{1+\alpha/\beta}} \int_z^\infty e^{ik^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}\tilde{\varepsilon}^{-2\alpha/\beta}} \left[\mathbb{E}_z^\varepsilon [\delta_\varepsilon V_s^\varepsilon] \mathcal{F}_2^{-1}\theta\right](\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{y}) ds. \end{aligned} \tag{100}$$

The corrector equation holds again:

$$\left[\tilde{\varepsilon}^{-2\alpha/\beta}\frac{\mathbf{p}}{k}\cdot\nabla_{\tilde{\mathbf{x}}}+\mathcal{A}^{\varepsilon}\right]\tilde{\mathcal{L}}_z^{\varepsilon}\theta=\tilde{\varepsilon}^{-2}\mathcal{L}_z^{\varepsilon}\theta. \tag{101}$$

Following the same argument as in the proof of Theorem 1 we have the following estimates:

Lemma 5.

$$\begin{aligned}&\mathbb{E}\left[\tilde{\mathcal{L}}_z^{\varepsilon}\theta\right]^2(\mathbf{x},\mathbf{p})\\&\leq\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\int_0^\infty\rho(s)ds\right]^2\int\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.\end{aligned}$$

Corollary 5

$$\begin{aligned}&\mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\right]^2(\mathbf{x},\mathbf{p})\\&\leq\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\int_0^\infty\rho(s)ds\right]^2\\&\quad\times\int\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\mathbf{p}\cdot\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)\right]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.\end{aligned}$$

Lemma 6.

$$\begin{aligned}\mathbb{E}\|\mathcal{L}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2^2\leq{}&\tilde{\varepsilon}^{4-4\alpha/\beta}8C\left(\int_0^\infty\rho(s)ds\right)^2\mathbb{E}[V_z]^2\\&\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\end{aligned}$$

$$\begin{aligned}\mathbb{E}\|\tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2^2\leq{}&\tilde{\varepsilon}^{4-4\alpha/\beta}8C\left(\int_0^\infty\rho(s)ds\right)^4\mathbb{E}[V_z]^2\\&\times\int[\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\end{aligned}$$

for some constant C independent of ε.

Corollary 6

$$\begin{aligned}&\mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^{\varepsilon}\tilde{\mathcal{L}}_z^{\varepsilon}\theta\|_2^2\\&\leq\tilde{\varepsilon}^{4-4\alpha/\beta}32C\left(\int_0^\infty\rho(s)ds\right)^4\\&\quad\times\left\{\mathbb{E}[\nabla_{\mathbf{y}}V_z]^2\int[\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\right.\\&\quad\left.+\mathbb{E}[V_z]^2\int[\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}+\mathbf{q}/2)-\nabla_{\mathbf{x}}\theta(\mathbf{x},\mathbf{p}-\mathbf{q}/2)]^2|\mathbf{p}|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\right\};\end{aligned}$$

$$\mathbb{E}\|\mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 \leq \tilde{\varepsilon}^{6-6\alpha/\beta} 32C \left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z]^4$$
$$\times \int [\theta(\mathbf{x}, \mathbf{p}+\mathbf{q}/2) - \theta(\mathbf{x}, \mathbf{p}-\mathbf{q}/2)]^2 \Phi(\xi, \mathbf{q}) d\xi d\mathbf{x} d\mathbf{q} d\mathbf{p}$$

for some constant C *independent of* ε.

As a consequence of the divergent factor $\tilde{\varepsilon}^{1-1/\beta}$ in the above estimates the previous proof of uniform integrability of $\mathcal{A}^\varepsilon[f(z) + \tilde{\varepsilon} f_1^\varepsilon]$ (e.g. Proposition 3) breaks down. For both the tightness and the identification we shall use the test function

$$f^\varepsilon(z) = f(z) + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z)$$

where

$$f_1^\varepsilon(z) = k\tilde{\varepsilon} f'(z) \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle \tag{102}$$

$$f_2^\varepsilon(z) = \frac{\tilde{\varepsilon}^2 k^2}{2} f''(z) \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right] \tag{103}$$

$$f_3^\varepsilon(z) = \frac{\tilde{\varepsilon}^2 k^2}{2} f'(z) \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle - A_1^{(2)}(W_z^\varepsilon)\right]. \tag{104}$$

with $A_1^{(2)}, A_2^{(2)}$ given as before.

Following the same procedure as in the proof of Theorem 1 we obtain

$$\mathcal{A}^\varepsilon f^\varepsilon(z) = \frac{1}{k} f'(z) \langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} \theta \rangle + k^2 f''(z) A_2^{(1)}(W_z^\varepsilon) + k^2 f' A_1^{(1)}(W_z^\varepsilon)$$
$$+ R_2^\varepsilon(z) + R_3^\varepsilon(z) + A_3^\varepsilon(z). \tag{105}$$

where

$$R_2^\varepsilon(z) = \tilde{\varepsilon}^2 k^2 \frac{f'''(z)}{2} \left[\frac{1}{k} \langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} \theta \rangle + \frac{k}{\tilde{\varepsilon}} \langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \theta \rangle\right]$$
$$\times \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle^2 - A_2^{(2)}(W_z^\varepsilon)\right]$$
$$+ \tilde{\varepsilon}^2 k^2 f''(z) \left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle \left[\frac{1}{k} \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} (\tilde{\mathcal{L}}_z^\varepsilon \theta) \right\rangle + \frac{k}{\tilde{\varepsilon}} \left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle\right]$$
$$- \tilde{\varepsilon}^2 k^2 f''(z) \left[\frac{1}{k} \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} (G_\theta^{(2)} W_z^\varepsilon) \right\rangle + \frac{k}{\tilde{\varepsilon}} \left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon \right\rangle\right] \tag{106}$$

$$R_3^\varepsilon(z) = \tilde{\varepsilon}^2 k^2 f'(z) \left[\frac{1}{k} \left\langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} (\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta) \right\rangle + \frac{k}{\tilde{\varepsilon}} \left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle\right] \tag{107}$$

$$+ \tilde{\varepsilon}^2 \frac{k^2}{2} f''(z) \left[\frac{1}{k} \langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} \theta \rangle + \frac{k}{\tilde{\varepsilon}} \langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \theta \rangle\right]$$
$$\times \left[\left\langle W_z^\varepsilon, \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta \right\rangle - A_1^{(2)}(W_z^\varepsilon)\right] \tag{108}$$
$$- \tilde{\varepsilon}^2 k^2 f'(z) \left[\frac{1}{k} \langle W_z^\varepsilon, \mathbf{p} \cdot \nabla_\mathbf{x} (\mathcal{Q}_2' \theta) \rangle + \frac{k}{\tilde{\varepsilon}} \langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \mathcal{Q}_2' \theta \rangle\right]$$

and $A_3^\varepsilon, A_2^{(1)}, A_1^{(1)}, G_\theta^{(2)}, \mathcal{Q}_2'$ all have the same expressions as in the proof of Theorem 1.

With the assumption $\alpha/\beta < 4/3$, Propositions 1, 2, 4 and 5 hold true. Let us remark that the most severe terms due to the divergent factor $\tilde{\varepsilon}^{1-\alpha/\beta}$ are

$$\sup_{z<z_0} \tilde{\varepsilon}\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon G_\theta^{(2)} W_z^\varepsilon\right\rangle\right| = O(\tilde{\varepsilon}^{4-3\alpha/\beta}), \tag{109}$$

$$\sup_{z<z_0} \tilde{\varepsilon}\mathbb{E}\left|\left\langle W_z^\varepsilon, \mathcal{L}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\right\rangle\right| = O(\tilde{\varepsilon}^{4-3\alpha/\beta}) \tag{110}$$

(cf. Corollary 6).

To satisfy (63) we need to show

Proposition 8

$$\lim_{\varepsilon\to 0}\sup_{z<z_0} |f_{j,z}^\varepsilon| = 0, \quad j = 2,3, \quad \textit{in probability.}$$

Proof. We have the estimates

$$\sup_{z<z_0} |f_{2,z}^\varepsilon| \le \sup_{z<z_0} \tilde{\varepsilon}^2 k^2 \|f''\|_\infty \left[\|W_0\|_2^2 \|\tilde{\mathcal{L}}_z^\varepsilon \theta\|_2^2 + A_2^{(2)}(W_z^\varepsilon)\right]$$

$$\sup_{z<z_0} |f_{3,z}^\varepsilon| \le \sup_{z<z_0} \tilde{\varepsilon}^2 k^2 \|f'\|_\infty \left[\|W_0\|_2 \|\tilde{\mathcal{L}}_z^\varepsilon \tilde{\mathcal{L}}_z^\varepsilon \theta\|_2 + A_1^{(2)}(W_z^\varepsilon)\right]$$

which vanish in probability by using Assumption 4, Lemma 5, 6 and Chebyshev's inequality. ■

Proposition 9

$$\lim_{\varepsilon\to 0}\sup_{z<z_0}\sup_{\|\psi\|_2=1} A_2^{(1)}(\psi) = 0.$$

Proof. As in (89) we have

$$A_2^{(1)}(\psi) = \frac{1}{2}\int \psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\, d\mathbf{x}d\mathbf{p}\, d\mathbf{y}d\mathbf{q}$$

with the symmetrized kernel

$$\begin{aligned}
&\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{y},\mathbf{q},\mathbf{x},\mathbf{p}) + \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \tilde{\varepsilon}^{2-2\alpha/\beta}\int_0^\infty\int \check{\Phi}(s,\mathbf{p}')e^{i\mathbf{p}'\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{p}'\tilde{\varepsilon}^{2-2\alpha/\beta}}\\
&\quad\times[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}'/2)]\,[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}'/2)]\;d\mathbf{p}'\;ds\\
&= \tilde{\varepsilon}^{2-2\alpha/\beta}\pi\int \Phi(k^{-1}\mathbf{p}\cdot\mathbf{p}'\tilde{\varepsilon}^{2-2\alpha/\beta},\mathbf{p}')e^{i\mathbf{p}'\cdot(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})}\\
&\quad\times[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}'/2)]\,[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}'/2)]\;d\mathbf{p}'.
\end{aligned}$$

Taking the L^2-norm and passing to the limit we have

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{q}\,|\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})|^2\\
&=\lim_{\varepsilon\to 0}\pi^2\int d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{q}\left|\int\delta(\frac{\mathbf{p}\cdot\mathbf{p}'}{k})\left[\int\Phi(w,\mathbf{p}')dw\right]e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\tilde{\varepsilon}^{2-2\alpha/\beta}}\right.\\
&\quad\left.\times\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}'/2)\right]\left[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}'/2)\right]\;d\mathbf{p}'\right|^2\\
&=\lim_{\varepsilon\to 0}\pi^2\int d\mathbf{x}d\mathbf{p}d\mathbf{y}d\mathbf{q}\left|\int\;\frac{k}{|\mathbf{p}|}\left[\int\Phi(w,\mathbf{p}_\perp)dw\right]e^{i\mathbf{p}_\perp\cdot(\mathbf{x}-\mathbf{y})/\tilde{\varepsilon}^{2-2\alpha/\beta}}\right.\\
&\qquad\times\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{p}_\perp/2)-\theta(\mathbf{x},\mathbf{p}-\mathbf{p}_\perp/2)\right]\\
&\qquad\left.\times\left[\theta(\mathbf{y},\mathbf{q}+\mathbf{p}_\perp/2)-\theta(\mathbf{y},\mathbf{q}-\mathbf{p}_\perp/2)\right]\;d\mathbf{p}_\perp\right|^2
\end{aligned}
$$

where $\mathbf{p}_\perp\cdot\mathbf{p}=0,\mathbf{p}_\perp\in\mathbb{R}^{d-1}$. In passing to the limit the only problem is at the point $\mathbf{p}=0$. But the integrand in the above integral is bounded by $c|\mathbf{p}|^{-2}, c=$ const., which is integrable in a neighborhood of zero if $d\geq 3$. Hence the L^2-norm of $\mathcal{Q}_1^s$ tends to zero by the dominated convergence theorem. ■

We have the following straightforward calculation: For any real-valued, L^2-weakly convergent sequence $\psi^\varepsilon\to\psi$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}A_1^{(1)}(\psi^\varepsilon)\\
&\quad=\lim_{\varepsilon\to 0}\int_0^\infty ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})\tilde{\varepsilon}^{2-2\alpha/\beta}\\
&\qquad\left[e^{-ik^{-1}s(\mathbf{p}+\mathbf{q}/2)\cdot\mathbf{q}\tilde{\varepsilon}^{2-2\alpha/\beta}}\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\right.\\
&\qquad\quad\left.-e^{-ik^{-1}s(\mathbf{p}-\mathbf{q}/2)\cdot\mathbf{q}\tilde{\varepsilon}^{2-2\alpha/\beta}}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\right]\\
&\quad=\lim_{\varepsilon\to 0}\pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\left[\Phi(k^{-1}(\mathbf{p}+\mathbf{q}/2)\cdot\mathbf{q}\tilde{\varepsilon}^{2-2\alpha/\beta},\mathbf{q})\right.\\
&\qquad\times\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad\quad\left.-\Phi(k^{-1}(\mathbf{p}-\mathbf{q}/2)\cdot\mathbf{q}\tilde{\varepsilon}^{2-2\alpha/\beta},\mathbf{q})\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\right]\\
&\quad=\lim_{\varepsilon\to 0}\pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\left[\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k}\right.\\
&\qquad\quad\times\tilde{\varepsilon}^{2-2\alpha/\beta},\mathbf{q}-\mathbf{p})\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad\quad\left.-\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k}\tilde{\varepsilon}^{2-2\alpha/\beta},\mathbf{q}-\mathbf{p})\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\right]\\
&\quad=2\pi\int d\mathbf{x}d\mathbf{p}\;\psi(\mathbf{x},\mathbf{p})\int d\mathbf{q}\delta(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k})\\
&\qquad\times\left[\int\Phi(w,\mathbf{q}-\mathbf{p})dw\right]\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\quad\equiv\bar{A}_1(\psi),\quad\forall\theta\in C_c^\infty(\mathbb{R}^{2d})
\end{aligned}
$$

following from the strong convergence of

$$\int \Big[\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k}\tilde{\varepsilon}^{2-2\alpha/\beta}, \mathbf{q}-\mathbf{p})\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]$$
$$-\Phi(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k}\tilde{\varepsilon}^{2-2\alpha/\beta}, \mathbf{q}-\mathbf{p})\tilde{\varepsilon}^{2-2\alpha/\beta}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\mathbf{q})\right]\Big]\, d\mathbf{q},$$

to the L^2-function

$$\int d\mathbf{q}\delta(\frac{|\mathbf{q}|^2-|\mathbf{p}|^2}{2k})\left[\int \Phi(w,\mathbf{q}-\mathbf{p})dw\right]\left[\theta(\mathbf{x},\mathbf{p}+\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]$$

which is square-integrable because of (10).

Proposition 10 *$\mathcal{A}^\varepsilon f^\varepsilon(z)$ is uniformly integrable.*

This, of course, follows from the fact that each term in (105) has a uniformly bounded second moment. Therefore we have completed the tightness argument. Moreover, we have also identified the limiting equation.

8. Proof of Theorem 4

Introducing a new parameter

$$\tilde{\varepsilon} = \varepsilon^\beta,$$

the fast variable

$$\tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-2\alpha}$$

and the rescaled process

$$\widehat{V}^\varepsilon_z(d\mathbf{q}) = \widehat{V}(\frac{z}{\tilde{\varepsilon}^2}, d\mathbf{q}), \quad V^\varepsilon_z(\mathbf{x}) = V(\frac{z}{\tilde{\varepsilon}^2}, \mathbf{x}) \tag{111}$$

we rewrite the equation as

$$\frac{\partial W^\varepsilon_z}{\partial z} + \frac{\mathbf{p}}{k}\cdot\nabla W^\varepsilon_z + \frac{k}{\tilde{\varepsilon}}\mathcal{L}^\varepsilon_z W^\varepsilon_z = 0 \tag{112}$$

with, in Case (i),

$$\begin{aligned}&\mathcal{L}^\varepsilon_z W^\varepsilon_z(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) \qquad (113)\\ &= i\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\varepsilon^{2\alpha-2}\left[W^\varepsilon_z(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2)-W^\varepsilon_z(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\right]\widehat{V}^\varepsilon_z(d\mathbf{q}),\end{aligned}$$

and in Case (ii)

$$\begin{aligned}\mathcal{L}^\varepsilon_z W^\varepsilon_z(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) &= i\tilde{\varepsilon}^{1-\alpha/\beta}\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}\varepsilon^{2\alpha-2} \qquad (114)\\ &\times\left[W^\varepsilon_z(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2)-W^\varepsilon_z(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\right]\widehat{V}^\varepsilon_z(d\mathbf{q}).\end{aligned}$$

Taking the partial inverse Fourier transform we get in Case (i)

$$\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = -i\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \tag{115}$$

and in Case (ii)

$$\mathcal{F}_2^{-1}\mathcal{L}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y}) = -i\tilde{\varepsilon}^{1-\alpha/\beta}\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y})\mathcal{F}_2^{-1}\theta(\mathbf{x},\mathbf{y}) \tag{116}$$

with

$$\delta_\varepsilon V_z^\varepsilon(\tilde{\mathbf{x}},\mathbf{y}) = \varepsilon^{2\alpha-2}\left[V_z^\varepsilon(\tilde{\mathbf{x}}+\varepsilon^{2-2\alpha}\mathbf{y}/2) - V_z^\varepsilon(\tilde{\mathbf{x}}-\varepsilon^{2-2\alpha}\mathbf{y}/2\right]. \tag{117}$$

The proof for Case (i) is entirely analogous to that for Theorem 2 and we will focus on Case (ii) in the sequel. And we will work with (114) and (116) and construct the perturbed test function in the power of $\tilde{\varepsilon}$.

First we note that

$$\limsup_{\varepsilon\to 0}\tilde{\varepsilon}^{-2+2\alpha/\beta}\mathbb{E}\left[\mathcal{L}_z^\varepsilon\theta\right]^2(\mathbf{x},\mathbf{p}) = \int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}. \tag{118}$$

As in the proof of Theorem 3, we carry the analysis in the power of $\tilde{\varepsilon}=\varepsilon^\beta$. We consider the rescaled process (97) and its sigma algebras.

We use the corrector

$$\begin{aligned}&\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})\\ &= \frac{i}{\tilde{\varepsilon}^{1+\alpha/\beta}}\int_z^\infty\int e^{i\mathbf{q}\cdot\tilde{\mathbf{x}}}e^{ik^{-1}(s-z)\mathbf{p}\cdot\mathbf{q}/\tilde{\varepsilon}^{2\alpha/\beta}}\varepsilon^{2\alpha-2}\\ &\qquad[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2)-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)]\mathbb{E}_z^\varepsilon\hat{V}_s^\varepsilon(d\mathbf{q})\end{aligned} \tag{119}$$

which after the partial Fourier inversion becomes

$$\begin{aligned}&\mathcal{F}_2^{-1}\tilde{\mathcal{L}}_z^\varepsilon\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})\\ &= -\frac{i}{\tilde{\varepsilon}^{1+\alpha/\beta}}\int_z^\infty e^{ik^{-1}(s-z)\nabla_{\mathbf{y}}\cdot\nabla_{\tilde{\mathbf{x}}}/\tilde{\varepsilon}^{2\alpha/\beta}}\mathbb{E}_z^\varepsilon[\delta_\varepsilon V_s^\varepsilon]\mathcal{F}_2^{-1}\theta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{y})ds.\end{aligned} \tag{120}$$

The corrector solves the corrector equation eq. (101).

Following the same argument as in the proof of Theorem 1 we have the following estimates:

Lemma 7.

$$\begin{aligned}&\limsup_{\tilde{\varepsilon}\to 0}\tilde{\varepsilon}^{-2+2\alpha/\beta}\mathbb{E}\left[\tilde{\mathcal{L}}_z^\varepsilon\theta\right]^2(\mathbf{x},\mathbf{p})\\ &\le\left[\int_0^\infty\rho(s)ds\right]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.\end{aligned} \tag{121}$$

Corollary 7

$$\begin{aligned}&\limsup_{\tilde{\varepsilon}\to 0}\tilde{\varepsilon}^{-2+2\alpha/\beta}\mathbb{E}\left[\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}_z^\varepsilon\theta\right]^2(\mathbf{x},\mathbf{p})\\ &\le\left[\int_0^\infty\rho(s)ds\right]^2\int[\mathbf{p}\cdot\nabla_{\mathbf{x}}\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{q}.\end{aligned} \tag{122}$$

Lemma 8.

$$
\limsup_{\tilde\varepsilon\to 0} \tilde\varepsilon^{-4+4\alpha/\beta}\mathbb{E}\|\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2 \tag{123}
$$
$$
\leq 8C\left(\int_0^\infty \rho(s)ds\right)^2 \mathbb{E}[V_z]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
$$

$$
\limsup_{\tilde\varepsilon\to 0} \tilde\varepsilon^{-4+4\alpha/\beta}\mathbb{E}\|\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2
$$
$$
\leq 8C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z]^2\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
$$

for some constant C independent of ε.

Corollary 8

$$
\limsup_{\tilde\varepsilon\to 0} \tilde\varepsilon^{-4+4\alpha/\beta}\mathbb{E}\|\mathbf{p}\cdot\nabla_{\mathbf{x}}\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2
$$
$$
\leq 32C\left(\int_0^\infty \rho(s)ds\right)^4 \Big\{\mathbb{E}[\nabla_{\mathbf{y}}V_z]^2\int[\nabla_{\mathbf{x}}\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
$$
$$
+\mathbb{E}[V_z]^2\int[\nabla_{\mathbf{x}}\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2|\mathbf{p}|^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}\Big\};
$$

$$
\limsup_{\tilde\varepsilon\to 0} \tilde\varepsilon^{-6+6\alpha/\beta}\mathbb{E}\|\mathcal{L}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\tilde{\mathcal{L}}^\varepsilon_z\theta\|_2^2
$$
$$
\leq 32C\left(\int_0^\infty \rho(s)ds\right)^4 \mathbb{E}[V_z]^4\int[\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})]^2\Phi(\xi,\mathbf{q})d\xi d\mathbf{x}d\mathbf{q}d\mathbf{p}
$$

for some constant C independent of ε.

The rest of the argument follows the general outline of that of Theorem 3 Case (ii).

Let us know verify that the quadratic variation vanishes in the limit.

Proposition 11

$$
\lim_{\varepsilon\to 0}\sup_{z<z_0}\sup_{\|\psi\|_2=1} A_2^{(1)}(\psi)=0.
$$

Proof. To keep the notation simple we use ε, instead of $\tilde\varepsilon$, in the following calculation. As in (89) we have

$$
A_2^{(1)}(\psi)=\frac{1}{2}\int\psi(\mathbf{x},\mathbf{p})\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\psi(\mathbf{y},\mathbf{q})\,d\mathbf{x}d\mathbf{p}\,d\mathbf{y}d\mathbf{q}
$$

with the symmetrized kernel

$$
\begin{aligned}
&\mathcal{Q}_1^s(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{y},\mathbf{q},\mathbf{x},\mathbf{p}) + \mathcal{Q}_1(\theta\otimes\theta)(\mathbf{x},\mathbf{p},\mathbf{y},\mathbf{q})\\
&= \varepsilon^{2\beta-2\alpha}\int_0^\infty\int \check{\Phi}(s,\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2\beta-2\alpha}}\\
&\qquad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\\
&\qquad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{y},\mathbf{q}+\varepsilon^{2-2\alpha}\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\;d\mathbf{p}'\;ds\\
&= \pi\int \varepsilon^{2\beta-2\alpha}\Phi(k^{-1}\mathbf{p}\cdot\mathbf{p}'\varepsilon^{2\beta-2\alpha},\mathbf{p}')e^{i\mathbf{p}'\cdot(\mathbf{x}-\mathbf{y})/\varepsilon^{2\alpha}}\\
&\qquad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{p}'/2)-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\\
&\qquad\times\varepsilon^{2\alpha-2}\left[\theta(\mathbf{y},\mathbf{q}+\varepsilon^{2-2\alpha}\mathbf{p}'/2)-\theta(\mathbf{y},\mathbf{q}-\varepsilon^{2-2\alpha}\mathbf{p}'/2)\right]\;d\mathbf{p}'
\end{aligned}
$$

whose L^2-norm has the following limit

$$
\begin{aligned}
0 = \lim_{\varepsilon\to 0}\pi^2\int\bigg|\int_{\mathbf{p}_\perp\cdot\mathbf{p}=0} & k|\mathbf{p}|^{-1}\left[\int\Phi(w,\mathbf{p}_\perp)dw\right]\\
&\times e^{i\mathbf{p}_\perp\cdot(\mathbf{x}-\mathbf{y})\varepsilon^{-2\alpha}}|\mathbf{q}\cdot\nabla_{\mathbf{p}}\theta(\mathbf{x},\mathbf{p})|^2 d\mathbf{p}_\perp\bigg|^2 d\mathbf{x}d\mathbf{p}
\end{aligned}
$$

if $d\geq 3$ by the dominated convergence theorem because the integrand is bounded by an integrable function behaving like $c|\mathbf{p}|^{-2}$ in a neighborhood of $\mathbf{p}=0$. ■

To identify the limit, we have the following straightforward calculation: For any real-valued, L^2-weakly convergent sequence $\psi^\varepsilon \to \psi$,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&= \lim_{\varepsilon\to 0}\int_0^\infty ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2\beta-2\alpha}}\varepsilon^{2\beta-2\alpha}\varepsilon^{4\alpha-4}\\
&\qquad\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{2+2\beta-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{2+2\beta-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\Big]\\
&= \lim_{\varepsilon\to 0}\int_0^\infty ds\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\;\psi^\varepsilon(\mathbf{x},\mathbf{p})\check{\Phi}(s,\mathbf{q})e^{-ik^{-1}s\mathbf{p}\cdot\mathbf{q}\varepsilon^{2\beta-2\alpha}}\varepsilon^{2\beta-2\alpha}\varepsilon^{4\alpha-4}\\
&\qquad\Big[e^{-ik^{-1}s|\mathbf{q}|^2\varepsilon^{2+2\beta-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad -e^{ik^{-1}s|\mathbf{q}|^2\varepsilon^{2+2\beta-4\alpha}/2}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\Big]\\
&= \lim_{\varepsilon\to 0}\pi\int\varepsilon^{2\alpha-2}\varepsilon^{2\beta-2\alpha}\Big[\Phi(k^{-1}(\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2)\cdot\mathbf{q}\varepsilon^{2\beta-2\alpha},\mathbf{q})\\
&\quad\times\varepsilon^{2-2\alpha}\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]-\Phi(k^{-1}(\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\cdot\mathbf{q}\varepsilon^{2\beta-2\alpha},\mathbf{q})\\
&\quad\times\varepsilon^{2-2\alpha}\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\Big]\psi^\varepsilon(\mathbf{x},\mathbf{p})d\mathbf{q}d\mathbf{x}d\mathbf{p}\\
&= \pi\int\mathbf{q}\cdot\nabla_{\mathbf{p}}\left[\delta(k^{-1}\mathbf{p}\cdot\mathbf{q})\left[\int\Phi(w,\mathbf{q})dw\right]\mathbf{q}\cdot\nabla_{\mathbf{p}}\right]\theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})d\mathbf{q}d\mathbf{x}d\mathbf{p}\\
&= \pi\int k|\mathbf{p}|^{-1}\left[\int\Phi(w,\mathbf{p}_\perp)dw\right](\mathbf{p}_\perp\cdot\nabla_{\mathbf{p}})^2\theta(\mathbf{x},\mathbf{p})\psi(\mathbf{x},\mathbf{p})d\mathbf{p}_\perp d\mathbf{x}d\mathbf{p}\\
&\equiv \bar{A}_1(\psi)
\end{aligned}
$$

where $\mathbf{p}_\perp\in\mathbb{R}^{d-1}, \mathbf{p}_\perp\cdot\mathbf{p}=0$. Note again that

$$
\int_{\mathbf{p}\cdot\mathbf{p}_\perp=0}\int\Phi(w,\mathbf{p}_\perp)\mathbf{p}_\perp\otimes\mathbf{p}_\perp dwd\mathbf{p}_\perp<\infty,\quad\forall\mathbf{p}\in\mathbb{R}^d
$$

because of (10) and that the function

$$
k|\mathbf{p}|^{-1}\int_{\mathbf{p}_\perp\cdot\mathbf{p}=0}\left[\int\Phi(w,\mathbf{p}_\perp)dw\right](\mathbf{p}_\perp\cdot\nabla_{\mathbf{p}})^2\theta(\mathbf{x},\mathbf{p})d\mathbf{p}_\perp
$$

is square-integrable because of $d\geq 3$.

Case (iii): $\alpha = \beta$.

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} A_1^{(1)}(\psi^\varepsilon)\\
&= \lim_{\varepsilon\to 0} \pi \int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi^\varepsilon(\mathbf{x},\mathbf{p})\\
&\quad \varepsilon^{4\alpha-4}\big[\Phi(k^{-1}(\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q}/2)\cdot\mathbf{q},\mathbf{q})\left[\theta(\mathbf{x},\mathbf{p}+\varepsilon^{2-2\alpha}\mathbf{q})-\theta(\mathbf{x},\mathbf{p})\right]\\
&\qquad -\Phi(k^{-1}(\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q}/2)\cdot\mathbf{q},\mathbf{q})\left[\theta(\mathbf{x},\mathbf{p})-\theta(\mathbf{x},\mathbf{p}-\varepsilon^{2-2\alpha}\mathbf{q})\right]\big]\\
&= \pi\int d\mathbf{q}d\mathbf{x}d\mathbf{p}\ \psi(\mathbf{x},\mathbf{p})\mathbf{q}\cdot\nabla_{\mathbf{p}}\left[\Phi(k^{-1}\mathbf{p}\cdot\mathbf{q},\mathbf{q})\mathbf{q}\cdot\nabla_{\mathbf{p}}\right]\theta(\mathbf{x},\mathbf{p})\\
&\equiv \bar{A}_1(\psi).
\end{aligned}
$$

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