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RADIATIVE TRANSFER OF SOUND WAVES IN A RANDOM FLOW: TURBULENT SCATTERING, STRAINING, AND MODE-COUPLING*

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Abstract. We study the sound wave propagation in a random flow, whose mean flow is large compared with its fluctuation, in the infinite three-dimensional space. We consider the intermediate regime, where the range of acoustic wave numbers overlaps with the range of turbulent wave numbers.

We use the multiscale expansions for the Wigner distributions to derive the radiative transport equations that describe the evolution of acoustic correlation and the turbulent scattering, straining, and mode-coupling of sound waves. We show that, because of the flow-straining term, the flow-acoustic scattering becomes nonconservative and, depending on the propagation direction, a sound wave can gain or lose energy. We calculate the attenuation/amplification coefficients due to mode-coupling and/or turbulent scattering with flow-straining. These coefficients depict interesting dependence on the propagating direction and the wave length of sound wave. We demonstrate numerically that the attenuation/amplification coefficients are enhanced significantly when *both* the straining and the mode-coupling effects are present.

We also obtain the diffusion equations on the physical space and, thus, further reduce the dimension of the flow-acoustic equations.

Key words. radiative transfer, flow acoustics, turbulence

AMS subject classifications. 76Q05, 82D30

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1. Introduction.

Flow acoustics concerns three different processes: propagation of sound through flow, generation of sound by flow, and generation of flow by sound [24], [26]. Correspondingly, there are three main effects: scattering (and refraction), acoustic radiation, and absorption. Refraction alters the direction of a beam of sound and is well studied in geometrical acoustics. Scattering is the redistribution of energy among different wave numbers and/or different components (modes) of the same wave number and results in spectral or directional broadening. Both refraction and scattering processes are commonly assumed to preserve the acoustic energy. This is not the case, however, when the effect of flow-straining is taken into account. As a sound wave propagates through an extensive body of the turbulence it may be attenuated or amplified through two mechanisms: wave mode-coupling and straining. These effects are usually small but can be important in a fluid turbulence with a strong mean flow (like a jet or a grid turbulence in a wind tunnel [4], [5], [27]), especially when both the straining and the mode-coupling are active (see Figures 5 and 6).

Turbulent scattering of sound waves has been much studied both theoretically and experimentally since the single-scattering theory of Blokhintzev [3], Lighthill [23], and

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Kraichnan [20] (see [28] and [17] and the references therein). The experimental studies have been mainly of sound propagation in the atmosphere or oceans, but relatively few experiments have been conducted under laboratory conditions. This is more so when it comes to the study of turbulent *absorption* of sound waves, with [18], [25], and [14] being notable exceptions. These experiments suggest loss of acoustic energy which cannot be accounted for by turbulent scattering or refraction caused by the variation of the mean flow. These experiments typically have turbulent intensity of 20%, turbulent Mach number on the order of 10^{-3}, the integral scale roughly 10 cm, and the Kolmogorov dissipation scale roughly 1 mm. The acoustic wave lengths used (e.g., 600 to 5000 Hz in [18]) in the experiments are typically larger than the integral scale. This is a *long* wave regime. The work [18] suggests a turbulent absorption mechanism which is frequency independent in the above frequency range in contrast to the frequency-square dependence of absorption by molecular dissipation. Theories have been proposed to explain this phenomenon qualitatively [27], [16]. The work [27] invokes the semiempirical viscoelasticity theory of turbulence (see also [8]), while [16] extends Lighthill's approach. Both assume certain temporal structure of the turbulence and ignore the effect of the mean flow, while acknowledging that the absorption can become significantly greater in the presence of a mean flow (see also [15]).

We have also seen significant advances in the study of the short wave regime where the geometrical acoustics or the parabolic approximation is valid [28], [32]. In the present study we consider the *intermediate* regime where the sound wave lengths are comparable to the sizes of the turbulent eddies under the influence of a mean flow. This regime would correspond to the high-end audible range or the low-end ultrasonic frequencies in the experimental setting of [18]. Under such a circumstance turbulent scattering and absorption of sound waves are expected to be more pronounced (Figures 3–8).

Our main goal is to derive the transport equation for the phase-space distribution of the sound field with the explicit formulas for the scattering cross section and the attenuation/amplification coefficient. The radiative transport equations describe naturally the aforementioned mechanisms of turbulent scattering, straining, and mode-coupling. We show that, because of the flow-straining term, the flow-acoustic scattering becomes nonconservative and, depending on the propagation direction, the flow can emit or absorb acoustic radiation. Moreover, this effect is most pronounced when *both* the straining and the mode-coupling are active (see Figures 3–8).

2. Phase-space formulation.

2.1. Flow-acoustic equations. Flow field and sound field are different modes of the total fluid motion described by the compressible fluid equations; they are distinguished by dispersion relation (see section 3.2). Although flow acoustics is nonlinear in general, the fraction of the total fluid energy contained in the acoustic field is very small in a subsonic flow (the ratio of sound-to-flow amplitude is of order 10^{-3} for the loudest sounds of interest). The linear mechanisms of the mode-coupling and the flow-straining are more important in this case.

We think of the sound (generated by the flow or other sources) as a small perturbation of a background flow field and study the sound propagation via the linearized Euler equations for weakly compressible, homentropic fluids:

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p_1 = 0, \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} + \mathbf{u} \cdot \nabla \mathbf{V} = 0, \tag{2}$$

where $p_1, \mathbf{V}$ are the pressure and the velocity fields of the underlying random, incompressible flow, $\rho_0 \approx$ const is the density, $\kappa_0 = 1/(c_0^2 \rho_0)$ is the compressibility of the fluid, and $p, \mathbf{u}$ are the pressure and velocity fluctuations due to the perturbation by sound waves. Here c_0 is the sound speed of the (still) fluid and assumed to be a constant.

When κ_0 is relatively small, we can make the following approximation:

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} = 0, \tag{3}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} + \mathbf{u} \cdot \nabla \mathbf{V} = 0. \tag{4}$$

That is, the turbulent pressure field is neglected. Further simplification can be made under suitable conditions by neglecting altogether the lowest order terms responsible for flow-straining effect:

$$\frac{\partial p}{\partial t} + \mathbf{V} \cdot \nabla p + \frac{1}{\kappa_0} \nabla \cdot \mathbf{u} = 0, \tag{5}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho_0} = 0. \tag{6}$$

This approximation is valid for very high frequency waves.

We do not consider the effect of solid structure (such as walls, edges, corners), so (1)–(2), (3)–(4), or (5)–(6) are studied in an infinite medium (see [17]).

The effect of molecular viscosity is neglected. Although viscosity is ultimately responsible for the conversion of mechanical energy into heat, it has a negligible influence on the transfer of energy between the sound and the turbulence. The incompressibility of the flow is not essential to our approach; we assume it to simplify the presentation of the method and the results.

Specifically, we assume the background flow field has a small turbulent intensity with a mean flow, $\bar{\mathbf{u}}$, and write the velocity field as

$$\mathbf{V}(\mathbf{x}) = \bar{\mathbf{u}} + \sqrt{\varepsilon}\mathbf{v}(\mathbf{x}), \tag{7}$$

where $\sqrt{\varepsilon}$ is the turbulent intensity ($\ll 1$, typical of wind tunnels) and $\sqrt{\varepsilon}\mathbf{v}(\mathbf{x})$ is the turbulent velocity fluctuation assumed to be a mean zero, time independent, space homogeneous, divergence-free random field with covariance matrix $\mathbf{R} = [R_{ij}(\mathbf{x})]$, $R_{ij}(\mathbf{x}) = \langle v_i(\cdot) v_j(\cdot + \mathbf{x}) \rangle$. The homogeneity of the turbulence is a reasonable assumption for, e.g., the core region of fully developed turbulent (pipe, jet, or grid) flows, or the outer region of a turbulent boundary layer. The incompressibility of the flow requires that the (turbulent) Mach number measured in the reference frame of the mean flow is infinitesimally small ($M_f \equiv \sqrt{\varepsilon \langle |\mathbf{v}|^2 \rangle}/c_0 \ll 1$), while the mean-flow Mach number ($M_a \equiv |\bar{\mathbf{u}}|/c_0$) is assumed to be small to moderate. We assume for simplicity a *uniform* mean flow, in which case the mean-flow Mach number can be arbitrary. Stratified wind mean velocity profile is an important factor in sound propagation in the atmosphere and the resulting refraction can be accounted for by geometrical acoustics or parabolic approximation. Such an effect is decoupled from the turbulent scattering because of separation of scales.

When the turbulent intensity is small, time dependence of the turbulent fluctuation is negligible compared to the pulsation seen from the moving reference frame of the mean flow. This is called the random sweeping effect [7, 31] and is closely related to the Taylor hypothesis of frozen turbulence [12]. Also, when the turbulent Mach number is small and the incident sound wave lengths are comparable to the eddy sizes, the frequencies of the turbulent pulsations are much smaller than those of the incident sound. Thus only the spatial correlations of the turbulence are pertinent to sound propagation and the turbulence is effectively frozen for our purpose.

The covariance function R_{ij} can be determined from its Fourier transform, which takes the following form in the case of locally isotropic turbulence:

$$\widehat{R}_{ij}(\mathbf{k}) := \mathcal{R}(|\mathbf{k}|)\left(\delta_{ij} - k_i k_j |\mathbf{k}|^{-2}\right)|\mathbf{k}|^{1-d} \quad \forall i, j.$$

The factor $\left(\delta_{ij} - k_i k_j |\mathbf{k}|^{-2}\right)$, resulting from the incompressibility assumption, is the projection onto the plane orthogonal to $\mathbf{k}$. A typical example is given by the power-law spectrum

$$\mathcal{R}(|\mathbf{k}|) = R_0|\mathbf{k}|^{-\nu}, \quad \ell_0^{-1} \ll |\mathbf{k}| \ll \ell_1^{-1}, \quad \text{some } \nu \in (-\infty, \infty),\ R_0 > 0, \tag{8}$$

and decaying rapidly elsewhere. Kolmogorov's spectrum corresponds to $\nu = 5/3$ with ℓ_1, ℓ_0 being the dissipation and the integral lengths, respectively [12].

The form of the velocity field (7) is more of mathematical convenience than necessity. $\bar{\mathbf{u}}$ can be seen as representing the flow component on the integral scale and $\sqrt{\varepsilon}\mathbf{v}$ as that in the inertial subrange. Small ε corresponds, in this connection, to the fact that the integral scale is the energy-containing scale. Indeed, the kinetic energy contained in the shell $[k, 2k]$ diverges as $k \to 0$ for any $\nu \geq 1$, including the Kolmogorov spectrum. In this view, (7) is a simple model of real turbulence which may not exhibit separation of scales between the integral scale and the inertial subrange as implied by (7).

2.2. The Wigner distribution, the Wigner equation, and transport scaling. Radiative transfer theory is well established [6], [2], [19], [29] for *strictly hyperbolic* waves, such as acoustic, electromagnetic, and elastic waves propagation in inhomogeneous media *at rest*, governed by equations of the form

$$\mathbf{C}\frac{\partial \mathbf{w}}{\partial t} + \mathbf{D}^j \frac{\partial \mathbf{w}}{\partial x_j} = 0, \quad \mathbf{x} = (x_1, x_2, x_3), \tag{9}$$

where the positive definite matrix $\mathbf{C} = \mathbf{C}(\mathbf{x})$ represents nonuniform materials properties pertaining to the speed of wave propagation and $\mathbf{D}^j$ are *constant*, symmetric matrices. The wave filed $\mathbf{w}$ may be a scalar, a vector, or a tensor. In the case of underwater sound propagation in a density stratified fluid, the medium fluctuation occurs in material properties, such as density and compressibility, as opposed to nonuniform movements of otherwise uniform medium, and is relatively small:

$$\mathbf{C}(\mathbf{x}) = \mathbf{C}_0 + \sqrt{\varepsilon}\mathbf{C}_1(\mathbf{x}). \tag{10}$$

Assuming the medium perturbation $\mathbf{C}_1$ is smooth, wave propagation on the scale of the inhomogeneities is only slightly perturbed. On larger scales, however, multiple scattering results in significant effects. If interactions between the wave and the medium inhomogeneities are incoherent, then we expect a Markovian-type of transport to take place on larger scales. This motivates the transport scaling

$$t \to t/\varepsilon, \quad \mathbf{x} \to \mathbf{x}/\varepsilon, \quad \mathbf{k} \to \mathbf{k}, \tag{11}$$

where $\mathbf{k}$ is the Fourier variable associated with the scale of the inhomogeneities. Incoherent scattering is a result of the randomness of medium fluctuation and the dimensionality *three.* For periodic medium fluctuation, the Bloch waves arise. In lower dimensions, coherent backscattering may be strong and results in localization, instead of transport, of waves.

Waves usually do not have a well-defined phase-space energy density, a fundamental distinction between waves and particles. However, a *pseudodensity* function, called the Wigner distribution, can be defined as

$$\mathbf{W}(t,\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^d} e^{\boldsymbol{i}\mathbf{k}\cdot\mathbf{y}} \mathbf{w}\left(t,\mathbf{x}-\frac{\mathbf{y}}{2}\right) \otimes \mathbf{w}^*\left(t,\mathbf{x}+\frac{\mathbf{y}}{2}\right)$$

before the scaling factor ε of the medium fluctuation is considered. As the Wigner distribution preserves all the wave properties, it usually *loses* positive semidefiniteness as it evolves in time. Here and below $*$ denotes the complex conjugate for scalars and the conjugate transpose for vectors and matrices, and $\boldsymbol{i}$ denotes the imaginary number $\sqrt{-1}$. In [29], the Wigner distribution was used to derive radiative transfer equations for hyperbolic waves [13].

In contrast, for the flow-acoustic problem at hand, the material properties are uniform and represented by a constant matrix

$$\mathbf{C} = \operatorname{diag}(\rho_0, \rho_0, \rho_0, \kappa_0).$$

(The case of variable $\mathbf{C}$ can be treated also, but we do not pursue it here for the sake of presentation.) Instead of material inhomogeneities, we have a nonuniform fluid flow $\mathbf{u}(\mathbf{x})$. We rewrite the linearized Euler equation as

$$\frac{\partial \mathbf{w}}{\partial t} + \mathbf{u}(\mathbf{x})\cdot\nabla\mathbf{w} + \mathbf{C}^{-1}\mathbf{D}^j\frac{\partial \mathbf{w}}{\partial x^j} + \mathbf{G}\mathbf{w} = 0, \quad \mathbf{w} = (\mathbf{u}, p) \tag{12}$$

with symmetric, constant matrices $\mathbf{D}^j$ and

$$\mathbf{G} = \begin{pmatrix} \tilde{\mathbf{G}} & 0 \\ (\nabla P)^* & 0 \end{pmatrix}, \quad \tilde{\mathbf{G}} = \left[\frac{\partial v_i}{\partial x_j}\right].$$

The lower-order term (not displayed) corresponds to the flow-straining effect. It should be noted that (12) is *not strictly hyperbolic* because of the presence of the nonpropagating vortical mode (see section 3.1).

By the transport scaling, the flow field (7) becomes

$$\bar{\mathbf{u}} + \sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}\right). \tag{13}$$

We also assume

$$p_1 = \sqrt{\varepsilon}P\left(\frac{\mathbf{x}}{\varepsilon}\right)$$

for some space-homogeneous random function P. The rescaled flow-sound field $\mathbf{w}_\varepsilon$ now satisfies the equation

$$\frac{\partial \mathbf{w}_\varepsilon}{\partial t} + \left(\bar{\mathbf{u}} + \sqrt{\varepsilon}\mathbf{v}\left(\frac{\mathbf{x}}{\varepsilon}\right)\right)\cdot\nabla\mathbf{w}_\varepsilon + \mathbf{C}^{-1}\mathbf{D}^j\frac{\partial \mathbf{w}_\varepsilon}{\partial x_j} + \frac{1}{\sqrt{\varepsilon}}\mathbf{G}\left(\frac{\mathbf{x}}{\varepsilon}\right)\mathbf{w}_\varepsilon = 0,$$
$$\mathbf{w}_\varepsilon(0,\mathbf{x}) = \mathbf{w}^0_\varepsilon(\mathbf{x}).$$

For the rescaled field $\mathbf{w}_\varepsilon$ the Wigner distribution should rescale accordingly as

$$\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k}) = \int \frac{d\mathbf{y}}{(2\pi)^{d/2}} e^{i\mathbf{k}\cdot\mathbf{y}} \mathbf{w}_\varepsilon\left(t, \mathbf{x} - \frac{\varepsilon\mathbf{y}}{2}\right) \otimes \mathbf{w}^*_\varepsilon\left(t, \mathbf{x} + \frac{\varepsilon\mathbf{y}}{2}\right), \tag{14}$$

which satisfies the Wigner equation

$$\begin{aligned}
&\frac{\partial \mathbf{W}_\varepsilon}{\partial t} + \bar{\mathbf{u}} \cdot \nabla_{\mathbf{x}} \mathbf{W}_\varepsilon + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j \frac{\partial \mathbf{W}_\varepsilon}{\partial x_j} + \frac{1}{2}\frac{\partial \mathbf{W}_\varepsilon}{\partial x_j}\mathbf{D}^j\mathbf{C}^{-1} \\
&\quad + \frac{i}{\varepsilon} k_j \mathbf{C}^{-1}\mathbf{D}^j \mathbf{W}_\varepsilon - \frac{i}{\varepsilon}\mathbf{W}_\varepsilon k_j \mathbf{D}^j \mathbf{C}^{-1} \\
&\quad = \frac{i}{\sqrt{\varepsilon}} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}) \left[\mathbf{W}_\varepsilon\left(\mathbf{k} + \frac{\mathbf{p}}{2}\right) - \mathbf{W}_\varepsilon\left(\mathbf{k} - \frac{\mathbf{p}}{2}\right)\right] \\
&\quad - \frac{\sqrt{\varepsilon}}{2} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \hat{v}_j(\mathbf{p}) \left[\frac{\partial \mathbf{W}_\varepsilon(\mathbf{k} - \frac{\mathbf{p}}{2})}{\partial x_j} + \frac{\partial \mathbf{W}_\varepsilon(\mathbf{k} + \frac{\mathbf{p}}{2})}{\partial x_j}\right] \\
&\quad - \frac{1}{\sqrt{\varepsilon}} \int \frac{d\mathbf{p}}{(2\pi)^d} e^{i\mathbf{p}\cdot\mathbf{x}/\varepsilon} \left[\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_\varepsilon\left(\mathbf{k} - \frac{\mathbf{p}}{2}\right) + \mathbf{W}_\varepsilon\left(\mathbf{k} + \frac{\mathbf{p}}{2}\right)\hat{\mathbf{G}}^*(\mathbf{p})\right]
\end{aligned} \tag{15}$$

with $\hat{\mathbf{G}}^*(\mathbf{p})$ the Fourier transform of $\mathbf{G}$:

$$\hat{\mathbf{G}}(\mathbf{p}) = i(\hat{\mathbf{v}}(\mathbf{p}), \hat{P}(\mathbf{p})) \otimes \tilde{\mathbf{p}}, \tag{16}$$

where $\tilde{\mathbf{p}} = (p_1, p_2, p_3, 0)$. For ease of notation here and below, we omit writing the independent variables of physical quantities as much as possible. For example, $\mathbf{W}_\varepsilon(\mathbf{k} + \frac{\mathbf{p}}{2})$ denotes $\mathbf{W}_\varepsilon(t, \mathbf{x}, \mathbf{k} + \frac{\mathbf{p}}{2})$ here. We also adopt the summation convention for repeated indices except for Greek letters in the superscript that distinguish different modes. Only the summation over *superscripts in Greek letters* will be displayed explicitly. The derivation of the Wigner equation from the wave equation is lengthy but straightforward, so we leave it to the reader.

Why do we use the Wigner distribution? Among other things, the energy density

$$\mathcal{E}(t,\mathbf{x}) = (\mathbf{C}\mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon)$$

and the energy flux

$$\mathcal{F}_j(t,\mathbf{x}) = u_j(\mathbf{x})(\mathbf{C}\mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon) + (\mathbf{D}^j\mathbf{w}_\varepsilon \cdot \mathbf{w}_\varepsilon)$$

can be recast in terms of the Wigner distribution as

$$\mathcal{E}(t,\mathbf{x}) = \mathrm{Tr} \int d\mathbf{k}\mathbf{C}\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k}), \tag{17}$$

$$\mathcal{F}_j(t,\mathbf{x}) = \mathrm{Tr} \int d\mathbf{k} \left[u_j(\mathbf{x})\mathbf{C}\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k}) + \mathbf{D}^j\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k})\right]. \tag{18}$$

The advantage of the phase-space formulation is that while a transport equation of the form

$$\frac{\partial \mathcal{E}}{\partial t} + \sum_j \frac{\partial \mathcal{F}_j}{\partial x_j} = \mathcal{S}, \tag{19}$$

where $\mathcal{S}$ is the source term accounting for the generation or absorption of sound [17], is not always valid, the Wigner equation is. Moreover, in the transport limit

$\varepsilon \to 0$, the nonpositiveness of the Wigner distribution disappears and the weak limit of the averaged Wigner distribution $\langle \mathbf{W}_\varepsilon \rangle$ satisfies the radiative transfer equation which corresponds to a *Markov* process in the phase space and preserves the positive semidefiniteness. In the following sections, we present and analyze such transport equations, which are derived in the appendices.

Before ending the section, we pause to note that the transport limit of the Wigner equation when $\mathbf{D}^j = \mathbf{G} = 0$ (i.e., the transport of a passive scalar) has been rigorously obtained by an entirely probabilistic method in [11]. Another case where the transport limit has also been rigorously obtained is the Schrödinger wave in a random potential of the form (10) [10] (see also [30] and [1]). In both cases, the unknowns are scalars. In addition, the inhomogeneity of the Schrödinger equation is in the lower-order term.

3. Radiative transfer.

3.1. Linear dispersion: Acoustical and vortical modes. To analyze (15) we consider a multiscale expansion

$$\mathbf{W}_\varepsilon(t,\mathbf{x},\mathbf{k}) = \mathbf{W}_0(t,\mathbf{x},\mathbf{k}) + \sqrt{\varepsilon}\mathbf{W}_1\left(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}\right) + \varepsilon\mathbf{W}_2\left(t,\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{k}\right) + \cdots. \tag{20}$$

We introduce the fast scale variable $\mathbf{z} = \dfrac{\mathbf{x}}{\varepsilon}$ and make the substitution

$$\nabla_\mathbf{x} \to \nabla_\mathbf{x} + \frac{1}{\varepsilon}\nabla_\mathbf{z}$$

in the Wigner equation (15). Inserting (20) we obtain in the leading order $O(\varepsilon^{-1})$:

$$\mathbf{L}(\mathbf{k})\mathbf{W}_0 - \mathbf{W}_0\mathbf{L}^*(\mathbf{k}) = 0, \tag{21}$$

where $\mathbf{L}(\mathbf{k})$ is the dispersion matrix given by

$$\mathbf{L}(\mathbf{k}) = \mathbf{C}^{-1}k_j\mathbf{D}^j.$$

As $\mathbf{L}$ is symmetric with regard to the scalar product $\mathbf{Cx}\cdot\mathbf{y}$, $\mathbf{L}$ has a complete set of eigenvectors and associated real eigenvalues. Let $\omega_\alpha(\mathbf{k})$ be the eigenvalues of $\mathbf{L}(\mathbf{k})$ with the multiplicities r_α, and let $\mathbf{b}^{\alpha,j}(\mathbf{k})$, $j = 1,\ldots,r_\alpha$, be the corresponding eigenvectors

$$\mathbf{L}(\mathbf{k})\mathbf{b}^{\alpha,i} = \omega_\alpha\mathbf{b}^{\alpha,i}, \tag{22}$$

normalized so that

$$\mathbf{C}\mathbf{b}^{\alpha,i}(\mathbf{k})\cdot\mathbf{b}^{\beta,j}(\mathbf{k}) = \delta_{ij}\delta_{\alpha\beta}. \tag{23}$$

For our system (5)–(6), the matrix $\mathbf{C} = \mathrm{diag}(\rho_0,\rho_0,\rho_0,\kappa_0)$. The dispersion matrix $\mathbf{L}(\mathbf{k})$ is

$$\mathbf{L}(\mathbf{k}) = \begin{pmatrix} 0 & 0 & 0 & k_1/\rho_0 \\ 0 & 0 & 0 & k_2/\rho_0 \\ 0 & 0 & 0 & k_3/\rho_0 \\ k_1/\kappa_0 & k_2/\kappa_0 & k_3/\kappa_0 & 0 \end{pmatrix}. \tag{24}$$

It has three eigenvalues:

$$\omega_1 = c_0|\mathbf{k}|, \quad \omega_2 = -c_0|\mathbf{k}|, \quad \omega_3 = 0,$$

where $c_0 = 1/\sqrt{\kappa_0 \rho_0}$ is the speed of sound. The eigenvalues $\omega_1(\mathbf{k}), \omega_2(\mathbf{k})$ are simple, whereas the zero eigenvalue has multiplicity two. The eigenvalues $\omega_1(\mathbf{k})$ and $\omega_2(\mathbf{k})$ correspond to the (acoustical) longitudinal modes (forward and backward, respectively) while the eigenvalue ω_3 corresponds to the (vortical) transverse mode. The corresponding eigenvectors are

$$\mathbf{b}^1 = \left(\frac{\hat{\mathbf{k}}}{\sqrt{2\rho_0}}, \frac{1}{\sqrt{2\kappa_0}}\right), \quad \mathbf{b}^2 = \left(\frac{\hat{\mathbf{k}}}{\sqrt{2\rho_0}}, -\frac{1}{\sqrt{2\kappa_0}}\right), \quad \mathbf{b}^{3,i} = \left(\frac{\hat{\mathbf{k}}_\perp^{(i)}}{\sqrt{\rho_0}}, 0\right), \quad i = 1, 2.$$

To fix the idea, we take the first coordinate axis to be in the direction of $\bar{\mathbf{u}}$, the second to be on the $\bar{\mathbf{u}} - \mathbf{k}$ plane, and the third to be orthogonal to the $\bar{\mathbf{u}} - \mathbf{k}$ plane. Let θ be the angle between $\bar{\mathbf{u}}$ and $\mathbf{k}$. Then we have

$$\hat{\mathbf{k}} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \hat{\mathbf{k}}_\perp^{(1)} = \begin{pmatrix} -\cos\phi\sin\theta \\ \cos\phi\cos\theta \\ \sin\phi \end{pmatrix}, \quad \hat{\mathbf{k}}_\perp^{(2)} = \begin{pmatrix} \sin\phi\sin\theta \\ -\sin\phi\cos\theta \\ \cos\phi \end{pmatrix}, \tag{25}$$

in terms of the angular coordinates θ, ϕ. Note that when $\kappa_0 \ll 1$, the acoustic wave modes point predominantly in the direction of the pressure variable.

The solution $\mathbf{W}_0$ of (21) is given by

$$\begin{aligned} \mathbf{W}_0(t, \mathbf{x}, \mathbf{k}) &= \sum_{\alpha,i,j} W_{ij}^\alpha(t, \mathbf{x}, \mathbf{k}) \mathbf{b}^{\alpha,i}(\mathbf{k}) \otimes \mathbf{b}^{\alpha,j}(\mathbf{k}) \\ &= W^1 \mathbf{b}^1 \otimes \mathbf{b}^1 + W^2 \mathbf{b}^2 \otimes \mathbf{b}^2 + \sum_{i,j=1}^{2} W_{ij}^3 \mathbf{b}^{3,i} \otimes \mathbf{b}^{3,j}, \end{aligned} \tag{26}$$

where $\mathbf{W}^\alpha = [W_{ij}^\alpha]$ are the $r_\alpha \times r_\alpha$ Wigner distribution matrices associated with $\omega_\alpha, \alpha = 1, 2, 3$.

The matrices $\mathbf{W}^\alpha$ provide a decomposition of the total fluid energy density on the phase space among different modes and polarizations so that

$$\mathcal{E}(t, \mathbf{x}) = \sum_\alpha \int d\mathbf{k} \mathrm{Tr} \mathbf{W}^\alpha(t, \mathbf{x}, \mathbf{k}).$$

Likewise, the flux $\boldsymbol{\mathcal{F}}$ can be decomposed as

$$\mathcal{F}_j = \sum_\alpha \int d\mathbf{k} \left[\bar{u}_j + \frac{\partial \omega_\alpha}{\partial k_j}\right] \mathrm{Tr} \mathbf{W}^\alpha(t, \mathbf{x}, \mathbf{k}).$$

Note that the first term corresponds to the mean flow, and the second to the phase speed.

3.2. The radiative transport equations. Throughout this section we use $\tilde{\mathbf{a}}$ to denote the 4-vector

$$\tilde{\mathbf{a}} = (a_1, a_2, a_3, 0)$$

for any given 3-vector $\mathbf{a} = (a_1, a_2, a_3)$. We denote by $\mathbf{R} = [R_{ij}]$ the 4×4 correlation tensor of the vector $(\mathbf{v}, P)$. Note that the divergence-free property of the flow implies the symmetry

$$\hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{k}} \cdot \tilde{\mathbf{k}} = \hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{k}} \cdot \tilde{\mathbf{p}} = \hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})\tilde{\mathbf{p}} \cdot \tilde{\mathbf{p}}. \tag{27}$$

Assuming the expansion (20) and (26) we derive, in Appendix A, the transport equations

$$(28)\quad \frac{\partial \langle W^\tau_{nm}\rangle}{\partial t} + (\bar{\mathbf{u}} + \nabla_{\mathbf{k}}\omega_\tau)\cdot\nabla_{\mathbf{x}}\langle W^\tau_{nm}\rangle + (\langle \mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^\tau(\mathbf{k}))_{nm} + (\Sigma^{\tau *}(\mathbf{k})\langle \mathbf{W}^\tau\rangle(\mathbf{k}))_{nm}$$
$$= \sum_\beta \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p}) S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p}) \langle W^\beta_{rs}\rangle(\mathbf{p})\delta(\omega_\tau(\mathbf{k}) - \omega_\beta(\mathbf{p}) + \bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})),$$

where the total scattering cross section matrices $\Sigma^\tau(\mathbf{k})$ are given by

$$(29)\qquad \Sigma^\tau_{sm}(\mathbf{k}) = -\sum_\beta i \int \frac{d\mathbf{p}}{(2\pi)^d} \frac{\hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S^{\tau\beta}_{jmr}(\mathbf{k},\mathbf{p}) S^{\beta\tau}_{lrs}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k}) - \omega_\beta(\mathbf{p}) + \bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p}) - i0}.$$

Note that $[S^{\tau\beta}_{lnr}]$ is rank-1 for $\tau,\beta = 1,2$, rank-2 if exactly one of two superscripts is 3, and rank-3 if both superscripts are 3. They can be rewritten more concretely as

$$\mathbf{S}^{11}(\mathbf{k},\mathbf{p}) = \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + 1)\tilde{\mathbf{k}} - \frac{1}{2}(|\mathbf{p}| - \mathbf{k}\cdot\hat{\mathbf{p}})\left(\hat{\mathbf{k}}, \sqrt{\frac{\kappa_0}{\rho_0}}\right),$$
$$\mathbf{S}^{22}(\mathbf{k},\mathbf{p}) = \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + 1)\tilde{\mathbf{k}} - \frac{1}{2}(|\mathbf{p}| - \mathbf{k}\cdot\hat{\mathbf{p}})\left(\hat{\mathbf{k}}, -\sqrt{\frac{\kappa_0}{\rho_0}}\right),$$
$$\mathbf{S}^{12}(\mathbf{k},\mathbf{p}) = \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - 1)\tilde{\mathbf{k}} - \frac{1}{2}(|\mathbf{p}| - \mathbf{k}\cdot\hat{\mathbf{p}})\left(\hat{\mathbf{k}}, \sqrt{\frac{\kappa_0}{\rho_0}}\right),$$
$$\mathbf{S}^{21}(\mathbf{k},\mathbf{p}) = \frac{1}{2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - 1)\tilde{\mathbf{k}} - \frac{1}{2}(|\mathbf{p}| - \mathbf{k}\cdot\hat{\mathbf{p}})\left(\hat{\mathbf{k}}, -\sqrt{\frac{\kappa_0}{\rho_0}}\right),$$
$$\mathbf{S}^{13}_{\cdot m}(\mathbf{k},\mathbf{p}) = \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp) + \frac{1}{\sqrt{2}}\mathbf{k}\cdot\hat{\mathbf{p}}^{(m)}_\perp\left(\hat{\mathbf{k}}, \sqrt{\frac{\kappa_0}{\rho_0}}\right),\quad m = 1,2,$$
$$\mathbf{S}^{23}_{\cdot m}(\mathbf{k},\mathbf{p}) = \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp) + \frac{1}{\sqrt{2}}\mathbf{k}\cdot\hat{\mathbf{p}}^{(m)}_\perp\left(\hat{\mathbf{k}}, -\sqrt{\frac{\kappa_0}{\rho_0}}\right),\quad m = 1,2,$$
$$\mathbf{S}^{31}_{\cdot m}(\mathbf{k},\mathbf{p}) = \frac{1}{\sqrt{2}}\tilde{\mathbf{k}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp) + \frac{1}{\sqrt{2}}(\mathbf{k}-\mathbf{p})\cdot\hat{\mathbf{p}}(\hat{\mathbf{k}}^{(m)}_\perp, 0) = \mathbf{S}^{32}_{\cdot m}(\mathbf{k},\mathbf{p}),\quad m = 1,2,$$
$$\mathbf{S}^{33}_{\cdot ml} = \tilde{\mathbf{k}}(\hat{\mathbf{k}}^{(m)}_\perp\cdot\hat{\mathbf{p}}^{(l)}_\perp) + \mathbf{k}\cdot\hat{\mathbf{p}}^{(l)}_\perp(\hat{\mathbf{k}}^{(m)}_\perp, 0),\quad m,l = 1,2,$$

with $\hat{\mathbf{k}}^{(m)}_\perp, \hat{\mathbf{p}}^{(l)}_\perp$ defined as in (25). $\mathbf{S}^{11}, \mathbf{S}^{22}, \mathbf{S}^{33}$ account for the self-coupling of acoustic and vortical modes, $\mathbf{S}^{12}, \mathbf{S}^{21}$ the coupling between the forward and backward acoustic modes, and $\mathbf{S}^{13}, \mathbf{S}^{23}$ the coupling between the acoustic and vortical modes. For weak compressibility $\kappa_0 \ll 1$, we have the following approximations:

$$(30)\qquad \mathbf{S}^{11}(\mathbf{k},\mathbf{p}) \approx \mathbf{S}^{22}(\mathbf{k},\mathbf{p}) \approx (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + 1/2 - |\mathbf{p}|/(2|\mathbf{k}|))\tilde{\mathbf{k}},$$
$$(31)\qquad \mathbf{S}^{12}(\mathbf{k},\mathbf{p}) \approx \mathbf{S}^{21}(\mathbf{k},\mathbf{p}) \approx (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - 1/2 - |\mathbf{p}|/(2|\mathbf{k}|))\tilde{\mathbf{k}},$$
$$(32)\qquad \mathbf{S}^{13}_{\cdot m}(\mathbf{k},\mathbf{p}) \approx \mathbf{S}^{23}_{\cdot m}(\mathbf{k},\mathbf{p}) \approx \sqrt{2}\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(m)}_\perp\tilde{\mathbf{k}},\quad m = 1,2.$$

Equation (28) is a coupled system of equations for the limiting distributions $\langle W^1\rangle(t,\mathbf{x},\mathbf{k})$, $\langle W^2\rangle(t,\mathbf{x},\mathbf{k})$ and $\langle \mathbf{W}^3\rangle(t,\mathbf{x},\mathbf{k})$ for the acoustic and the vortical modes, respectively. The coupling occurs between the level surfaces (ellipsoids) of the (Doppler) acoustic frequency and the level surfaces (planes) of the vortical frequency (see the next section).

Equation (28) preserves the positive definiteness of the initial data so that if initially the matrices $\langle \mathbf{W}^\tau\rangle(0,\mathbf{x},\mathbf{k})$ are positive definite, the solution $\langle \mathbf{W}^\tau\rangle(t,\mathbf{x},\mathbf{k})$ will remain positive definite. However, it does not in general conserve the energy

$$E(t) = \int \mathcal{E}(t,\mathbf{x})d\mathbf{x} = \text{const}$$

(of the perturbation) due to the flow-straining.

To see how the flow-straining term has affected the transport equation, we derive, in Appendix A, the radiative transfer equations by neglecting the term involving $\hat{\mathbf{G}}$ in (15):

$$\begin{aligned}(33)\qquad &\frac{\partial\langle \mathbf{W}^\tau\rangle}{\partial t} + (\bar{\mathbf{u}} + \nabla_{\mathbf{k}}\omega_\tau)\cdot\nabla_{\mathbf{x}}\langle \mathbf{W}^\tau\rangle + \Sigma^\tau\langle \mathbf{W}^\tau\rangle(\mathbf{k}) + \langle \mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^{\tau *}(\mathbf{k})\\ &= \sum_\beta \int \frac{d\mathbf{p}}{(2\pi)^{d-1}}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\mathbf{T}^{\tau\beta}(\mathbf{k},\mathbf{p})\langle \mathbf{W}^\beta\rangle(\mathbf{p})\\ &\times\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k})\delta(\omega_\tau(\mathbf{k}) + \bar{\mathbf{u}}\cdot\mathbf{k} - \omega_\beta(\mathbf{p}) - \bar{\mathbf{u}}\cdot\mathbf{p}),\end{aligned}$$

where the total scattering cross section matrices $\Sigma^\tau(\mathbf{k})$ are given by

$$\Sigma^\tau(\mathbf{k}) = i\sum_\beta\int\frac{d\mathbf{p}}{(2\pi)^d}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{\mathbf{T}^{\tau\beta}(\mathbf{k},\mathbf{p})\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k}) + \bar{\mathbf{u}}\cdot\mathbf{k} - \omega_\beta(\mathbf{p}) - \bar{\mathbf{u}}\cdot\mathbf{p} + i0}$$

with

$$(34)\qquad T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{p}) = \mathbf{C}\mathbf{b}^{\beta,m}(\mathbf{k})\cdot\mathbf{b}^{\alpha,j}(\mathbf{p}),$$

or, more explicitly,

$$\begin{aligned}(35)\qquad & T^{11}(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)/2 = T^{22}(\mathbf{k},\mathbf{p}),\\ (36)\qquad & T^{12}(\mathbf{k},\mathbf{p}) = (\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)/2 = T^{21}(\mathbf{k},\mathbf{p}),\\ (37)\qquad & T^{13}_i(\mathbf{k},\mathbf{p}) = \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}^{(i)}_\perp/\sqrt{2} = T^{23}_i(\mathbf{k},\mathbf{p}),\quad i=1,2,\\ (38)\qquad & T^{33}_{ij}(\mathbf{k},\mathbf{p}) = \hat{\mathbf{k}}^{(i)}_\perp\cdot\hat{\mathbf{p}}^{(j)}_\perp,\qquad i,j=1,2.\end{aligned}$$

We know from (23) that $T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{k}) = \delta_{\alpha\beta}\delta_{mj}$. It is interesting to compare (35)–(38) with (30)–(32) to see the effect of flow-straining. (Note that the square of $\mathbf{k}$ has been factored out of $\mathbf{T}^{ij}$.)

In addition, thanks to the reciprocity relations

$$\mathbf{T}^{\beta\tau}(\mathbf{p},\mathbf{k}) = \mathbf{T}^{\tau\beta *}(\mathbf{k},\mathbf{p})$$

(33) preserves positive definiteness, as well as the averaged total energy

$$\langle E(t)\rangle = \sum_\tau \mathrm{Tr}\int\langle \mathbf{W}^\tau\rangle(\mathbf{x},\mathbf{k})d\mathbf{x}d\mathbf{k} = \text{const.}$$

Energy conservation is due to the absence of the straining term in (5)–(6).

3.3. Resonant interaction: Mode-coupling and conversion. Because of the presence of Dirac's delta-function in the kernels of the transport equations, acoustic/vortical waves interaction is best described in terms of the Doppler frequency surfaces $\mathcal{S}_h^{\pm}$ and $\mathcal{P}_h$, for $h \in R$,

$$\mathcal{S}_h^{\pm} = \{\mathbf{k}|\omega_a^{\pm}(\mathbf{k}) = h\},$$
$$\mathcal{P}_h = \{\mathbf{k}|\omega_v(\mathbf{k}) = h\}$$

with

$$\omega_a^{\pm}(\mathbf{p}) = \pm c_0|\mathbf{p}| + \bar{\mathbf{u}} \cdot \mathbf{p}, \quad \omega_v(\mathbf{p}) = \bar{\mathbf{u}} \cdot \mathbf{p}$$

being the Doppler frequencies of the forward, backward acoustic modes and the vortical mode, respectively.

We have the relation

$$\mathcal{S}_{-h}^{-} = \{-\mathbf{k}|\mathbf{k} \in \mathcal{S}_h^{+}\}, \quad h \in R,$$

between the frequency surfaces $\mathcal{S}_h^{+}$ and $\mathcal{S}_h^{-}$ associated with the forward and backward acoustic modes.

In addition to self-coupling the acoustic modes can interact with the vortical mode through resonance

$$\pm c_0|\mathbf{k}| + \bar{\mathbf{u}} \cdot \mathbf{k} = \bar{\mathbf{u}} \cdot \mathbf{p}$$

and, as a result, the transport equations for the acoustic energy densities $W^i(t, \mathbf{x}, \mathbf{k})$, $i = 1, 2$, couple with that for the vortical Wigner distribution matrix $\mathbf{W}^3(t, \mathbf{x}, \mathbf{p})$.

The surfaces $\mathcal{P}_h$ for the vortical mode are planar. Thus the vortical energy cascades to high wave numbers regardless of coupling with the acoustic mode. The vortical energy cascade does not introduce long-range correlation in velocity fluctuations and, hence, maintains the validity of the short-range correlation assumption in (13).

The frequency surfaces $\mathcal{S}_h^{\pm}$ are hyperboloids for supersonic flows $\bar{\mathbf{u}} > c_0$ and ellipsoids for subsonic flows $\bar{\mathbf{u}} < c_0$. In the supersonic regime the acoustic energy cascades to high wave numbers since $\mathcal{S}_h^{\pm}, h \in R$, also have infinite surface area. The hyperboloids defined by

$$c_0^2|\mathbf{k}|^2 = |h - \bar{\mathbf{u}} \cdot \mathbf{k}|^2, \quad h > 0,$$

have two branches: the forward branch $\mathcal{S}_h^{+}$ and the backward branch $\mathcal{S}_h^{-}$, corresponding to the forward acoustic mode $\omega_1(\mathbf{k}) = c_0|\mathbf{k}|$ and the backward acoustic mode $\omega_2(\mathbf{k}) = -c_0|\mathbf{k}|$, respectively, with the major axis of symmetry in the direction of $\bar{\mathbf{u}}$. The normal form of the hyperboloids is

$$(|\bar{\mathbf{u}}|^2 - c_0^2)\left(\xi_1 - \frac{h|\bar{\mathbf{u}}|}{|\bar{\mathbf{u}}|^2 - c_0^2}\right)^2 - c_0^2(\xi_2^2 + \xi_3^2) = h^2 \frac{c_0^2}{|\bar{\mathbf{u}}|^2 - c_0^2},$$

where ξ_1 is the coordinate in the direction of $\bar{\mathbf{u}}$ and ξ_2, ξ_3 the coordinates in the orthogonal directions. In the long times the acoustic energy is driven in the directions of the asymptotes which form two (forward and backward) cones of semiangle

$$\gamma = \arccos \frac{c_0}{|\bar{\mathbf{u}}|},$$

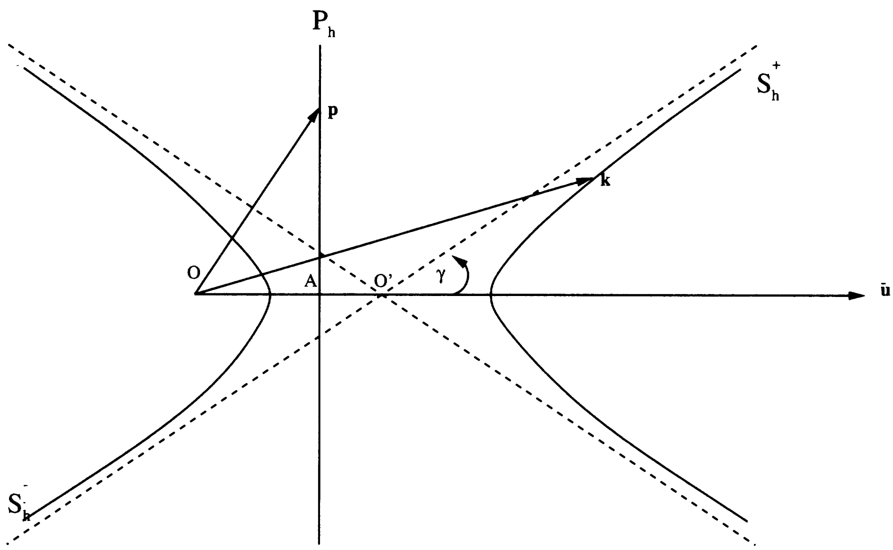

FIG. 1. *The supersonic diagram. The point O is the origin in the wave vector space. The length of $\overline{OA}$ is $|h|/|\bar{\mathbf{u}}|$.*

with the axis of symmetry parallel to $\bar{\mathbf{u}}$, joining their tips at

$$O' = \frac{h\bar{\mathbf{u}}}{\bar{\mathbf{u}}^2 - c_0^2}$$

(Figure 1). Since the slope of the asymptotes is independent of h, the acoustic energy gradually transfers to higher and higher wave vectors parallel to the asymptotes regardless of the initial distribution and eventually dissipates into heat.

Since the subsonic regime $\bar{\mathbf{u}} < c_0$ is probably more relevant, our attention is restricted to this regime in what follows.

The frequency surfaces $\mathcal{S}_h^{\pm}$ are football-like ellipsoids with the major axis parallel to $\bar{\mathbf{u}}$. The normal form of the ellipsoids is

$$\frac{(c_0^2 - |\bar{\mathbf{u}}|^2)^2}{c_0^2 h^2}\left(\xi_1 + \frac{h|\bar{\mathbf{u}}|}{c_0^2 - |\bar{\mathbf{u}}|^2}\right)^2 + \frac{c_0^2 - |\bar{\mathbf{u}}|^2}{h^2}(\xi_2^2 + \xi_3^2) = 1, \tag{39}$$

where the coordinate ξ_1 is in the direction of $\bar{\mathbf{u}}$ and ξ_2, ξ_3 are in the orthogonal directions. The major and minor radii are $hc_0/(c_0^2 - |\bar{\mathbf{u}}|^2)$ and $h/\sqrt{c_0^2 - |\bar{\mathbf{u}}|^2}$, respectively. Note that $\mathcal{S}_h^+$ and $\mathcal{S}_h^-$ exist only for positive and negative h, respectively. This means that the coupling of forward and backward acoustic modes disappears in the subsonic regime (Figure 2). Furthermore, the center of $\mathcal{S}_h^+$ is shifted backward ($h > 0$) and the center of $\mathcal{S}_h^-$ forward ($h < 0$) due to the stronger backscattering of wave vectors in the direction of the mean flow. This has an important implication on the Stokes drift (see section 4.3).

In the long times, equidistribution of energy takes place between the coupled surfaces $\mathcal{S}_h^{\pm}$ and $\mathcal{P}_h$, and within themselves, in the wave vector space. Since the surface area of $\mathcal{S}_h^{\pm}$ is finite the coupled acoustic energy gradually converts into vortical energy

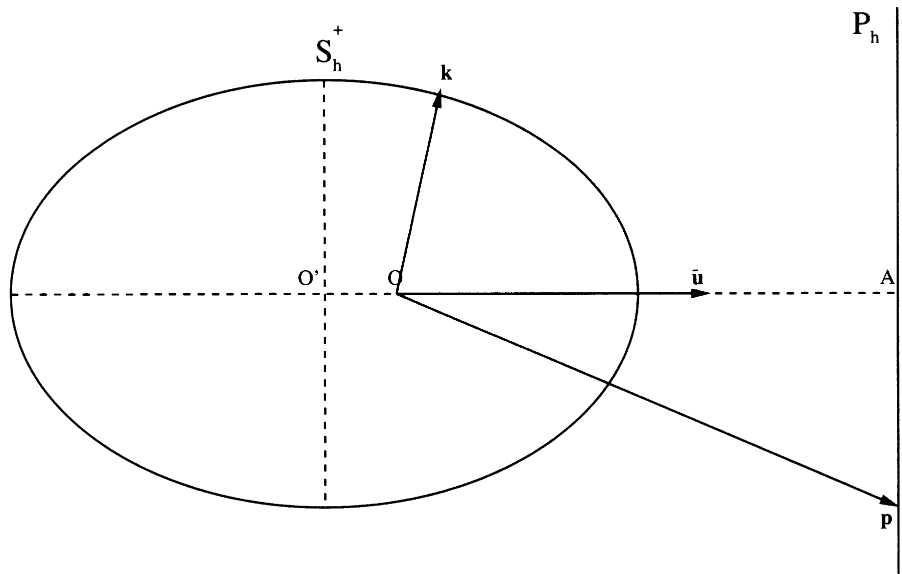

FIG. 2. *The subsonic diagram. The point O is the origin in the wave vector space. The length of $\overline{OA}$ is $h/|\bar{\mathbf{u}}|$.*

which, in turn, is spread further and further over eddies of increasingly smaller scales and eventually is dissipated as heat by molecular viscosity. This is the turbulent dissipation resulting in the absorption of sound waves. Conversely, the background turbulent flow energy can convert into acoustic energy by the mode-coupling mechanism and emit acoustic radiation. The flow-generated sound waves are called aero- or hydrodynamic sound [21], [22].

3.4. Subcritical wave numbers. The acoustic wave vector $\mathbf{k}$ is effectively decoupled from the vortical mode $\mathbf{p}$ if the sound-flow scattering cross section is much smaller than $1/\tau$, where τ is the observation time unit. In this case, the mode-coupling between acoustic and vortical modes does not occur. Such acoustic wave numbers are called *subcritical*; otherwise, they are called *supercritical.*

For simplicity of discussion, let us assume the correlation matrix $\mathbf{R}$ is isotropic and is band-limited with support of $\hat{\mathbf{R}}$ enclosed by the ball of radius r_c which roughly equals $1/\ell_1$ for the power-law spectrum (8). In the high-frequency end, wave vector $\mathbf{k}$ is subcritical if the distance from $\mathcal{S}_h^+$ to $\mathcal{P}_h$ for $h = \omega_a^+(\mathbf{k})$ is greater than r_c. This leads to the explicit condition

$$\omega_a^+(\mathbf{k}) \geq r_c(c_0 + |\bar{\mathbf{u}}|)|\bar{\mathbf{u}}|/c_0 \tag{40}$$

after some calculation or, equivalently,

$$|\mathbf{k}| \geq r_c(1 + M_a)M_a, \tag{41}$$

where $M_a \equiv \bar{\mathbf{u}}/c_0$ is the mean-flow Mach number.

In the low-frequency end, we make the following observation. The ellipsoids $\mathcal{S}_h^\pm$ are nearly spheres centered at $-h\bar{\mathbf{u}}/c_0^2$ with radius $|h|/c_0$. The distance between $\mathcal{P}_h$ and $\mathcal{S}_h^\pm$ is roughly $h/|\bar{\mathbf{u}}|$. For fixed $\mathbf{p} \in \mathcal{P}_h$, $\hat{\mathbf{R}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}$ is of the order $\mathrm{Tr}[\hat{\mathbf{R}}(|\mathbf{p}|)]h^2/c_0^2$.

Subcritical wave vectors $\mathbf{k}$ at the low-frequency end can then be characterized as

$$|\mathbf{k}| \ll \sqrt{\frac{|\bar{\mathbf{u}}|}{\tau \int \mathrm{Tr}[\hat{\mathbf{R}}(r)] r dr}}. \tag{42}$$

4. Subsonic regime.

4.1. Absorption/emission due to mode-coupling. In the subsonic regime the forward and backward acoustic modes are decoupled. We have the (forward) acoustic transport equation:

$$\frac{\partial \langle W^1 \rangle}{\partial t} + (\bar{\mathbf{u}} + \nabla_{\mathbf{k}} \omega_1) \cdot \nabla_{\mathbf{x}} \langle W^1 \rangle + (\Sigma^1(\mathbf{k}) + \Sigma^{1*}(\mathbf{k})) \langle W^1 \rangle (\mathbf{k}) \tag{43}$$

$$= \sum_\beta \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S_{js}^{1\beta}(\mathbf{k},\mathbf{p}) S_{lr}^{1\beta}(\mathbf{k},\mathbf{p}) \langle W_{rs}^\beta \rangle (\mathbf{p}) \delta(c_0|\mathbf{k}| - \omega_\beta(\mathbf{p}) + \bar{\mathbf{u}} \cdot (\mathbf{k}-\mathbf{p})),$$

which is coupled with the vortical mode through $\mathbf{S}^{13}_{\cdot m}, \mathbf{S}^{31}_{\cdot l}$.

As the sound waves propagate through the random flow, in addition to being scattered by the turbulent flow, the *supercritical* acoustic wave numbers can convert into the vortical mode by mode-coupling, resulting in the absorption of these wave numbers and the generation of flow which can occur with or without flow-straining. Conversely, the supercritical acoustic wave numbers can cause the background turbulent fluctuation to emit acoustic radiation via mode-coupling, resulting in the amplification of these wave numbers and sound generation. This latter effect, however, requires the presence of flow-straining.

The absorption/emission coefficient $L(\mathbf{k})$ due to mode-coupling is twice the real part of Σ^1. In the absence of flow-straining, we have

$$\begin{aligned} L(\mathbf{k}) &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}\right] T_r^{13}(\mathbf{k},\mathbf{p}) T_r^{31}(\mathbf{p},\mathbf{k}) \\ &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}\right] \frac{1}{2}(1-|\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}|^2), \end{aligned} \tag{44}$$

which is always nonnegative. Here $d\Omega(\mathbf{p})$ is the area element of the surface $\{\mathbf{p}|\omega_a^+(\mathbf{k}) = \omega_v(\mathbf{p})\}$.

In the presence of flow-straining, we have

$$\begin{aligned} L(\mathbf{k}) &= \int \frac{d\mathbf{p}}{(2\pi)^d} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S_{jr}^{13}(\mathbf{k},\mathbf{p}) S_{lr}^{31}(\mathbf{p},\mathbf{k}) \delta(c_0|\mathbf{k}| + \bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})) \\ &= \frac{1}{|\bar{\mathbf{u}}|(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_v(\mathbf{p})} d\Omega(\mathbf{p}) \left\{ \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}\right](1-|\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}|^2) \right. \\ &\qquad \left. + \sum_{m=1}^{2} \left[\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\hat{\mathbf{p}}_\perp^{(m)}\right]\left[(\mathbf{p}-\mathbf{k})\cdot\hat{\mathbf{k}}\right]\left[\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}_\perp^{(m)}\right] \right\}. \end{aligned} \tag{45}$$

We recall the convention that the lower index r in (45) is summed over $r = 1, 2$ (see (30)–(32)). In addition to the factor 2 in the first term, the second term in (45) is entirely due to flow-straining and can be positive or negative depending on the direction of propagation and, as shown in Figures 3–6, dominates in magnitude over

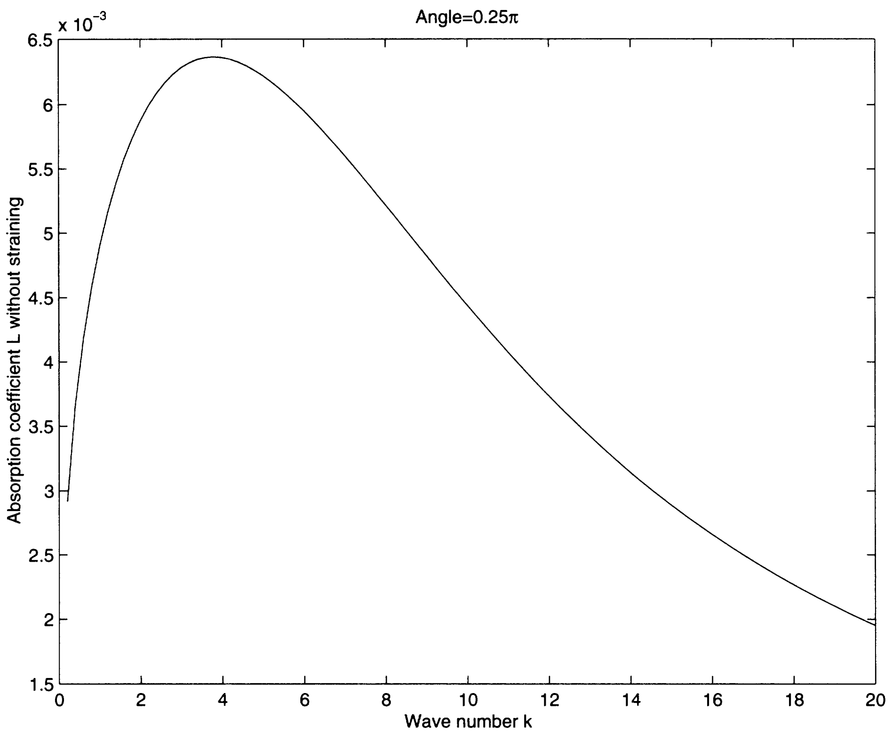

FIG. 3. *$L(k)$ without straining at $\theta = 0.25\pi$.*

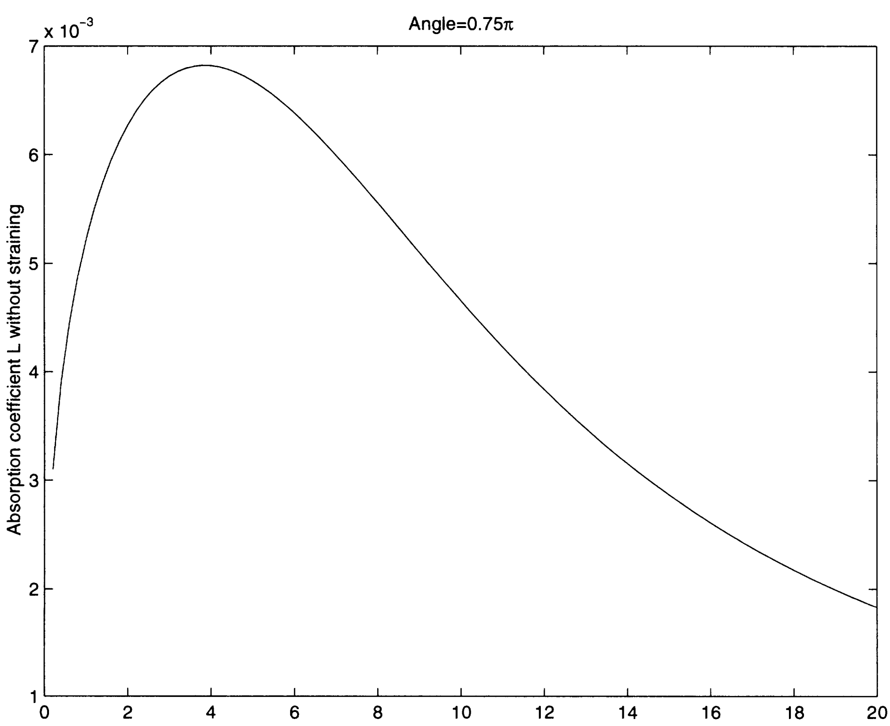

FIG. 4. *$L(k)$ without straining at $\theta = 0.75\pi$.*

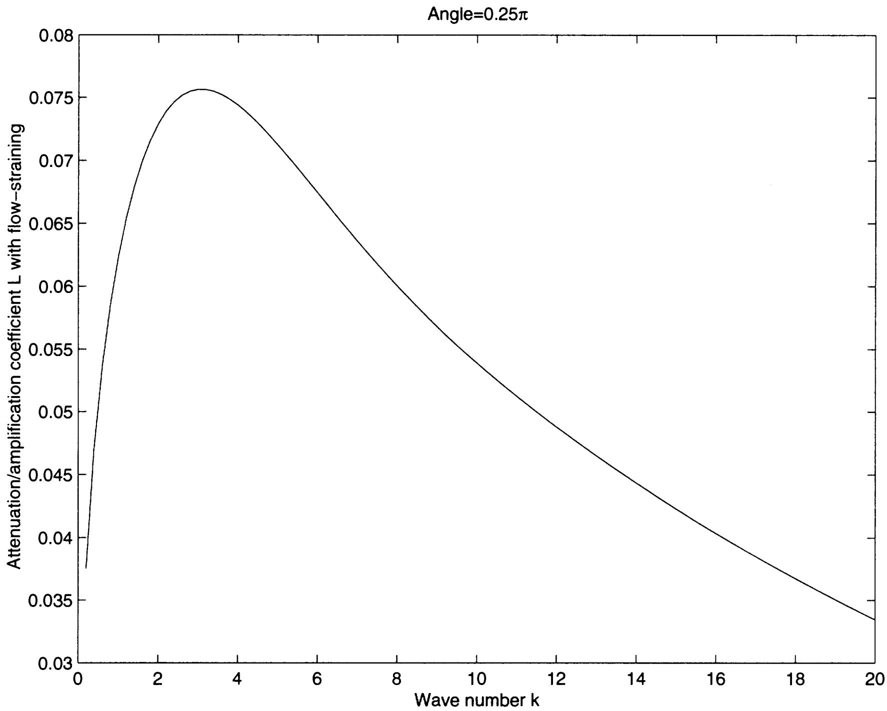

FIG. 5. *$L(k)$ with straining at $\theta = 0.25\pi$.*

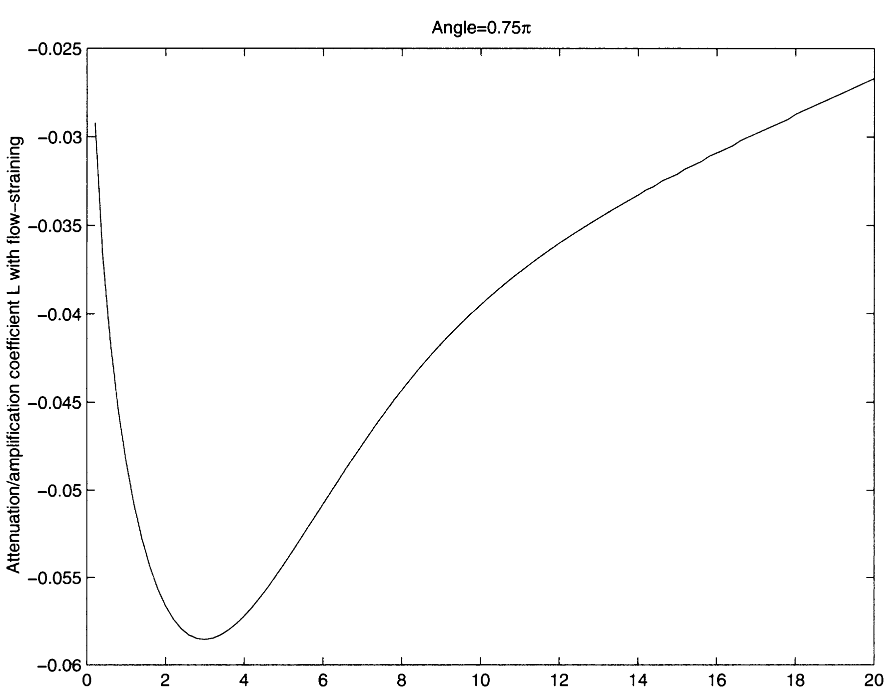

FIG. 6. *$L(k)$ with straining at $\theta = 0.75\pi$.*

the first term at small or medium Mach numbers. The vanishing of $L(\mathbf{k})$ for small and large k corresponds to the conditions for subcritical wave numbers (40)–(42). Note that the numbers in Figures 3 and 4 when the straining is absent are significantly lower in magnitude than those in Figures 5 and 6 when both the flow-straining and the mode-coupling are active.

For the numerical calculations of $L(\mathbf{k})$ (as well as $A(\mathbf{k})$ in the next section) we have used, instead of the Kolmogorov spectrum, the Von Kármán spectrum

$$\mathcal{R}(|\mathbf{k}|) = R_0|\mathbf{k}|^2(|\mathbf{k}|^2 + k_0^2)^{-17/6}, \quad |\mathbf{k}| \le k_1 = 1/\ell_1,$$

and taken $c_0 = 1, \bar{\mathbf{u}} = 0.1, R_0 = 1, k_0 = 1/\ell_0 = 0.1, \ell_1 = 0.01$. The Von Kármán spectrum provides a good approximation to the turbulence spectrum but avoids the abrupt transition at k_0 in the Kolmogorov spectrum [28], and so is more convenient for computation. The angle θ is between the mean flow $\bar{\mathbf{u}}$ and $\mathbf{k}$ (cf. (25)).

4.2. Attenuation/amplification due to scattering with flow-straining. In the simpler case of subcritical regime (i.e., the mode-coupling is absent) we have a single closed equation for $\langle W\rangle(t,\mathbf{x},\mathbf{k})$ (dropping the superscript 1)

$$\frac{\partial\langle W\rangle}{\partial t} + (\bar{\mathbf{u}} + c_0\hat{\mathbf{k}})\cdot\nabla_{\mathbf{x}}\langle W\rangle - A(\mathbf{k})\langle W\rangle = \mathcal{Q}\langle W\rangle, \tag{46}$$

where $\mathcal{Q}$ is a conservative scattering operator with a nonsymmetric kernel:

$$\mathcal{Q}\langle W\rangle = \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|}\left[\sigma(\mathbf{k},\mathbf{p})\langle W\rangle(\mathbf{p}) - \sigma(\mathbf{p},\mathbf{k})\langle W\rangle(\mathbf{k})\right] \tag{47}$$

and $A(\mathbf{k})$ is the attenuation/amplification coefficient given by

$$A(\mathbf{k}) = \frac{1}{(2\pi)^{d-1}}\int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|}\left[\sigma'(\mathbf{k},\mathbf{p}) - \sigma(\mathbf{k},\mathbf{p}) + \sigma(\mathbf{p},\mathbf{k})\right] \tag{48}$$

with

$$\begin{aligned}\sigma(\mathbf{k},\mathbf{p}) &= (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2 \\ &+\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right) \\ &+\frac{1}{4}\frac{\kappa_0}{\rho_0}\hat{R}_p(\mathbf{p}-\mathbf{k})|\mathbf{k}|^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right)^2\end{aligned}$$

and

$$\begin{aligned}\sigma'(\mathbf{k},\mathbf{p}) &= (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\frac{|\mathbf{k}|}{2|\mathbf{p}|} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right) \\ &+\frac{1}{2}\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(1 + \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\right)\left(1 - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right) \\ &+\frac{1}{2}\sqrt{\frac{\kappa_0}{\rho_0}}(\hat{\mathbf{R}}_{\mathbf{v}p}(\mathbf{p}-\mathbf{k})\cdot\mathbf{k})|\mathbf{k}|\left(\frac{|\mathbf{k}|}{2|\mathbf{p}|} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right) \\ &+\frac{1}{4}\frac{\kappa_0}{\rho_0}\hat{R}_p(\mathbf{p}-\mathbf{k})|\mathbf{k}|^2\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right)\left(1 + \hat{\mathbf{k}}\cdot\hat{\mathbf{p}}\right)\left(1 - \frac{|\mathbf{p}|}{|\mathbf{k}|}\right).\end{aligned}$$

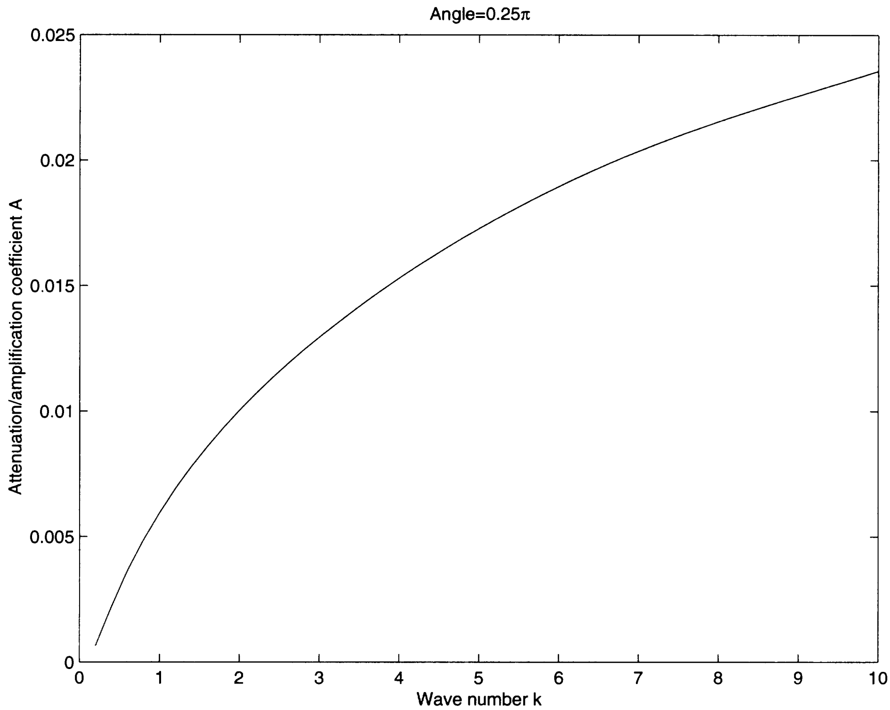

FIG. 7. $A(k)$ *at* $\theta = 0.25\pi$.

Note that $\sigma(\mathbf{k}, \mathbf{p})$ is a quadratic form associated with the nonnegative-definite matrix $\hat{\mathbf{R}}(\mathbf{p} - \mathbf{k})$ for all $\mathbf{k}, \mathbf{p}$ and, thus, is nonnegative for all $\mathbf{k}, \mathbf{p}$. The rate of pumping or draining energy $A(\mathbf{k})$ vanishes, however, if $\bar{\mathbf{u}} = 0$. Both the straining mechanism and the mean flow are necessary for destroying energy conservation in the radiative transfer equation when the temporal variation of the turbulent fluctuation is negligible.

For small $\kappa_0 \ll 1$, the scattering cross sections become

$$\sigma(\mathbf{k}, \mathbf{p}) = (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p} - \mathbf{k})\mathbf{k} \cdot \mathbf{k}) \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2, \tag{49}$$

$$\sigma'(\mathbf{k}, \mathbf{p}) = (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p} - \mathbf{k})\mathbf{k} \cdot \mathbf{k}) \left(\hat{\mathbf{k}} \cdot \hat{\mathbf{p}} + \frac{1}{2} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right) \left(\frac{|\mathbf{k}|}{2|\mathbf{p}|} - \frac{|\mathbf{p}|}{2|\mathbf{k}|}\right). \tag{50}$$

Then, we have $\sigma'(\mathbf{k}, \mathbf{p}) - \sigma'(\mathbf{p}, \mathbf{k}) = \sigma(\mathbf{k}, \mathbf{p}) - \sigma(\mathbf{p}, \mathbf{k})$ and, consequently,

$$A(\mathbf{k}) = \frac{1}{(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k}) = \omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{p}}|} \sigma'(\mathbf{p}, \mathbf{k}). \tag{51}$$

Figures 7 and 8 show the dependence of A, as given by (51), on the wave numbers and the direction of propagation. Note that the numbers in Figures 7 and 8 when the mode-coupling is absent are considerably smaller in magnitude than those in Figures 5 and 6 when both the straining and the mode-coupling are active.

Setting $\bar{\mathbf{u}} = P = 0$ we recover the transport equation of Howe [15]:

$$\frac{\partial \langle W \rangle}{\partial t} + c_0 \hat{\mathbf{k}} \cdot \nabla_{\mathbf{x}} \langle W \rangle = \mathcal{Q}_0 \langle W \rangle \tag{52}$$

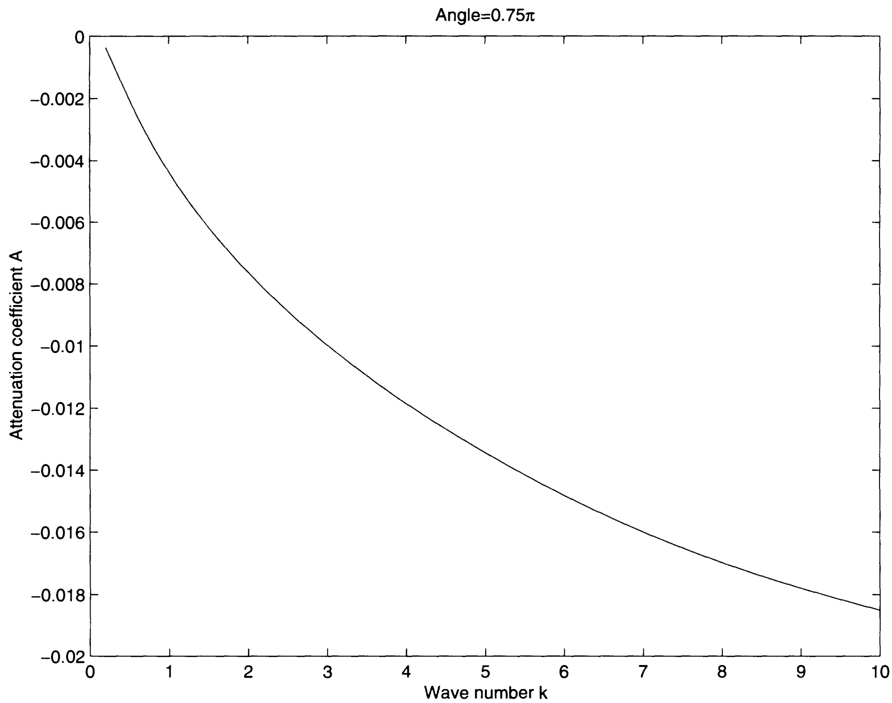

FIG. 8. $A(k)$ at $\theta = 0.75\pi$.

with

$$\mathcal{Q}_0 f := \frac{1}{c_0(2\pi)^{d-1}} \int d\Omega(\mathbf{p})(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\,[f(\mathbf{p})-f(\mathbf{k})]\,. \tag{53}$$

The geometric factor in (53) prohibits orthogonal scattering which would result in $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}} = 0$.

4.3. Diffusion approximation without flow-straining. In the absence of flow-straining, the transport equation for the subcritical wave numbers becomes, after dropping the superscript,

$$\frac{\partial\langle W\rangle}{\partial t} + (\bar{\mathbf{u}} + c_0\hat{\mathbf{k}})\cdot\nabla_{\mathbf{x}}\langle W\rangle = \mathcal{Q}\langle W\rangle \tag{54}$$

with

(55)
$$\mathcal{Q}f := \frac{1}{(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|c_0\hat{\mathbf{p}}+\bar{\mathbf{u}}|}(\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k})\frac{1}{4}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)^2\,[f(\mathbf{p})-f(\mathbf{k})]\,.$$

Notice that the geometric factor changes from $(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2$ in (53) to $(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1)^2/4$ in (55) due to the absence of flow-straining. In contrast to that in (53), the scattering kernel prohibits backscattering which would result in $\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+1=0$.

Next we consider a situation where energy transfer *per frequency* can be adequately described by a physical-space transport equation like (19): sound propagation over distances much longer than the mean free path. In this case the transport process can be decomposed into an *effective* transport and a fluctuation.

Since we expect the fluctuation to be diffusion-like, we make the diffusive scaling after subtracting the effective part

$$t = \frac{s}{\epsilon^2}, \quad \mathbf{x} - \bar{\mathbf{u}}_{\text{eff}} t = \frac{\mathbf{y}}{\epsilon}, \quad \mathbf{k} \to \mathbf{k} \tag{56}$$

with the transport velocity

$$\bar{\mathbf{u}}_{\text{eff}}(h) = \bar{\mathbf{u}} + \frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} c_0 \hat{\mathbf{k}} \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{k}}|}, \quad \Omega_h = \int_{\mathcal{S}_h^+} \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{k}}|}, \tag{57}$$

where the *Stokes drift*

$$\frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} c_0 \hat{\mathbf{k}} \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{k}}|}$$

accounts for the difference between the transport velocity and the Eulerian mean flow. Note that $\bar{\mathbf{u}}_{\text{eff}}$ is parallel to $\bar{\mathbf{u}}$ but is smaller in magnitude since the ellipsoid $\mathcal{S}_h^+$ is shifted to the left from the origin (Figure 2). Thus turbulent scattering reduces the transport velocity of sound resulting in *negative* Stokes drift. This can be explained physically as follows: the wave vectors parallel to the mean flow are scattered more intensely than those antiparallel to the mean flow due to the difference in speed relative to the turbulent fluctuation. Notice also that the reduction in speed is linearly proportional to $\bar{\mathbf{u}}$ for low Mach number and is roughly independent of frequency.

In the multiscale expansion $\langle W \rangle = \bar{W}_0 + \varepsilon \bar{W}_1 + \cdots$, the leading order term $\bar{W}_0$ satisfies a diffusion equation on the physical space (see Appendix B for derivation)

$$\frac{\partial \bar{W}_0}{\partial t} = \nabla_{\mathbf{y}} \cdot \mathbf{D}(\omega_a^+(\mathbf{k})) \nabla_{\mathbf{y}} \bar{W}_0, \tag{58}$$

where the anisotropic diffusion matrix $\mathbf{D} = [D_{ij}]$ is given by

$$\mathbf{D}(h) = \frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0 \hat{\mathbf{k}}) \otimes \chi \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{k}}|}$$

with χ being the vector-valued solution of the equation

$$\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0 \hat{\mathbf{k}} = \mathcal{Q} \chi. \tag{59}$$

By (59) the diffusion matrix $\mathbf{D}$ may be expressed as

$$\mathbf{D}(h) = -\frac{1}{\Omega_h} \int_{\mathcal{S}_h^+} \mathcal{Q} \chi \otimes \chi \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0 \hat{\mathbf{k}}|}.$$

Thus the matrix $\mathbf{D}$ is nonnegative since the operator $\mathcal{Q}$ is nonpositive definite [9].

4.4. Diffusion approximation with flow-straining. In this section we consider the long time limit of the perturbation of Howe's equation (52): the case of small mean-flow Mach number, $M_a = \epsilon \ll 1$. We write the mean flow as $\bar{\mathbf{u}} = \varepsilon c_0 \hat{\mathbf{u}}$ where $\hat{\mathbf{u}} = \bar{\mathbf{u}}/|\bar{\mathbf{u}}|$. The appropriate equation is (46) with (49)–(50).

As shown in Appendix C, the long-time amplification/attenuation effect due to a small mean flow is a second-order effect. To capture the second-order effect we consider the diffusive scaling

$$t \to \frac{t}{\epsilon^2}, \quad \mathbf{x} \to \frac{\mathbf{x}}{\epsilon}, \quad \mathbf{k} \to \mathbf{k} \tag{60}$$

for (46) and derive a phase-space diffusion equation for the leading order term in the expansion $\langle W\rangle = \bar{W}_0 + \varepsilon\bar{W}_1 + \cdots$:

$$\frac{\partial \bar{W}_0}{\partial t} + \mathbf{B}(\omega)\cdot\nabla_{\mathbf{x}}\bar{W}_0 + A(\omega)\bar{W}_0 = \nabla_{\mathbf{x}}\cdot\mathbf{D}\nabla_{\mathbf{x}}\bar{W}_0 \tag{61}$$

with $\omega = c_0|\mathbf{k}|$. The diffusion matrix $\mathbf{D} = [D_{ij}]$ is given by

$$D_{ij}(\omega) = -\frac{1}{4\pi}\int d\Omega(\hat{\mathbf{k}})\chi_i \mathcal{Q}_0 \chi_j \tag{62}$$

and is positive definite. Here $\mathcal{Q}_0$ is defined by (53). The drift $\mathbf{B}$ is given by

$$\begin{aligned}\mathbf{B} = &-\frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[\sigma(\mathbf{k},\mathbf{p})(\chi(\mathbf{p})-\chi(\mathbf{k}))\right]\\ &-\frac{1}{3}c_0\hat{\mathbf{u}} + \frac{1}{2\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{k}}\phi(\mathbf{k})\end{aligned} \tag{63}$$

and the attenuation/amplification coefficient is

$$\begin{aligned}A = &-\frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[\sigma(\mathbf{k},\mathbf{p})(\phi(\mathbf{p})-\phi(\mathbf{k}))\right]\\ &+\frac{1}{4\pi}\int d\Omega(\hat{\mathbf{k}})\phi(\mathbf{k})\mathcal{Q}_0\phi(\mathbf{k}) + \frac{1}{4\pi}\int d\Omega(\hat{\mathbf{k}})(5\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}A_1(\mathbf{k}) - A_2(\mathbf{k})).\end{aligned} \tag{64}$$

Here σ is given by (49) and $\phi, \chi = [\chi_j]$ are solutions of the following equations, respectively:

$$-A_1(\mathbf{k}) = \mathcal{Q}_0\phi(\mathbf{k}), \tag{65}$$

$$c_0\hat{k}_j = \mathcal{Q}_0\chi_j(\mathbf{k}) \tag{66}$$

with

$$A_1(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{c_0(2\pi)^{d-1}}\int d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{R}}_{\mathbf{v}}(|\mathbf{k}|(\hat{\mathbf{p}}-\hat{\mathbf{k}}))\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}}),$$

which is the leading-order term in the small Mach number expansion of $A(\mathbf{k})$, (48), and

$$\begin{aligned}A_2(\mathbf{k}) = &\frac{|\mathbf{k}|^{d-1}}{2c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})((\hat{\mathbf{k}}-\hat{\mathbf{p}})\cdot\hat{\mathbf{u}})\\ &\times\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}(3\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1) + \hat{\mathbf{k}}\cdot\hat{\mathbf{u}}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\right]\\ &-\frac{|\mathbf{k}|^{d-2}}{c_0}\int_{|\mathbf{p}|=|\mathbf{k}|}\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}})^2\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\right].\end{aligned}$$

Equation (61) describes the transport mechanisms as projected on the physical space. This reduction is possible because of the decoupling of different acoustic frequencies due to a vanishing Mach number. Equation (61) is simpler to solve than (46) because the dimension is halved once the two auxiliary equations (65) and (66) are solved in the phase space. Finally we remark that although the derivation of the diffusion equation is lengthy, it can be done rigorously because we start with the transport equation rather than the linearized Euler equation.

5. Conclusion. We study the sound wave propagation in a random flow whose mean flow is large compared with its fluctuation in the infinite three-dimensional space. In such a case, the temporal variation of turbulent fluctuation is negligible. We consider the intermediate regime, where the range of acoustic wave numbers overlaps with the range of turbulent wave numbers, between the short wave regime of geometric acoustics or paraxial approximation and the long wave regime of homogenization.

We have used the multiscale expansions for the Wigner distribution to derive the radiative transport equations that describe the evolution of the correlation of acoustic field and the turbulent scattering, straining, and mode-coupling of sound waves. We have shown that, because of the flow-straining term, the flow-acoustic scattering becomes nonconservative.

Further, we have calculated the attenuation/amplification coefficients due to mode-coupling and the scattering with flow-straining, respectively. We show that the absorption or the emission of sound waves is significantly enhanced when *both* the straining and the mode-coupling are active. The anisotropy of the coefficients is due to the presence of mean flow.

Finally, we have obtained the diffusion equations which describe the transport process in the physical space, and, thus, further reduced the dimension of the flow-acoustic equations.

Appendix A. Derivation of the radiative transport equations.

We derive the radiative transport equations (28) in this appendix. The order $O(\varepsilon^{-1})$ terms in (15) imply that the leading term in the asymptotic expansion (20) has the form (26). The order $O(\varepsilon^{-1/2})$ terms imply that

$$\begin{aligned}
(67)\quad & \bar{\mathbf{u}}\cdot\nabla_{\mathbf{z}}\mathbf{W}_1 + i\mathbf{C}^{-1}k_j\mathbf{D}^j\mathbf{W}_1 - i\mathbf{W}_1k_j\mathbf{D}^j\mathbf{C}^{-1}\frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j\frac{\partial\mathbf{W}_1}{\partial z_j} + \frac{1}{2}\frac{\partial\mathbf{W}_1}{\partial z_j}\mathbf{D}^j\mathbf{C}^{-1}\\
&= i\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))\left[\mathbf{W}_0\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)-\mathbf{W}_0\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right]\\
&\quad -\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\left[\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_0\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)+\mathbf{W}_0\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)\hat{\mathbf{G}}^*(\mathbf{p})\right],
\end{aligned}$$

and in the order ε^0 we get

$$\begin{aligned}
&\frac{\partial\mathbf{W}_0}{\partial t} + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\mathbf{W}_0 + \bar{\mathbf{u}}\cdot\nabla_{\mathbf{z}}\mathbf{W}_2 + i\mathbf{C}^{-1}k_j\mathbf{D}^j\mathbf{W}_2 - i\mathbf{W}_2k_j\mathbf{D}^j\mathbf{C}^{-1}\\
&+\frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j\frac{\partial\mathbf{W}_0}{\partial x_j} + \frac{1}{2}\frac{\partial\mathbf{W}_0}{\partial x_j}\mathbf{D}^j\mathbf{C}^{-1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{D}^j\frac{\partial\mathbf{W}_2}{\partial z_j} + \frac{1}{2}\frac{\partial\mathbf{W}_2}{\partial z_j}\mathbf{D}^j\mathbf{C}^{-1}\\
&= i\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))\left[\mathbf{W}_1\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)-\mathbf{W}_1\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right]\\
&\quad -\frac{1}{2}\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\hat{v}_j(\mathbf{p})\left[\frac{\partial\mathbf{W}_1(\mathbf{k}-\frac{\mathbf{p}}{2})}{\partial z_j}+\frac{\partial\mathbf{W}_1(\mathbf{k}+\frac{\mathbf{p}}{2})}{\partial z_j}\right]\\
(68)\quad &\quad -\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\left[\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_1\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)+\mathbf{W}_1\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)\hat{\mathbf{G}}^*(\mathbf{p})\right].
\end{aligned}$$

The term $\mathbf{W}_1$ can be solved for in terms of $\mathbf{W}_0$ from (67). Inserting the resulting expression into (68) and averaging we can get a closed equation for $\langle\mathbf{W}_0\rangle$. This is done as follows. We solve (67) for $\mathbf{W}_1(t,\mathbf{x},\mathbf{z},\mathbf{k})$ by taking the Fourier transform of this equation in $\mathbf{z}$ and representing

$$\hat{\mathbf{W}}_1(t,\mathbf{x},\mathbf{p},\mathbf{k}) = \sum_{\alpha,\beta,n,m}F^{\alpha\beta}_{nm}(t,\mathbf{x},\mathbf{p},\mathbf{k})\mathbf{b}^{\alpha,n}\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)\otimes\mathbf{b}^{\beta,m}\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right),$$

where $\mathbf{b}^{\alpha,n}(\mathbf{k})$ are defined by (22). We insert (69) and (26), multiply the resulting equation by the matrix $\mathbf{Cb}^{\beta,m}(\mathbf{k}-\frac{\mathbf{p}}{2})\otimes(\mathbf{Cb}^{\alpha,n}(\mathbf{k}+\frac{\mathbf{p}}{2}))$, and take the trace. Now we get

$$
\begin{aligned}
(69) \qquad & F^{\alpha\beta}_{nm}(t,\mathbf{x},\mathbf{p},\mathbf{k}) \\
&= \frac{W^{\alpha}_{nj}(\mathbf{k}+\frac{\mathbf{p}}{2})\left[(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))T^{\beta\alpha}_{mj}(\mathbf{k}-\frac{\mathbf{p}}{2},\mathbf{k}+\frac{\mathbf{p}}{2})+i(\hat{\mathbf{G}}^*(\mathbf{p})\mathbf{Cb}^{\beta,m}(\mathbf{k}-\frac{\mathbf{p}}{2})\cdot\mathbf{b}^{\alpha,j}(\mathbf{k}+\frac{\mathbf{p}}{2}))\right]}{\omega_\alpha(\mathbf{k}+\frac{\mathbf{p}}{2})-\omega_\beta(\mathbf{k}-\frac{\mathbf{p}}{2})+\bar{\mathbf{u}}\cdot\mathbf{p}-i\theta} \\
&\quad -\frac{W^{\beta}_{jm}(\mathbf{k}-\frac{\mathbf{p}}{2})\left[(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))T^{\beta\alpha}_{jn}(\mathbf{k}-\frac{\mathbf{p}}{2},\mathbf{k}+\frac{\mathbf{p}}{2})-i(\mathbf{C}\hat{\mathbf{G}}(\mathbf{p})\mathbf{b}^{\beta,j}(\mathbf{k}-\frac{\mathbf{p}}{2})\cdot\mathbf{b}^{\alpha,n}(\mathbf{k}+\frac{\mathbf{p}}{2}))\right]}{\omega_\alpha(\mathbf{k}+\frac{\mathbf{p}}{2})-\omega_\beta(\mathbf{k}-\frac{\mathbf{p}}{2})+\bar{\mathbf{u}}\cdot\mathbf{p}-i\theta}.
\end{aligned}
$$

$T^{\alpha\beta}_{mj}(\mathbf{k},\mathbf{p})=\mathbf{Cb}^{\beta,m}(\mathbf{k})\cdot\mathbf{b}^{\alpha,j}(\mathbf{p})$ as in (35)–(38). To avoid a vanishing denominator, a regularization parameter θ is introduced. The limit $\theta\to 0$ will be taken at the end of the derivation.

To get an effective equation we average equations (68), multiply throughout the averaged equation by the matrix $\mathbf{C}^{\tau,mn}(\mathbf{k})=\mathbf{Cb}^{\tau,m}(\mathbf{k})\otimes(\mathbf{Cb}^{\tau,n}(\mathbf{k}))$, and take the trace. We assume that $\mathbf{W}_2(t,\mathbf{x},\mathbf{z},\mathbf{k})$ is sublinear in the fast spatial variable $\mathbf{z}$ and thus $\langle\nabla_{\mathbf{z}}\mathbf{W}_2\rangle=0$. We note that

$$\mathrm{Tr}\mathbf{C}^{\tau,mn}(\mathbf{k})(\mathbf{L}(\mathbf{k})\mathbf{W}_2-\mathbf{W}_2\mathbf{L}^*(\mathbf{k}))=0$$

because of (22).

We get the left side of the equation:

$$LHS=\frac{\partial\langle W^{\tau}_{nm}\rangle}{\partial t}+(\bar{\mathbf{u}}+\nabla_{\mathbf{k}}\omega_\tau)\cdot\nabla_{\mathbf{x}}\langle W^{\tau}_{nm}\rangle.$$

The right side is a sum of several terms:

$$(70) \qquad RHS=I^{\tau}_{1,nm}+I^{\tau}_{2,nm}+I^{\tau}_{3,nm},$$

where

$$
\begin{aligned}
I^{\tau}_{1,nm}&=-\frac{1}{2}\left\langle\mathrm{Tr}\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\hat{v}_j(\mathbf{p})\left(\frac{\partial\mathbf{W}_1(\mathbf{k}-\frac{\mathbf{p}}{2})}{\partial z_j}+\frac{\partial\mathbf{W}_1(\mathbf{k}+\frac{\mathbf{p}}{2})}{\partial z_j}\right)\mathbf{C}^{\tau,mn}(\mathbf{k})\right\rangle,\\
I^{\tau}_{2,nm}&=\left\langle\mathrm{Tr}\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\left(i(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))\mathbf{W}_1\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)-\mathbf{W}_1\left(\mathbf{k}+\frac{\mathbf{p}}{2}\right)\hat{\mathbf{G}}^*(\mathbf{p})\right)\mathbf{C}^{\tau,mn}(\mathbf{k})\right\rangle,
\end{aligned}
$$

and

$$I^{\tau}_{3,nm}=-\left\langle\mathrm{Tr}\int\frac{d\mathbf{p}}{(2\pi)^d}e^{i\mathbf{p}\cdot\mathbf{z}}\left(i(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))\mathbf{W}_1\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)+\hat{\mathbf{G}}(\mathbf{p})\mathbf{W}_1\left(\mathbf{k}-\frac{\mathbf{p}}{2}\right)\right)\mathbf{C}^{\tau,mn}(\mathbf{k})\right\rangle.$$

First we observe that $I^{\tau}_{1,nm}$ contains (after averaging) terms of the form $p_j\hat{R}_{jl}(\mathbf{p})$ which vanish because $\mathbf{v}(\mathbf{x})$ is incompressible. Thus we have

$$I^{\tau}_{1,nm}=0.$$

We also have the symmetry

$$(71) \qquad I^{\tau}_{2,nm}=I^{\tau *}_{3,mn}.$$

So we have only $I^{\tau}_{2,nm}$ to compute. We have

(72)

$$I^{\tau}_{2,nm} = \sum_{\alpha,\beta}\Big\langle \iint \frac{d\mathbf{p}d\mathbf{q}e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{z}}}{(2\pi)^d} F^{\alpha\beta}_{n_1,m_1}\Big(\mathbf{q},\mathbf{k}+\frac{\mathbf{p}}{2}\Big) \times\Big[i(\mathbf{k}\cdot\hat{\mathbf{v}}(\mathbf{p}))\Big(\mathbf{C}\mathbf{b}^{\tau,m}(\mathbf{k})\cdot\mathbf{b}^{\beta,m_1}\Big(\mathbf{k}+\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2}\Big)\Big)\Big(\mathbf{C}\mathbf{b}^{\tau,n}(\mathbf{k})\cdot\mathbf{b}^{\alpha,n_1}\Big(\mathbf{k}+\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2}\Big)\Big) - \Big(\hat{\mathbf{G}}^*(\mathbf{p})\mathbf{C}\mathbf{b}^{\tau,m}(\mathbf{k})\cdot\mathbf{b}^{\beta,m_1}\Big(\mathbf{k}+\frac{\mathbf{p}}{2}-\frac{\mathbf{q}}{2}\Big)\Big)\Big(\mathbf{C}\mathbf{b}^{\tau,n}(\mathbf{k})\cdot\mathbf{b}^{\alpha,n_1}\Big(\mathbf{k}+\frac{\mathbf{p}}{2}+\frac{\mathbf{q}}{2}\Big)\Big)\Big]\Big\rangle.$$

We insert expression (69) for $F^{\alpha\beta}_{n_1,m_1}$ and (16) for $\hat{\mathbf{G}}(\mathbf{p})$ into (72). We evaluate this average by assuming the hypothesis of scales separation: W^{α}_{ij} varies on a scale much slower than the velocity fluctuation so that, up to the leading order, $\langle W^{\alpha}_{ij}\rangle$ factors out. We get

(73)

$$I^{\tau}_{2,nm} = \sum_{\alpha,\beta}\int\frac{d\mathbf{p}}{(2\pi)^d}\delta_{\tau\alpha}\delta_{nn_1}\hat{R}_{sl}(\mathbf{p})\Big[i\tilde{k}_s T^{\tau\beta}_{mm_1}(\mathbf{k},\mathbf{k}+\mathbf{p}) - i(\tilde{\mathbf{p}}\cdot\mathbf{b}^{\beta m_1}(\mathbf{k}+\mathbf{p}))(\mathbf{C}\mathbf{b}^{\tau m}(\mathbf{k}))_s\Big] \times\Bigg\{\frac{\langle W^{\alpha}_{n_1 j}\rangle(\mathbf{k})[\tilde{k}_l T^{\beta\alpha}_{m_1 j}(\mathbf{k}+\mathbf{p},\mathbf{k}) + (\tilde{\mathbf{p}}\cdot\mathbf{b}^{\alpha j}(\mathbf{k}))(\mathbf{C}\mathbf{b}^{\beta m_1})_l(\mathbf{k}+\mathbf{p})]}{\omega_\alpha(\mathbf{k})-\omega_\beta(\mathbf{k}+\mathbf{p})-\bar{\mathbf{u}}\cdot\mathbf{p}-i\theta} - \frac{\langle W^{\beta}_{jm_1}\rangle(\mathbf{k}+\mathbf{p})[\tilde{k}_l T^{\beta\alpha}_{jn_1}(\mathbf{k}+\mathbf{p},\mathbf{k}) - (\tilde{\mathbf{p}}\cdot\mathbf{b}^{\beta j}(\mathbf{k}+\mathbf{p}))(\mathbf{C}\mathbf{b}^{\alpha n_1})_l(\mathbf{k})]}{\omega_\alpha(\mathbf{k})-\omega_\beta(\mathbf{k}+\mathbf{p})-\bar{\mathbf{u}}\cdot\mathbf{p}-i\theta}\Bigg\}.$$

We define the tensor

$$S^{\tau\beta}_{jml}(\mathbf{k},\mathbf{p}) = \tilde{k}_j T^{\tau\beta}_{ml}(\mathbf{k},\mathbf{p}) - ((\tilde{\mathbf{p}}-\tilde{\mathbf{k}})\cdot\mathbf{b}^{\beta,l}(\mathbf{p}))(\mathbf{C}\mathbf{b}^{\tau,m}(\mathbf{k}))_j,$$

which, like $T^{\tau\beta}_{ml}(\mathbf{k},\mathbf{p})$ in (34), is completely determined by the dispersion matrix $\mathbf{L}(\mathbf{k})$ but independent of the flows. For the system (1)–(2) we have the explicit expression for $S^{\tau\beta}_{jml}$ as in section 3.2. Then we may rewrite (73), using also incompressibility of the fluctuations $\mathbf{v}(\mathbf{y})$, as

$$I^{\tau}_{2,nm} = \sum_{\beta} i\int\frac{d\mathbf{p}}{(2\pi)^d}\frac{\langle W^{\tau}_{ns}\rangle(\mathbf{k})\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jmr}(\mathbf{k},\mathbf{p})S^{\beta\tau}_{lrs}(\mathbf{p},\mathbf{k})}{\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})-i\theta} - i\int\frac{d\mathbf{p}}{(2\pi)^d}\frac{\langle W^{\beta}_{rs}\rangle(\mathbf{p})\hat{R}_{jl}(\mathbf{p}-\mathbf{k})S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p})S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p})}{\omega_\tau(\mathbf{k})-\omega_\beta(\mathbf{p})+\bar{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p})-i\theta}. \tag{74}$$

Note that because $\mathbf{v}$ is incompressible, the tensor $\hat{\mathbf{R}}$

$$\hat{\mathbf{R}} = \begin{pmatrix}\hat{\mathbf{R}}_{\mathbf{v}} & \hat{\mathbf{R}}_{\mathbf{v},p}\\ \hat{\mathbf{R}}^*_{\mathbf{v},p} & \hat{R}_p\end{pmatrix}$$

satisfies

$$\hat{\mathbf{R}}(\mathbf{k})\tilde{\mathbf{k}} = \begin{pmatrix}\mathbf{0}\\ \hat{\mathbf{R}}_{\mathbf{v},p}(\mathbf{k})\cdot\mathbf{k}\end{pmatrix},$$

and

$$(\hat{\mathbf{R}}_{\mathbf{v},p}(\mathbf{k}) \cdot \mathbf{k})\delta(\mathbf{k}+\mathbf{q}) = \langle \hat{P}(\mathbf{k})\hat{\mathbf{v}}(\mathbf{q})\rangle \cdot \mathbf{k} = -\langle \hat{P}(\mathbf{k})\hat{\mathbf{v}}(\mathbf{q}) \cdot \mathbf{q}\rangle = 0.$$

We combine (74) and the symmetry (71) to evaluate the right side of (70) in the limit $\theta \to 0$ and obtain

$$\begin{aligned} RHS = \sum_\beta \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \hat{R}_{jl}(\mathbf{p}-\mathbf{k}) S^{\tau\beta}_{jms}(\mathbf{k},\mathbf{p}) S^{\tau\beta}_{lnr}(\mathbf{k},\mathbf{p}) \langle W^\beta_{rs}\rangle(\mathbf{p}) \delta(\omega_\tau(\mathbf{k}) \\ - \omega_\beta(\mathbf{p}) + \bar{\mathbf{u}} \cdot (\mathbf{k}-\mathbf{p})) \\ - ((\langle \mathbf{W}^\tau\rangle(\mathbf{k})\Sigma^\tau(\mathbf{k}))_{nm} - (\Sigma^{\tau *}(\mathbf{k})\langle \mathbf{W}^\tau\rangle(\mathbf{k}))_{nm} \end{aligned}$$

with Σ^τ given by (29).

Appendix B. Derivation of the diffusion equation without flow-straining. Here we derive the diffusion equation (58). Equation (54), after the change of coordinates (56), becomes

$$\begin{aligned} &\epsilon^2 \frac{\partial \langle W\rangle}{\partial s} + \epsilon(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \langle W\rangle \\ &= \frac{1}{(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|} (\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}) \frac{1}{4}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 [\langle W\rangle(\mathbf{p}) - \langle W\rangle(\mathbf{k})]. \end{aligned} \tag{75}$$

Inserting the expansion $\langle W\rangle = \bar{W}_0 + \epsilon\bar{W}_1 + \epsilon^2\bar{W}_2 + \cdots$ into (75) we obtain, in the order $O(1)$,

$$\frac{1}{(2\pi)^{d-1}} \int_{\omega_a^+(\mathbf{k})=\omega_a^+(\mathbf{p})} \frac{d\Omega(\mathbf{p})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{p}}|} (\hat{\mathbf{R}}(\mathbf{p}-\mathbf{k})\mathbf{k}\cdot\mathbf{k}) \frac{1}{4}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 [\bar{W}_0(\mathbf{p}) - \bar{W}_0(\mathbf{k})] = 0.$$

Then it follows from the Krein–Rutman theorem [9] that $\bar{W}_0$ is constant on the ellipsoids $\mathcal{S}_h^\pm$ for all $h \in R$:

$$\bar{W}_0 = \bar{W}_0(s, \mathbf{y}, c_0|\mathbf{k}| + \bar{\mathbf{u}} \cdot \mathbf{k}). \tag{76}$$

The order $O(\epsilon)$ term in (54) gives

$$(\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_0 = \mathcal{Q}\bar{W}_1(\mathbf{k}) \tag{77}$$

with the operator $\mathcal{Q}$ as defined by (55). The solvability condition for $\bar{W}_1$ from (77) is

$$\int_{\omega_a^+(\mathbf{k})=h} (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}}(h) + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_0(\mathbf{k}) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0 \quad \forall h \in R.$$

Using (76) we obtain the effective drift $\bar{\mathbf{u}}_{\text{eff}}$ as given in (57).

To solve (77) we first solve for the corrector $\chi(\mathbf{k})$

$$\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}} = \mathcal{Q}\chi. \tag{78}$$

Then the first-order term $\bar{W}_1$ is given by

$$\bar{W}_1(t, \mathbf{x}, \mathbf{k}) = \chi(\mathbf{k}) \cdot \nabla_{\mathbf{y}} \bar{W}_0. \tag{79}$$

The order $O(\varepsilon^2)$ terms in (54) give

$$\frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_1 = \mathcal{Q}\bar{W}_2,$$

which is solvable provided that

$$\int_{S_h^+} \left(\frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} \bar{W}_1 \right) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0 \quad \forall h \in R.$$

Using (79) we have

$$\int_{S_h^+} \left(\frac{\partial \bar{W}_0}{\partial s} + (\bar{\mathbf{u}} - \bar{\mathbf{u}}_{\text{eff}} + c_0\hat{\mathbf{k}}) \cdot \nabla_{\mathbf{y}} [\chi \cdot \nabla_{\mathbf{y}} \bar{W}_0] \right) \frac{d\Omega(\mathbf{k})}{|\bar{\mathbf{u}} + c_0\hat{\mathbf{k}}|} = 0 \quad \forall h \in R.$$

Thus the limiting diffusion equation for $\bar{W}_0$ is (58).

Appendix C. Derivation of the diffusion approximation with flow-straining. We derive in this section the diffusion approximation with straining (61). In the long run the deviation from the equilibrium state with equidistribution of energy on frequency surfaces is small. It is more convenient to use a slightly different form of the transport equation (46), which, after rescaling, becomes

$$\begin{aligned} (80) \qquad & \epsilon^2 \frac{\partial \langle W \rangle}{\partial t} + \epsilon c_0 (\hat{\mathbf{k}} + \epsilon \hat{\mathbf{u}}) \cdot \nabla_{\mathbf{x}} \langle W \rangle - L'_\epsilon(\mathbf{k}) \langle W \rangle \\ & = \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \sigma(\mathbf{k}, \mathbf{p}) (\langle W \rangle(\mathbf{p}) - \langle W \rangle(\mathbf{k})) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}} \cdot (\mathbf{k} - \mathbf{p})), \end{aligned}$$

where

$$(81) \qquad L'_\epsilon(\mathbf{k}) = \int \frac{d\mathbf{p}}{(2\pi)^{d-1}} \sigma'(\mathbf{k}, \mathbf{p}) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}} \cdot (\mathbf{k} - \mathbf{p})),$$

which is different from (51) since $\sigma(\mathbf{k}, \mathbf{p}) \neq \sigma(\mathbf{p}, \mathbf{k})$ in general. The cross sections $\sigma(\mathbf{k}, \mathbf{p})$ and $\sigma'(\mathbf{k}, \mathbf{p})$ are given by (49) and (50). Let us first consider the coefficient $L'_\epsilon(\mathbf{k})$ defined above.

We use the delta function expansion

$$\begin{aligned} (82) \qquad & \int d\mathbf{p} f(\mathbf{p}) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}| + \epsilon c_0 \hat{\mathbf{u}} \cdot (\mathbf{k} - \mathbf{p})) \\ & = \int d\mathbf{p} f(\mathbf{p}) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}|) - \epsilon \int d\mathbf{p} f(\mathbf{p}) (\hat{\mathbf{p}} \cdot \hat{\mathbf{u}}) \delta(c_0|\mathbf{k}| - c_0|\mathbf{p}|) \\ & \quad + \epsilon \frac{|\mathbf{k}|}{c_0} \int_{|\mathbf{p}|=|\mathbf{k}|} d\Omega(\hat{\mathbf{p}}) (\hat{\mathbf{u}} \cdot (\hat{\mathbf{k}} - \hat{\mathbf{p}})) \hat{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \left(|\mathbf{p}|^2 f(\mathbf{p}) \right) + O(\epsilon^2) \end{aligned}$$

for any smooth, fast decaying test functions $f(\mathbf{p})$. Using (82) we write

$$(83) \qquad L'_\epsilon(\mathbf{k}) = \epsilon A_1 + \epsilon^2 A_2 + \cdots.$$

The scalars $A_{1,2}$ are given by

$$A_1(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{c_0} \int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}} (\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{p} - \mathbf{k}) \mathbf{k} \cdot \mathbf{k}) (\hat{\mathbf{k}} \cdot \hat{\mathbf{p}}) ((\hat{\mathbf{p}} - \hat{\mathbf{k}}) \cdot \hat{\mathbf{u}})$$

and

$$A_2(\mathbf{k}) = \frac{|\mathbf{k}|^{d-1}}{2c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})((\hat{\mathbf{k}}-\hat{\mathbf{p}})\cdot\hat{\mathbf{u}})$$
$$\times\left[\hat{\mathbf{p}}\cdot\hat{\mathbf{u}}(3\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}-1)+\hat{\mathbf{k}}\cdot\hat{\mathbf{u}}(1+\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})\right]$$
$$-\frac{|\mathbf{k}|^{d-2}}{c_0}\int_{|\mathbf{p}|=|\mathbf{k}|}\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})((\hat{\mathbf{p}}-\hat{\mathbf{k}})\cdot\hat{\mathbf{u}})^2\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\right].$$

Note that we have

$$\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k}) = 0 \tag{84}$$

and hence the amplification/attenuation effect is only second order in the Mach number ϵ, and hence diffusive scaling leads to a nontrivial limit. Similarly, we expand the right-hand side of (80).

We assume the expansion $\langle W\rangle = \bar{W}_0^\epsilon + \epsilon\bar{W}_1 + \epsilon^2\bar{W}_2 + \cdots$ with the ansatz for the leading term

$$\bar{W}_0^\epsilon = \bar{W}_0(t,\mathbf{x},c_0|\mathbf{k}|+\epsilon c_0\hat{\mathbf{u}}\cdot\mathbf{k}) = \bar{W}_0(t,\mathbf{x},\omega)+\epsilon c_0\hat{\mathbf{u}}\cdot\mathbf{k}\frac{\partial\bar{W}_0(c_0|\mathbf{k}|)}{\partial\omega}+\cdots.$$

Here $\omega = c_0|\mathbf{k}|$ denotes the frequency variable. Then the contribution of the leading term $\bar{W}_0^\epsilon$ to the right side of (80) vanishes identically,

$$\int\frac{d\mathbf{p}}{(2\pi)^{d-1}}\sigma(\mathbf{k},\mathbf{p})\left[\bar{W}_0^\epsilon(\mathbf{p})-\bar{W}_0^\epsilon(\mathbf{k})\right]\delta(c_0|\mathbf{k}|-c_0|\mathbf{p}|+\epsilon c_0\hat{\mathbf{u}}\cdot(\mathbf{k}-\mathbf{p}))=0,$$

since $\bar{W}_0^\epsilon$ is constant on the frequency ellipsoids. The leading-order term in (80) is $O(\epsilon)$:

$$c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_0(\omega)-A_1(\mathbf{k})\bar{W}_0(c_0|\mathbf{k}|)=\mathcal{Q}_0\bar{W}_1 \tag{85}$$

with $\mathcal{Q}_0$ as defined by (53). The solvability condition for (85) is that the integral of the left side over the frequency sphere $\mathcal{S}_h^+ = \{\mathbf{k} : c_0|\mathbf{k}| = h\}$ for all $h > 0$ vanishes. We note that

$$\int c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_0(c_0|\mathbf{k}|)\delta(c_0|\mathbf{k}|-\omega)dk = 0 \quad \forall\omega\geq 0$$

and also (84) holds. Thus, (85) is solvable by the Fredholm alternative.

Let $\phi(\mathbf{k})$ and $\chi_j(\mathbf{k})$ be solutions of (65) and (66), respectively. Then $\bar{W}_1$ is given by

$$\bar{W}_1 = \sum_{j=1}^d\chi_j(\mathbf{k})\frac{\partial\bar{W}_0}{\partial x_j}+\phi(\mathbf{k})\bar{W}_0. \tag{86}$$

Substituting (83), (C) in (80), we find the order $O(\epsilon^2)$ term

$$\frac{\partial\bar{W}_0}{\partial t}+c_0\hat{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\bar{W}_0+c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_1+c_0^2\hat{\mathbf{u}}\cdot\mathbf{k}\hat{k}_j\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega}-c_0\hat{\mathbf{u}}\cdot\mathbf{k}A_1(\mathbf{k})\frac{\partial\bar{W}_0}{\partial\omega}$$
$$-A_1\bar{W}_1-A_2\bar{W}_0-\mathcal{Q}_1\bar{W}_1=\mathcal{Q}_0\bar{W}_2, \tag{87}$$

since the terms on the right that involve $\bar{W}_0$ and its derivatives vanish identically by construction. The operator $\mathcal{Q}_1$ on the left side of (87) is defined by

$$\begin{aligned}\mathcal{Q}_1 f = &-\frac{|\mathbf{k}|^{d-1}}{c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(f(\mathbf{p})-f(\mathbf{k}))\\ &+\frac{|\mathbf{k}|^{d-2}}{c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\\ &\times\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(f(\mathbf{p})-f(\mathbf{k}))\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\right].\end{aligned}$$

As before $\bar{W}_2$ is solvable from (87) if the left side of (87) has the zero mean on every frequency surface $c_0|\mathbf{k}|=\omega$. Let us now compute all the terms in (87) after averaging over this sphere of wave vectors:

$$\int d\mathbf{k}\frac{\partial\bar{W}_0}{\partial t}\delta(c_0|\mathbf{k}|-\omega)=\frac{4\pi|\mathbf{k}|^2}{c_0}\frac{\partial\bar{W}_0(\omega)}{\partial t}$$

and

$$\int d\mathbf{k}c_0\hat{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\bar{W}_0\delta(c_0|\mathbf{k}|-\omega)=4\pi|\mathbf{k}|^2\hat{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\bar{W}_0=\frac{4\pi|\mathbf{k}|^2}{c_0}\mathbf{B}_0\cdot\nabla_{\mathbf{x}}\bar{W}_0.$$

By (86),

$$\begin{aligned}&\int d\mathbf{k}c_0\hat{\mathbf{k}}\cdot\nabla_{\mathbf{x}}\bar{W}_1(\mathbf{k})\delta(c_0|\mathbf{k}|-\omega)\\ &\quad=|\mathbf{k}|^2\int d\Omega(\hat{\mathbf{k}})\hat{k}_n\chi_j(\mathbf{k})\frac{\partial^2\bar{W}_0}{\partial x_j\partial x_n}+|\mathbf{k}|^2\int d\Omega(\hat{\mathbf{k}})\hat{k}_n\phi(\mathbf{k})\frac{\partial\bar{W}_0}{\partial x_n}\\ &\quad:=-\frac{4\pi|\mathbf{k}|^2}{c_0}D_{nj}\frac{\partial^2\bar{W}_0}{\partial x_n\partial x_j}+\frac{4\pi|\mathbf{k}|^2}{c_0}\mathbf{B}_1\cdot\nabla_{\mathbf{x}}\bar{W}_0,\end{aligned}$$

and

$$\int d\mathbf{k}c_0^2\hat{\mathbf{u}}\cdot\mathbf{k}\hat{k}_j\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega}\delta(c_0|\mathbf{k}|-\omega)=\frac{4\pi c_0|\mathbf{k}|^3}{3}\hat{u}_j\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega}:=\frac{4\pi|\mathbf{k}|^2}{c_0}F_j^0\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega}. \tag{88}$$

So we have $\mathbf{B}_1=[B_{1,j}], A_0$ with

$$\begin{aligned}B_{1,j}&=\frac{1}{4\pi}\int d\Omega(\hat{\mathbf{p}})\phi\mathcal{Q}_0\chi_j,\\ A_0&=\frac{1}{4\pi}\int d\Omega(\hat{\mathbf{p}})\phi\mathcal{Q}_0\phi.\end{aligned}$$

The next term on the left side is

$$\begin{aligned}-c_0\int d\mathbf{k}\hat{\mathbf{u}}\cdot\mathbf{k}A_1(\mathbf{k})\delta(c_0|\mathbf{k}|-\omega)\frac{\partial\bar{W}_0}{\partial\omega}&=-|\mathbf{k}|^3\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}A_1(\mathbf{k})\frac{\partial\bar{W}_0}{\partial\omega}\\ &:=\frac{4\pi|\mathbf{k}|^2}{c_0}C_1\frac{\partial\bar{W}_0}{\partial\omega}.\end{aligned} \tag{89}$$

The next term on the left in (87) is

$$
\begin{aligned}
-\int d\mathbf{k} A_1(\mathbf{k})\bar{W}_1(\mathbf{k})\delta(c_0|\mathbf{k}|-\omega) &= -\frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k})\chi_j(\mathbf{k})\frac{\partial \bar{W}_0}{\partial x_j} \\
&\quad -\frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k})\phi(\mathbf{k})\bar{W}_0 \\
&:= \frac{4\pi|\mathbf{k}|^2}{c_0}\mathbf{B}_2\cdot\nabla_{\mathbf{x}}\bar{W}_0 + \frac{4\pi|\mathbf{k}|^2}{c_0}A_0\bar{W}_0.
\end{aligned}
$$

Thus, $\mathbf{B}_2 = [B_{2,j}]$ with

$$
B_{2,j} = \frac{1}{4\pi}\int d\Omega(\hat{\mathbf{p}})\mathcal{Q}_0\chi_j\phi.
$$

The second-to-last term on the left of (87) is

$$
-\frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{k}})A_2(\mathbf{k})\bar{W}_0(\omega) := \frac{4\pi|\mathbf{k}|^2}{c_0}A_1\bar{W}_0
$$

where A_1 becomes part of the attenuation/amplification coefficient. The last term on the left side of (87) that we have to compute is

$$
\begin{aligned}
-\int d\mathbf{k}\mathcal{Q}_1[\bar{W}_1]\delta(c_0|\mathbf{k}|-\omega) &= -\int d\mathbf{k}\mathcal{Q}_1\left[\chi_j\frac{\partial \bar{W}_0}{\partial x_j}\right]\delta(c_0|\mathbf{k}|-\omega) \\
&\quad -\int d\mathbf{k}\mathcal{Q}_1[\phi\bar{W}_0]\delta(c_0|\mathbf{k}|-\omega) \\
&:= I + II.
\end{aligned}
$$

The first term may be written as

$$
\begin{aligned}
I &= \frac{|\mathbf{k}|^4}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{R}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\chi_j(\mathbf{p})-\chi_j(\mathbf{k}))\frac{\partial \bar{W}_0}{\partial x_j}(\omega) \\
&\quad -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}}) \\
&\quad \times\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{R}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\chi_j(\mathbf{p})\frac{\partial \bar{W}_0}{\partial x_j}(c_0|\mathbf{p}|)-\chi_j(\mathbf{k})\frac{\partial \bar{W}_0}{\partial x_j}(c_0|\mathbf{k}|)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\right] \\
&:= I_1 + I_2.
\end{aligned}
$$

Using (66) we have

$$
\begin{aligned}
I_1 &= -\frac{|\mathbf{k}|^2}{c_0}\int\frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}\mathcal{Q}_0\chi_j(\mathbf{p})(\hat{\mathbf{p}}\cdot\hat{\mathbf{u}})\frac{\partial \bar{W}_0}{\partial x_j} \\
&= -|\mathbf{k}|^2\int d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{x}}\bar{W}_0 = -\frac{4\pi|\mathbf{k}|^2}{3}\hat{\mathbf{u}}\cdot\nabla_{\mathbf{x}}\bar{W}_0
\end{aligned}
$$

and

(90)

$$
\begin{aligned}
I_2 &= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Bigg[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \\
&\quad\times\left(\chi_j(\mathbf{p})\frac{\partial\bar{W}_0(c_0|\mathbf{p}|)}{\partial x_j}-\chi_j(\mathbf{k})\frac{\partial\bar{W}_0(c_0|\mathbf{k}|)}{\partial x_j}\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\Bigg] \\
&= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\Bigg\{|\mathbf{k}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\chi_j(\mathbf{p})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 c_0\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega} \\
&\quad+\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\left(\chi_j(\mathbf{p})-\chi_j(\mathbf{k})\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\right]\frac{\partial\bar{W}_0(c_0|\mathbf{k}|)}{\partial x_j}\Bigg\} \\
&:= \frac{4\pi|\mathbf{k}|^2}{c_0}\left[F_j^1\frac{\partial^2\bar{W}_0}{\partial x_j\partial\omega}+\mathbf{B}_3\cdot\nabla_{\mathbf{x}}\bar{W}_0\right].
\end{aligned}
$$

Next we look at

$$
II = -\int d\mathbf{k}\mathcal{Q}_1[\phi(\mathbf{k})\bar{W}_0(\mathbf{k})]\delta(c_0|\mathbf{k}|-\omega) = II_1 + II_2.
$$

By (65) we have

$$
\begin{aligned}
II_1 &= -\frac{|\mathbf{k}|^2}{c_0}\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}\mathcal{Q}_0\chi_j(\mathbf{p})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\bar{W}_0 \\
&= \frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{p})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\bar{W}_0(\omega) = \frac{|\mathbf{k}|^2}{c_0}\int d\Omega(\hat{\mathbf{p}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})A_1(\mathbf{p})\bar{W}_0 \\
&= -\frac{4\pi|\mathbf{k}|^2}{c_0}\frac{C_1}{\omega}\bar{W}_0 \\
&:= \frac{4\pi|\mathbf{k}|^2}{c_0}A_2\bar{W}_0
\end{aligned}
$$

and

(91)

$$
\begin{aligned}
II_2 &= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\Bigg[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k}) \\
&\quad\times\left(\phi(\mathbf{p})\bar{W}_0(c_0|\mathbf{p}|)-\phi(\mathbf{k})\bar{W}_0(c_0|\mathbf{k}|)\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\Bigg] \\
&= -\frac{|\mathbf{k}|^3}{c_0^2}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}-\hat{\mathbf{u}}\cdot\hat{\mathbf{p}})\Bigg\{|\mathbf{k}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\phi(\mathbf{p})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2 c_0\frac{\partial\bar{W}_0}{\partial\omega} \\
&\quad+\hat{\mathbf{p}}\cdot\nabla_{\mathbf{p}}\left[|\mathbf{p}|^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})\left(\phi(\mathbf{p})-\phi(\mathbf{k})\right)\left(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}}+\frac{1}{2}-\frac{|\mathbf{p}|}{2|\mathbf{k}|}\right)^2\right]\bar{W}_0(c_0|\mathbf{k}|)\Bigg\} \\
&:= \frac{4\pi|\mathbf{k}|^2}{c_0}\left\{C_2\frac{\partial\bar{W}_0}{\partial\omega}+A_3\bar{W}_0\right\}.
\end{aligned}
$$

Now we start simplifying these expressions and putting them together. The terms involving the mixed second-order derivative in x_j and ω appear in (88) and (90). Their total contribution is zero:

$$\begin{aligned}
F_j &= F_j^0 + F_j^1 \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\chi_j(\mathbf{p}) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\int \frac{d\Omega(\mathbf{p})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0|\mathbf{k}|}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\mathcal{Q}_0\chi_j(\mathbf{k}) \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0^2|\mathbf{k}|}{4\pi}\int d\Omega(\hat{\mathbf{k}})\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}\hat{k}_j \\
&= \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j - \frac{c_0^2|\mathbf{k}|}{3}\hat{u}_j = 0,
\end{aligned}$$

where we used (66). The drift in frequency $\partial\bar{W}_0/\partial\omega$ appears in (89) and (91). Its total contribution is also zero:

$$\begin{aligned}
&C_1 + C_2 \\
&= C_1 - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2\phi(\mathbf{p}) \\
&= C_1 - \frac{|\mathbf{k}|^3}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\phi(\mathbf{p}) - \phi(\mathbf{k})) \\
&= C_1 + \frac{|\mathbf{k}|c_0}{4\pi}\int d\Omega(\hat{\mathbf{k}})A_1(\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}}) = 0,
\end{aligned}$$

where we used (65) and (89). Therefore different frequencies are decoupled.

Now we look at the drift in the spatial variable $\mathbf{x}$ and the amplification/attenuation coefficient. The following term in the expression of $\mathbf{B}_3$ can be simplified:

$$\begin{aligned}
&-\frac{2|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \\
&= -\frac{4|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\chi_j(\mathbf{p}) - \chi_j(\mathbf{k})) \\
&= -\frac{1}{\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\mathcal{Q}_0\chi_j \\
&= -\frac{4}{3}c_0\hat{u}_j,
\end{aligned}$$

as well as a similar term in the expression of A_3,

$$\begin{aligned}
&-\frac{2|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}} - \hat{\mathbf{u}}\cdot\hat{\mathbf{p}})(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\phi(\mathbf{p}) - \phi(\mathbf{k})) \\
&= -\frac{4|\mathbf{k}|^2}{4\pi c_0}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})\int \frac{d\Omega(\hat{\mathbf{p}})}{(2\pi)^{d-1}}(\hat{\mathbf{k}}\cdot\hat{\mathbf{p}})^2(\hat{\mathbf{R}}_{\mathbf{v}}(\mathbf{k}-\mathbf{p})\mathbf{k}\cdot\mathbf{k})(\phi(\mathbf{p}) - \phi(\mathbf{k})) \\
&= \frac{4}{4\pi}\int d\Omega(\hat{\mathbf{k}})(\hat{\mathbf{u}}\cdot\hat{\mathbf{k}})A_1(\mathbf{k}) = -4C_1/\omega.
\end{aligned}$$

Note also that $\mathbf{B}_2 = \mathbf{B}_1$. Then we collect the rest of the terms. In summary we get a diffusion equation for $\bar{W}_0$:

$$\frac{\partial \bar{W}_0}{\partial t} + \mathbf{B}(\omega) \cdot \nabla_{\mathbf{x}} \bar{W}_0 + A(\omega)\bar{W}_0 = D_{ij} \frac{\partial^2 \bar{W}_0}{\partial x_i \partial x_j}$$

with $\mathbf{B}, A, D_{ij}$ given by (63), (64), (62), respectively.

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REFERENCES

[1] G. BAL, A. FANNJIANG, G. PAPANICOLAOU, AND L. RYZHIK, *Radiative transport in a periodic structure*, J. Statist. Phys., 95 (1999), pp. 479–494.

[2] I. M. BESIERIS AND W. E. KOHLER, *Underwater sound wave propagation in the presence of a randomly perturbed sound speed profile.* I, SIAM J. Appl. Math., 34 (1978), pp. 423–436.

[3] D. BLOKHINTZEV, *The propagation of sound in an inhomogeneous and moving medium* II, J. Acoust. Soc. Amer., 18 (1946), pp. 329–334.

[4] E. H. BROWN AND S. F. CLIFFORD, *On the attenuation of sound by turbulence*, J. Acoust. Soc. Amer., 60 (1976), pp. 788–794.

[5] L. M. B. C. CAMPOS, *On waves in gases. Part* I: *Acoustics of jets, turbulences and ducts*, Rev. Modern Phys., 58 (1986), pp. 117–182.

[6] S. CHANDRASEKHAR, *Radiative Transfer*, Dover, New York, 1960.

[7] S. CHEN AND R. H. KRAICHNAN, *Sweeping decorrelation in isotropic turbulence*, Phys. Fluids A, 1 (1989), pp. 2019–2024.

[8] S. C. CROW, *Visco-elastic character of fine-grained isotropic turbulence*, Phys. Fluids, 10 (1967), pp. 1587–1589.

[9] R. DAUTRAY AND J. L. LIONS, *Mathematical Analysis and Numerical Methods for Science and Technology*, Springer-Verlag, New York, 1990.

[10] L. ERDÖS AND H.-T. YAU, *Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation*, Comm. Pure Appl. Math., 53 (2000), pp. 667–735.

[11] A. FANNJIANG, L. RYZHIK, AND G. PAPANICOLAOU, *Evolution of trajectory correlations in steady random flows*, in Recent Advances in Partial Differential Equations, Proc. Sympos. Appl. Math. 54, R. Spigler and S. Venakides, eds., AMS, Providence, RI, 1998, pp. 105–130.

[12] U. FRISCH, *Turbulence: The Legacy of A.N. Kolmogorov*, Cambridge University Press, Cambridge, UK, 1995.

[13] P. GÉRARD, P. MARKOVICH, N. MAUSER, AND F. POUPAUD, *Homogenization limits and Wigner transforms*, Comm. Pure Appl. Math., 50 (1997), pp. 323–380.

[14] A. GUÉDEL, *Scattering of an acoustic field by a free jet shear layer*, J. Sound Vibration, 100 (1985), pp. 285–304.

[15] M. HOWE, *Multiple scattering of sound by turbulence and other inhomogeneities*, J. Sound Vibration, 27 (1973), pp. 455–476.

[16] M. S. HOWE, *On the absorption of sound by turbulence and other hydrodynamic flows*, IMA J. Appl. Math., 32 (1984), pp. 187–209.

[17] M. S. HOWE, *Acoustics of Fluid-Structure Interactions*, Cambridge University Press, Cambridge, UK, 1998.

[18] G. H. HUNTER AND M. V. LOWSON, *Reverberation time measurement of the scattering of sound by turbulence*, J. Acoust. Soc. Amer., 55 (1974), pp. 937–940.

[19] W. KOHLER AND G. C. PAPANICOLAOU, *Wave propagation in a randomly inhomogenous ocean*, in Wave Propagation and Underwater Acoustics (Workshop, Mystic, Conn., 1974), Lecture Notes in Phys. 70, Springer-Verlag, Berlin, 1977, pp. 153–223.

[20] R. H. KRAICHNAN, *Scattering of sound in a turbulent medium*, J. Acoust. Soc. Amer., 25 (1953), pp. 1096–1104.

[21] M. J. LIGHTHILL, *On sound generated aerodynamically.* I. *General theory*, Proc. Roy. Soc. London Sect. A, 211 (1952), pp. 564–587.

[22] M. J. LIGHTHILL, *On sound generated aerodynamically.* II. *Turbulence as a source of sound*, Proc. Roy. Soc. London Sect. A, 222 (1954), pp. 1–32.

[23] M. J. LIGHTHILL, *On the energy scattered from the interaction of turbulence with sound or shock waves*, Proc. Cambridge Philos. Soc., 49 (1953), pp. 531–551.

[24] J. LIGHTHILL, *Waves in Fluids*, Cambridge University Press, Cambridge, UK, 1978.
[25] P. A. LUSH, *Measurements of subsonic jet noise and comparison with theory*, J. Fluid Mech., 46 (1971), pp. 477–500.
[26] W. MÖHRING, E.-A. MÜLLER, AND F. OBERMEIER, *Problems in flow acoustics*, Rev. Modern Phys., 55 (1983), pp. 707–724.
[27] D. T. NOIR AND A. R. GEORGE, *Absorption of sound by homogeneous turbulence*, J. Fluid Mech., 86 (1978), pp. 593–608.
[28] V. E. OSTASHEV, *Acoustics in Moving Inhomogeneous Media*, E & FN Spon, London, 1997.
[29] L. RYZHIK, G. PAPANICOLAOU, AND J. KELLER, *Transport equations for elastic and other waves in random media*, Wave Motion, 24 (1996), pp. 327–370.
[30] H. SPOHN, *Derivation of the transport equation for electrons moving through random impurities*, J. Statist. Phys., 17 (1979), pp. 385–412.
[31] H. TENNEKES, *Eulerian and Lagrangian microscales in isotropic turbulence*, J. Fluid Mech., 67 (1975), pp. 561–567.
[32] R. VEDANTHAM AND J. HUNTER, *Sound wave propagation through incompressible flows*, Wave Motion, 26 (1997), pp. 319–328.