Turbulent Diffusion in Markovian Flows

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Abstract We prove turbulent diffusion theorems for Markovian velocity fields which either are mixing in time or have stationary vector potentials.

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1 Introduction.

One of the central questions about the motion in random flows described by

$$d\mathbf{x}(t) = \mathbf{b}(t, \mathbf{x}(t))dt + \sqrt{2\kappa}d\mathbf{w}(t), \quad \mathbf{x}(0) = \mathbf{0}, \tag{1}$$

with the molecular diffusivity $\kappa \geq 0$, the standard Brownian motion $\{\mathbf{w}(t)\}_{t\geq 0}$ and a zero mean, jointly stationary, incompressible (i.e. $\nabla \cdot \mathbf{b}(t, \mathbf{x}) = 0$) velocity field $\mathbf{b}(t, \mathbf{x})$, is whether and when the motion has a long time diffusive limit which remains valid in the absence of molecular diffusion ($\kappa = 0$) or in the limit of vanishing molecular diffusion ($\kappa \to 0$). More specifically, one wants to find conditions under which the rescaled processes

$$\mathbf{x}_\varepsilon(t) = \varepsilon \mathbf{x}(t/\varepsilon^2), \quad \varepsilon > 0 \tag{2}$$

converge in law, as $\varepsilon \to 0$, to a Brownian Motion with an enhanced diffusivity $\mathbf{D}(\kappa)$ called the *effective diffusivity* which has a nonzero limit as κ tends to zero. The limit $\lim_{\kappa\to 0} \mathbf{D}(\kappa) = \mathbf{D}(0)$ is known as the *eddy diffusivity* or *turbulent diffusivity* for it is mainly a result of turbulent eddies. This question is referred to as the *turbulent diffusion* problem.

Diffusive limit exits in the presense of molecular diffusion ($\kappa > 0$) when the velocity field $\mathbf{b}$ has a jointly stationary vector potential (Fannjiang and Komorowski (1997)). Stationarity of velocity vector potential requires certain velocity decorrelations in *space* but not in *time*. Nonstationary vector potentials may result in nondiffusive limits (Fannjiang (1998), Fannjiang and Komorowski (1998)). Although previous numerical simulations (Kraichnan (1970)) suggests positive eddy diffusivity for three dimensional Gaussian flows with fast decorrelation in space, an invariance principle, however, is unlikely to hold in this case with $\kappa = 0$ due to possible trapping by flow-invariant domains unless there is also sufficient velocity decorrelation in time. It remains open if the effective diffusivity has a positive limit as κ tends to zero for three dimensional steady flows. In this connection, Komorowski and Papanicolaou (1997) has proved the diffusive limit for $\kappa = 0$ and stationary Gaussian velocity fields which become independent after a finite time. Conceptually this is a generalization of the corresponding turbulent diffusion result for white-noise velocity fields, but technically it is much more involved.

Generalizing in another direction one may consider the turbulent diffusion problem for time mixing flows, in particular, for time mixing Markovian velocities for which time mixing property can be conveniently characterized by a spectral gap. Markovian fields have been commonly used in fluid dynamics to model turbulent fluid velocity (McComb (1990), Frisch (1996)). A turbulent diffusion theorem was obtained by Carmona and Xu (1997) for Ornstein-Uhlenbeck velocities with finite Fourier modes.

In this paper, we prove an invariance principle for deterministic motion ($\kappa = 0$) as well as diffusive motion ($\kappa > 0$) in a general class of time mixing, Markovian velocity fields with no decorrelation in space (Theorem A). Moreover, small molecular diffusion acts as a regular perturbation to the positive eddy diffusity.

Theorem A *Let $\mathbf{b}(t, \mathbf{x})$ be a stationary Markovian field that is square integrable and ϱ - mixing in t. Then $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ converge in law, as $\varepsilon \downarrow 0$, to a Brownian Motion with the effective diffusivity $\mathbf{D}(\kappa) > 0, \kappa \geq 0$. Moreover, the limit exists*

$$\lim_{\kappa\downarrow 0} \mathbf{D}(\kappa) = \mathbf{D}(0) > 0. \tag{3}$$

The mixing property of the Markovian field $\mathbf{b}$ enables us to construct a *stationary "corrector"* as contrary to the usual non-stationary correctors in homogenization theory possessing stationary derivatives.

For our second result, we relax the time mixing condition to the ellipticity condition (Section 2) which allows velocity modes of small wave numbers to have long correlation time but we compensate the lack of time decorrelation with decorrelation in space by assuming stationary vector potentials for velocity.

Theorem B *Let $\mathbf{b}(t,\mathbf{x})$ be a bounded velocity field with a bounded Markovian vector potential and satisfy the ellipticity condition (L2) (Section 2). Then $\mathbf{x}_\varepsilon(t)$ converge in law, as $\varepsilon \downarrow 0$, to a Brownian Motion with the effective diffusivity $\mathbf{D}(\kappa) > 0$ for any $\kappa \geq 0$ and (3) holds.*

In proving Theorem B, we adopt the approach of Kipnis and Varadhan (1986) who established the central limit theorem for additive functionals of reversible, ergodic Markov processes.

The key object in our problem is the environment process viewed from the particle which is irreversible. Because of irreversibility of the process we cannot apply the argument of Kipnis and Vardhan. To overcome this problem, we construct a family of "asymptotic correctors" which converges in the norm generated by the *symmetric part* of the Dirichlet form associated with the environment process. To control the antisymmetric part of the process, it requires the boundedness of the velocity and the stream matrix. With that, using a perturbation argument, we then show that the crucial elements of Kipnis and Varadhan's approach remain valid in this case.

Theorems A and B are independent of dimension and, to certain degree, tight, except the boundedness assumption in Theorem B for technical reasons (Fannjiang and Komorowski (1998)). These two theorems are stated in a more precise form as Theorem 1 and 2.

2 Formulation and Results

Let $(\mathcal{X}, \mathcal{H}, P_0)$ be a probability space. Let $\tau_{\mathbf{x}}$, $\mathbf{x} \in R^d$ be a stochastically continuous, jointly measurable group of measure preserving transformations of $\mathcal{X}$with the following properties.

(T1) $\tau_{\mathbf{0}} = Id_{\mathcal{X}}$ and $\tau_{\mathbf{x}+\mathbf{y}} = \tau_{\mathbf{x}}\tau_{\mathbf{y}}$, for all $\mathbf{x}, \mathbf{y} \in R^d$.

(T2) The mapping $(x, \mathbf{x}) \mapsto \tau_{\mathbf{x}}(x)$ is jointly measurable.

(T3) $P_0[\tau_{\mathbf{x}}(A)] = P_0[A]$, for $\mathbf{x} \in R^d,\ A \in \mathcal{H}$.

(T4) $\lim_{\mathbf{x}\downarrow 0} P_0[x :\mid f \circ \tau_{\mathbf{x}}(x) - f(x) \mid \geq \eta] = 0, \quad \forall f \in L^2(\mathcal{X})$ and $\forall \eta > 0$.

(T5) If $P_0[A \triangle \tau_{\mathbf{x}}(A)] = 0$, for all $\mathbf{x} \in R^d$ then A is a trivial event i.e. $P_0(A)$ is either 0 or 1.

It is well known that $\tau_{\mathbf{x}}$ induces a strongly continuous group of unitary mappings $U^{\mathbf{x}}$ on $L^2(\mathcal{X})$:

$$U^{\mathbf{x}} f(x) = f(\tau_{\mathbf{x}}(x)),\ f \in L^2(\mathcal{X}),\ \ \mathbf{x} \in R^d. \tag{4}$$

The group has d independent, skew-adjoint generators $D_k : \mathcal{D}_k \to L^2(\mathcal{X})$ corresponding to the directions $\mathbf{e}_k,\ \ k = 1, \cdots, d$.

Let $C_b^m(\mathcal{X})$, $m=1,\cdots,\infty$ be the space of functions f in the intersection of the domains of D_k^n with $\|D_k^n f\|_{L^\infty(\mathcal{X})}<+\infty,\ k=1,\cdots,d,\ n=1,\cdots,m$. It is well known that $C_b^\infty(\mathcal{X})=\bigcap_{m\geq 1} C_b^m(\mathcal{X})$ is dense in $L^p(\mathcal{X})$, $1\leq p<+\infty$ (Dedik and Šubin (1982)).

Let $L_0^2(\mathcal{X})$ be the space of functions $f\in L^2(\mathcal{X})$ such that $\int f dP_0=0$ and let $\tilde{H}_0^1(\mathcal{X})$ be the space $\bigcap_{k=1}^d \mathcal{D}_k \bigcap L_0^2(\mathcal{X})$ equipped with the scalar product

$$(f,g)_{\tilde{H}_0^1(\mathcal{X})}=\sum_{k=1}^d \int D_k f D_k g dP_0, \qquad f,\ g\in \tilde{H}_0^1(\mathcal{X}).$$

$\tilde{H}_0^1(\mathcal{X})$ is a pre-Hilbert space and can be completed under the scalar product $(f,g)_{\tilde{H}^1(\mathcal{X})}$. Denote that completion by $H_0^1(\mathcal{X})$.

Let Ω be the space of $\mathcal{X}$-valued continuous functions $C([0,+\infty);\mathcal{X})$ and let $\mathcal{C}$ be its Borel σ-algebra. Let $P^t,\ t\geq 0$, be a strongly continuous Markov semigroup on $L^2(\mathcal{X})$ with the following properties.

(P1) $P^t\mathbf{1}=\mathbf{1}$ and $P^t f\geq 0$, if $f\geq 0$.

(P2) $\int P^t f dP_0=\int f dP_0$, for all $f\in L^2(\mathcal{X}),\ t\geq 0$.

(P3) $\mathbf{E}_x[f(\theta_{t+h}(\omega))|\mathcal{C}_{\leq t}]=P^h F(\omega(t))$, with $F(x):=\mathbf{E}_x f$. for any $f\in L^1(\Omega),\ t,h\geq 0$, $x\in\mathcal{X}$.

Here $\mathbf{E}_x$ the expectations associated with the probability measures P_x the Markov family corresponding to P^t. $\mathcal{C}_{\leq t}$ are the σ-algebras generated by events measurable up to time t. $\theta_t(\omega)(\cdot):=\omega(\cdot+t),\ t\geq 0$ is the standard shift operator on the path space $(\Omega,\mathcal{C})$.

Remark. 1 Conditions (P1), (P2) imply that all $P^t,\ t\geq 0$ are contractions in any $L^p(\mathcal{X})$, for all $1\leq p\leq+\infty$.

Let P be a Markovian measure on the path space $(\Omega,\mathcal{C})$

$$P(A)=\int P_x(A)P_0(dx),\ A\in\mathcal{C} \tag{5}$$

and let $\mathbf{E}$ be the corresponding expectation.

As a direct consequence of (T3) and (P2) we casn conclude that P is stationary. More spaecifically.

(S) P is invariant under the action of θ_t and $\tau_{\mathbf{x}}$ for any $(t,\mathbf{x})\in R^+\times R^d$.

Denote the space-time translates of a path by $T_{t,\mathbf{x}}(\omega)=\theta_t(\tau_{\mathbf{x}}(\omega))=\tau_{\mathbf{x}}(\theta_t(\omega)),\forall t,\mathbf{x}$. Following condition (S) Propositions 1 and 2 are well known.

Proposition. 1 *P^t and $U^{\mathbf{x}}$ commute, for all $t\geq 0$, $\mathbf{x}\in R^d$.*

Proposition. 2 $\mathbf{E}_x f(T_{0,\mathbf{x}}(\omega))=\mathbf{E}_{\tau_{\mathbf{x}}(x)}f(\omega)$, *for any bounded, $\mathcal{C}$-measurable f.*

We assume the time relaxation properties (L1) and (L2) for Theorem 1 and Theorem 2, respectively.

(L1) Spectral gap: $-(Lf,f)_{L^2(\mathcal{X})} \geq c_1 \|f\|^2_{L^2(\mathcal{X})}$ for some constant $c_1 > 0$ and for all $f \in \mathcal{D}(L) \cap L^2_0(\mathcal{X})$.

(L2) Ellipticity: $-(Lf,f)_{L^2(\mathcal{X})} \geq c_2 \|f\|^2_{H^1_0(\mathcal{X})}$ for some constant $c_2 > 0$ and all $f \in \mathcal{D}(L) \cap H^1_0(\mathcal{X})$.

Remark. 2 Condition (L1) is equivalent to the exponential decay property

$$\|P^t f\|_{L^2(\mathcal{X})} \leq e^{-ct} \|f\|_{L^2(\mathcal{X})}, \text{ for any } f \in L^2_0(\mathcal{X}) \tag{6}$$

(Rosenblatt (1971)). Inequality (6) is in fact equivalent to $\varrho-$mixing of the process $X(t)$, $t \geq 0$ (Doukhan (1994), p. 3) i.e. $\lim_{h \uparrow \infty} \varrho(h) = 0$, where

$$\varrho(h) = \sup \{ \mathrm{Cor}(X,Y) : X \text{ is } \mathcal{C}_{\geq t+h} \text{ measurable }, \ Y \text{ is } \mathcal{C}_{\leq t} \text{ measurable } \}$$

with $\mathrm{Cor}(X,Y)$ being the correlation of X, Y.

For Theorem 2 we also assume a reasonably general condition (L3) that the symmetric part of the Dirichlet form controls the antisymmetric part (Ma-Röckner (1992)).

(L3) The sector condition:

$$| (Lf,g)_{L^2(\mathcal{X})} | \leq K \mid (Lf,f)_{L^2(\mathcal{X})} |^{\frac{1}{2}} | (Lg,g)_{L^2(\mathcal{X})} |^{\frac{1}{2}} \tag{7}$$

for some constant $K > 0$ and all $f, g \in \mathcal{D}(L)$. Here $L : \mathcal{D}(L) \to L^2(\mathcal{X})$ is the generator of semigroup P^t, $t \geq 0$.

Remark. 3 All reversible measures P_0 such as stationary Ornstein-Uhlenbeck processes satisfy (L3) (Example 1 below).

(B1) The random field $\mathbf{b} = (b_1, \cdots, b_d) \in (L^2(\mathcal{X}))^d$ is jointly continuous in $(t, \mathbf{x})$, locally lipschitzian in $\mathbf{x}$, with finite second moments and of divergence free (i.e. $\sum_{i=1}^d \int b_i D_i \varphi dP_0 = 0, \ \forall \varphi \in C_b^\infty(\mathcal{X})$).

Here and in the sequel we denote a random vector f on $\mathcal{X}$ by $f(t, \mathbf{x}; \omega) = f(T_{t,\mathbf{x}}(\omega)(0))$, $(t, \mathbf{x}) \in R^+ \times R^d$.

For technical reasons in proving Theorem 2 we need the stronger assumption (B2).

(B2) $\mathbf{b}$ and its *stream matrix* Ψ are bounded i.e. $\|\mathbf{b}\|_{L^\infty(\mathcal{X})} + \|\Psi\|_{L^\infty(\mathcal{X})} < +\infty$.

A stream matrix of $\mathbf{b}$ is a real, $d \times d$ skew-symmetric matrix-valued process $\Psi = [\Psi_{i,j}]$ with $\Psi_{i,j} \in H^1_0(\mathcal{X})$ such that and $b_i = \sum_{j=1}^d D_j \Psi_{i,j}$. In three dimensions, Ψ is related to the vector potential $\boldsymbol{v} = (v_1, v_2, v_3)$ in the following way

$$\Psi = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}.$$

Due to skew symmetry of Ψ, $\mathbf{b}$ is of divergence-free.

Let $\mathbf{x}^{s,\mathbf{x}}(t)$ be the process given by

$$d\mathbf{x}^{s,\mathbf{x}}(t) = \mathbf{b}(t, \mathbf{x}^{s,\mathbf{x}}(t))dt + \sqrt{2\kappa}d\mathbf{w}(t-s), \tag{8}$$

$$\mathbf{x}^{s,\mathbf{x}}(s) = \mathbf{x}$$

where $\mathbf{w}(t)$, $t \geq 0$ is a standard Brownian Motion starting at the origin. Its underlying proability space shall be denoted by $(\Sigma, \mathcal{B}, Q)$ with the corresponding expectation $\mathbf{M}$. Denote the corresponding filtration by $\mathcal{B}_t$, $t \geq 0$. Thanks to the stationarity (S), (B1) implies the global existence and uniqueness of $\mathbf{x}^{s,\mathbf{x}}(t; \omega, \mathbf{w})$, $t \geq s$ for P a.s. ω and Q a.s. $\mathbf{w}$ without the usual linear growth condition at far fields (Fannjiang-Komorowski (1997)).

The main objectives of this article are the following two theorems.

Theorem. 1 *In addition to the general assumptions (T1)-(T4), (P1)-(P3), (S) and (B1), we assume (L1). Then the processes $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ converge weakly as $\varepsilon \downarrow 0$ to a Brownian Motion. Moreover,*

$$\lim_{\kappa \downarrow 0} \mathbf{D}(\kappa) = \mathbf{D}(0) > 0. \tag{9}$$

Theorem. 2 *In addition to the general assumptions (T1)-(T5), (P1)-(P3), (S), (L3) and (B1), we assume (L2) and (B2). Then the processes $\mathbf{x}_\varepsilon(t)$, $t \geq 0$ converge weakly as $\varepsilon \downarrow 0$ to a Brownian Motion and (9) holds.*

Example. An example of velocity satisfying assumptions of Theorem 1 is the stationary, divergence free, Ornstein-Uhlenbeck vector field $\mathbf{b}(t, \mathbf{x}; \omega)$ with the spectral measure

$$e^{-r(\mathbf{k})|t|}\Gamma(\mathbf{k})\left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2}\right)$$

where $\Gamma(\mathbf{k})$ is integrable and decays fast for large $\mathbf{k}$. For this velocity field ϱ-mixing is equivalent to α-mixing (Rosenblatt (1971), Doukhan (1994)) and amounts to

$$r(\mathbf{k}) > c_0, \quad \forall \mathbf{k} \in R^d$$

for some positive constant c_0.

Particularly interesting is the power-law spectral measure with $r(\mathbf{k}) = |\mathbf{k}|^{2\beta}$, $\Gamma(\mathbf{k}) = 1/|\mathbf{k}|^{2\alpha+d-2}$ with ultraviolet cutoff $|\mathbf{k}| \leq K < \infty$. The integrability of Γ requires $\alpha < 1$ and the spectral gap condition (L1) now becomes $\beta \leq 0$.

The boundedness requirement in (B2) already rules this class of velocity fields out of the scope of Theorem 2. Strictly speaking, to apply Theorem 2, we need to take smooth truncation on the velocity and its stream matrix. However, this is only due to the limitation of current techniques. The essential part of (B2) is the existence of a stationary stream matrix which requires

$$\int_{R^d} \frac{1}{|\mathbf{k}|^2}\Gamma(\mathbf{k})d\mathbf{k} < \infty$$

or, equivalently, $\alpha < 0$. The ellipticity condition (L2), which is weaker than (L1), is satisfied for $\beta \leq 1$. We believe that Theorem 2 holds for the above O-U velocity with $\alpha < 0$ and $\beta \leq 1$.

3 Preliminaries

Consider the environment process as viewed from the particle at any instant of time $\eta : [0, +\infty) \times \Omega \times \Sigma \to \mathcal{X}$ given by

$$\eta(t; \omega, \mathbf{w}) = \tau_{\mathbf{x}(t;\omega,\mathbf{w})}(\omega(t)),\ t \geq 0. \tag{10}$$

The rescaled process $\mathbf{x}_\varepsilon(t)$, given by (2), induces a probability measure Q_ε on a Frechét space $C([0, +\infty); R^d)$. $\mathbf{x}_\varepsilon(t)$ is said to *converge weakly* to a Brownian Motion if Q_ε converge weakly to a certain Wiener Measure. Denote the covariance matrix of the limiting Brownian Motion by $\mathbf{D}(\kappa)$, $\kappa \geq 0$.

Set

$$S^t f(x) = \mathbf{M}\mathbf{E}_x f(\eta(t)),\ t \geq 0,\ \text{for } f \in L^\infty(\mathcal{X}) \tag{11}$$

where η is given by (10).

Proposition. 3 *If (T1)-(T4), (P1)-(P3), (S) and (B1) hold, then*

(i) S^t, $t \geq 0$ *is a strongly continuous, Markov semigroup of contractions on* $L^2(\mathcal{X})$.

(ii) S^t, $t \geq 0$ *is measure-preserving i.e.*

$$\int S^t f dP_0 = \int f dP_0,\ t \geq 0,\ f \in L^2(\mathcal{X}). \tag{12}$$

Set

$$D_1 = \mathcal{D}(L) \cap C_b^2(\mathcal{X}). \tag{13}$$

Denote the generator of the semigroup S^t, $t \geq 0$ by $\mathcal{L}$:

$$\mathcal{L}f = Lf + \kappa\Delta f + (\mathbf{b}, \nabla f),\ \text{for } f \in D_1. \tag{14}$$

The following results are standard.

Proposition. 4 *If (T1)-(T4), (P1)-(P3), (S) and (B1) hold, then*

(i) D_1 *is dense in* $L^2(\mathcal{X})$ *and is invariant under the semigroup* P^t, $t \geq 0$ *(i.e.* $P^t(D_1) \subseteq D_1$ *for all* $t \geq 0$*).*

(ii) If (L2) and (L3) hold then $\mathcal{D}(L) \subseteq H_0^1(\mathcal{X})$.

(iii) Assume that the velocity field is bounded. Then D_1 *is invariant under the semigroup* S^t, $t \geq 0$ *(i.e.* $S^t(D_1) \subseteq D_1$ *for all* $t \geq 0$*).*

Define

$$S_0^t f(x, \mathbf{x}) = \mathbf{M}_{\mathbf{x}}\mathbf{E}_x f(\omega(t), \mathbf{w}(t)), \quad \forall f \in L^2(\mathcal{X} \times R^d). \tag{15}$$

Proposition. 5 *(i)* S_0^t, $t \geq 0$ *is a strongly continuous, Markov semigroup on* $L^2(\mathcal{X} \times R^d)$.

(ii) The semigroup S_0^t is measure preserving i.e.

$$\int\int S_0^t f dP_0 dm = \int\int f dP_0 dm, \text{ for all } f \in L^2(\mathcal{X} \times R^d), \ t \geq 0 \tag{16}$$

where dm is the Lebesgue measure.

(iii) Suppose that, in addition, either (L1) or all three conditions (L2), (L3) and (T5) holds. Then any $f \in L^2(\mathcal{X} \times R^d)$ such that $S_0^t f = f$ for a certain $t > 0$ is constant.

Proof. We only sketch the proof of (iii). (16) is a consequence of the invariance of the Lebesgue measure under Brownian motion.

Suppose that (L2), (L3) and (T5) hold. (L3) implies that the semigroup P^t is holomorphic (Ma-Röckner (1992) Corollary 2.21 p. 25) hence

$$\frac{d}{ds}\|P^s \tilde{f}_{A,B}\|^2_{L^2(\mathcal{X})} = 2(-LP^s\tilde{f}_{A,B}, P^s\tilde{f}_{A,B})_{L^2(\mathcal{X})} \leq -2c_2\|P^s\tilde{f}_{A,B}\|^2_{\tilde{H}^1(\mathcal{X})} \tag{17}$$

where $\tilde{f}_{A,B} = f_{A,B} - \int f_{A,B} dP_0$. Since $\|P^t\tilde{f}_{A,B}\|_{L^2(\mathcal{X})} = \|\tilde{f}_{A,B}\|_{L^2(\mathcal{X})}$ we have $\|P^s\tilde{f}_{A,B}\|_{\tilde{H}^1(\mathcal{X})} = 0$ for all $0 < s < t$. This, along with (T5), implies that $\tilde{f}_{A,B}$ is constant for any Borel set $B \subseteq P^d$. Thus χ_A must be constant.

Suppose that (L1) holds. By Remark 2, $\|P^t\tilde{f}_{A,B}\|_{L^2(\mathcal{X})} \leq e^{-c_1 t}\|\tilde{f}_{A,B}\|_{L^2(\mathcal{X})}$ which clearly implies that any $\tilde{f}_{A,B}$ and thus in consequence χ_A must be constant ∎

Propositions 3 and 5 are standard for $\kappa > 0$ and can be extended easily to the case $\kappa = 0$. The reason is that under, either (L1) or (L2), (L3), the molecular diffusion term $\kappa\Delta$ in $\mathcal{L}$ is negligible for small κ, compared to L.

4 Proof of Theorem 1

Proposition. 6 *Under the same assumptions as in Theorem 1, we have*

$$\|S^t f\|_{L^2(\mathcal{X})} \leq e^{-c_1 t}\|f\|_{L^2(\mathcal{X})}, \quad t \geq 0 \tag{18}$$

for any $f \in L^2_0(\mathcal{X})$.

Proof. First assume that $\mathbf{b}$ is bounded and $f \in D_1 \subseteq \mathcal{D}(\mathcal{L})$. Then we have

$$(-\mathcal{L}f, f)_{L^2_0(\mathcal{X})} \geq (-Lf, f)_{L^2_0(\mathcal{X})} \geq c_2\|f\|^2_{H^1_0(\mathcal{X})} \tag{19}$$

for all $f \in D_1 \cap L^2_0(\mathcal{X})$. By Proposition 4 $S^t f \in D_1$, $t \geq 0$ for any $f \in D_1$. Consequently

$$\frac{d}{dt}\|S^t f\|^2_{L^2(\mathcal{X})} = -(\mathcal{L}S^t f, S^t f)_{L^2(\mathcal{X})} \leq -c_1\|S^t f\|^2_{L^2(\mathcal{X})}$$

thus

$$\|S^t f\|_{L^2(\mathcal{X})} \leq e^{-c_1 t}\|f\|_{L^2(\mathcal{X})}, \forall t \geq 0 \tag{20}$$

and $f \in D_1 \cap L^2_0(\mathcal{X})$. (20) is then extended to $L^2_0(\mathcal{X})$ by using an approximation argument. Likewise the boundedness of the velocity is removed by another approximation argument ∎

Thanks to Proposition 6 we can define

$$\psi_k = \int_0^{+\infty} S^t b_k dt. \tag{21}$$

The following lemma is quite elementary.

Lemma. 1 *$\psi_k \in \mathcal{D}(\mathcal{L})$ and is the unique solution of the equation*

$$-\mathcal{L}\psi_k = b_k, \qquad k = 1, \cdots, d. \tag{22}$$

First observe that

$$(\mathbf{v}, \mathbf{x}_\varepsilon(t)) = N_{\varepsilon,\psi_\mathbf{v}}(t; \omega, \mathbf{w}) + R_\varepsilon(t; \omega, \mathbf{w})$$

for any $\mathbf{v} = (v_1, \cdots, v_d) \in R^d$ where $\psi_\mathbf{v} = \sum_{i=1}^d \psi_i v_i$, $R_\varepsilon(t; \omega, \mathbf{w}) = -\varepsilon\psi_\mathbf{v}(\eta(t/\varepsilon^2; \omega, \mathbf{w})) + \varepsilon\psi_\mathbf{v}(\eta(0; \omega, \mathbf{w}))$ and

$$N_{\varepsilon,\psi_\mathbf{v}}(t; \omega, \mathbf{w}) = \varepsilon\sqrt{2\kappa}(\mathbf{v}, \mathbf{w}(t/\varepsilon^2)) + \varepsilon M_{\psi_\mathbf{v}}(\frac{t}{\varepsilon^2}; \eta), \tag{23}$$

with

$$M_{\psi_\mathbf{v}}(t; \eta) = \psi_\mathbf{v}(\eta(t)) - \psi_\mathbf{v}(\eta(0)) - \int_0^t \mathcal{L}\psi_\mathbf{v}(\eta(s))ds. \tag{24}$$

By the stationarity of $\eta(t)$ we have that

$$\begin{aligned} P \otimes Q[\sup_{0 \le t \le T} |R_\varepsilon(t)| \ge \delta] &\le (\frac{1}{\varepsilon^2} + 1) P \otimes Q[\sup_{0 \le t \le \varepsilon^2 T} |R_\varepsilon(t)| \le \delta] \\ &\le (\frac{1}{\varepsilon^2} + 1) P \otimes Q[\sup_{0 \le t \le T} |N_{1,\psi_\mathbf{v}}(t)| \ge \frac{\delta}{\varepsilon}]. \end{aligned}$$

Since $\sup\limits_{0 \le t \le T} |N_{1,\psi_\mathbf{v}}(t)|$ has finite second moment the last expression tends to zero as $\varepsilon \downarrow 0$, i.e.

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le t \le T} |R_\varepsilon(t)| = 0 \tag{25}$$

in probability w.r.t. $P \otimes Q$.

Define $S_\varepsilon(t; \omega, \mathbf{w}) = \varepsilon \sum_{k=1}^{[t/\varepsilon^2]} \xi_k(\omega, \mathbf{w})$, for $t \ge 0$, where $\xi_k(\omega, \mathbf{w}) = N_{1,\psi_\mathbf{v}}((k+1)t_1; \omega, \mathbf{w}) - N_{1,\psi_\mathbf{v}}(kt_1; \omega, \mathbf{w})$. We prove in Lemma 2 that ξ_n, $n \ge 0$ is stationary and ergodic. Hence, the processes $S_\varepsilon(t), t \ge 0$ converge weakly in the Skorochod space $D[0, +\infty)$ to a Brownian Motion with diffusion coefficient $|\mathbf{v}|^2 + (-\mathcal{L}\psi_\mathbf{v}, \psi_\mathbf{v})_{L^2(\mathcal{X})}$ (Billingsley (1968), Theorem 23.1, p. 206). By (25),

$$\lim_{\varepsilon \downarrow 0} \sup_{0 \le t \le T} |N_{\varepsilon,\psi_\mathbf{v}}(t) - (\mathbf{v}, \mathbf{x}_\varepsilon(t))| = 0 \tag{26}$$

in probability.

Eq. (26) implies that $\lim\limits_{\varepsilon \downarrow 0} \sup\limits_{0 \le t \le T} |S_\varepsilon(t) - N_{\varepsilon,\psi_\mathbf{v}}(t)| = 0$ in probability, which shows the weak convergence of $\mathbf{x}_\varepsilon(t)$, $t \ge 0$ in $D[0, +\infty)$ (Helland (1982), Theorem 5.1 and (5.9)). Since $\mathbf{x}_\varepsilon$ are continuous processes this in turn implies weak convergence in $C[0, +\infty)$.

Lemma. 2 *The sequence ξ_n, $n \ge 0$ is stationary and ergodic.*

Proof. First assume $\kappa = 0$. Suppose that $A \in \mathcal{C}$ is invariant, $\theta_t^{-1}(A) = A$. We now prove that $g(x) = \mathbf{E}_x \chi_A(\eta)$, $x \in \mathcal{X}$ satisfying $S^t g = g$ is constant by showing that $\mathcal{C}_i$, the σ-algebra of invariant sets, is trivial (Foguel (1969)).

If χ_C is not the constant 1, then, by Proposition 6,

$$\|\chi_C\|_{L^2(\mathcal{X})} = \|S^t \chi_C\|_{L^2(\mathcal{X})} \leq e^{-c_1 t}\|\chi_C\|_{L^2(\mathcal{X})}, \quad C \in \mathcal{C}_i, \tag{27}$$

and $\chi_C = 0$ since t is arbitrary. Or else χ_C is the constant 1. Hence C has measure zero or one.

For $\kappa > 0$, stationarity of $(\xi_p(\omega, \mathbf{w}))_{p\geq 0}$ follows from the divergence free property of the velocity (Port-Stone (1976), Theorem 3, p. 500).

We turn to the proof of ergodicity. Suppose that $\chi_A((\xi_{k+1})_{k\geq 0}) = \chi_A((\xi_k)_{k\geq 0})$ for some A in the σ-algebra of cylindrical sets in the space of real-valued sequences $(x_n)_{n\geq 0}$. We have $S_0^t g = g$, where $g(x, \mathbf{x}) = \mathbf{M}_{\mathbf{x}}\mathbf{E}_x \chi_A((\xi_p)_{p\geq 0})$, which, by Proposition 5, implies that g is a constant c. For any $n \geq 1$ and a Borel measurable and bounded function $h : R^n \to R$ we write

$$\mathbf{ME}\chi_A((\xi_p)_{p\geq 0})h(\xi_{1,f}, \cdots, \xi_{n,f}) = \mathbf{ME}\chi_A((\xi_{p+n+1,f})_{p\geq 0})h(\xi_{1,f}, \cdots, \xi_{n,f}) =$$

$$\mathbf{ME}g(\omega(nt_1), \mathbf{w}(nt_1))h(\xi_{1,f}, \cdots, \xi_{n,f}) = c\mathbf{ME}h(\xi_{1,f}, \cdots, \xi_{n,f}).$$

Hence χ_A must be a constant. ∎

5 Proof of Theorem 2.

Proposition. 7

$$|(\mathcal{L}f, g)_{L^2(\mathcal{X})}| \leq K'|(\mathcal{L}f, f)_{L^2(\mathcal{X})}|^{\frac{1}{2}}|(\mathcal{L}g, g)_{L^2(\mathcal{X})}|^{\frac{1}{2}} \tag{28}$$

and

$$(-\mathcal{L}f, f)_{L^2(\mathcal{X})} \geq (-Lf, f)_{L^2(\mathcal{X})} \geq c_2\|f\|^2_{\tilde{H}^1(\mathcal{X})} \tag{29}$$

for some constant $K' > 0$ and any $f, g \in \mathcal{D}(\mathcal{L})$.

Proof. From (14), we know that (29) holds for all $f \in D_1$. The sector condition (28) holds therefore for the operator $\mathcal{L}_{|D_1}$ for all $\kappa \geq 0$. Indeed, for $f, g \in D_1$

$$(-\mathcal{L}f, g)_{L^2(\mathcal{X})} = (-Lf, g)_{L^2(\mathcal{X})} + \kappa(f, g)_{\tilde{H}^1(\mathcal{X})} - \sum_{p,q=1}^{d}(\Psi_{p,q}D_q f, D_p g)_{L^2(\mathcal{X})}.$$

The condition (L2) and formula (29), used for $f, g \in D_1$ imply that

$$|\kappa(f, g)_{\tilde{H}^1(\mathcal{X})}| \leq \frac{\kappa}{c_2}|(Lf, f)_{L^2(\mathcal{X})}|^{\frac{1}{2}}|(Lg, g)_{L^2(\mathcal{X})}|^{\frac{1}{2}} \leq \frac{\kappa}{c_2}|(\mathcal{L}f, f)_{L^2(\mathcal{X})}|^{\frac{1}{2}}|(\mathcal{L}g, g)_{L^2(\mathcal{X})}|^{\frac{1}{2}} \tag{30}$$

and likewise

$$|\sum_{p,q=1}^{d}(\Psi_{p,q}D_q f, D_p g)_{L^2(\mathcal{X})}| \leq \frac{\|\Psi\|_{L^\infty(\mathcal{X})}}{c_2}|(Lf, f)_{L^2(\mathcal{X})}|^{\frac{1}{2}}|(Lg, g)_{L^2(\mathcal{X})}|^{\frac{1}{2}} \tag{31}$$

$$\leq \frac{\|\Psi\|_{L^\infty(\mathcal{X})}}{c_2}|(\mathcal{L}f, f)_{L^2(\mathcal{X})}|^{\frac{1}{2}}|(\mathcal{L}g, g)_{L^2(\mathcal{X})}|^{\frac{1}{2}}. \tag{32}$$

The sector condition (L3) for L together with (30) and (31) imply that (28) holds for $f, g \in D_1$.

Both (28) and (29) can be extended from D_1 to the entire $\mathcal{D}(\mathcal{L})$ via a standard Dirichlet form argument (see, e.g., Ma-Röckner (1992)) ∎

Define

$$H_0^1(\mathcal{L}^s) = \{f \in L_0^2(\mathcal{X}) : \int_0^{+\infty} \lambda e_f(d\lambda) < +\infty\},$$

$$H_0^{-1}(\mathcal{L}^s) = \{f \in L_0^2(\mathcal{X}) : \int_0^{+\infty} \frac{1}{\lambda} e_f(d\lambda) < +\infty\}$$

where

$$e_f(A) = (E(A)f, f)_{L^2(\mathcal{X})}, \tag{33}$$

with $E(A)$ being the spectral resolution of $-\mathcal{L}^s$, the symmetric part of $\mathcal{L}$ corresponding to the asymmetric form $\mathcal{E}^s(f,g) := \frac{1}{2}[\mathcal{E}(f,g) + \mathcal{E}(g,f)]$. Observe that $\mathcal{D}(\mathcal{E}) = H_0^1(\mathcal{L}^s) \oplus R$ i.e. $f \in \mathcal{D}(\mathcal{E})$ iff $\tilde{f} \in H_0^1(\mathcal{L}^s)$, where

$$\tilde{f} = f - \overline{f}, \quad \overline{f} = \int f dP_0.$$

Denote by $H^1(\mathcal{L}^s)$ and $H^{-1}(\mathcal{L}^s)$ the completions of $H_0^1(\mathcal{L}^s)$ and $H_0^{-1}(\mathcal{L}^s)$ in the norms

$$\|f\|_{1,\mathcal{L}^s} = \mathcal{E}^s(f,f)^{\frac{1}{2}} = \left(\int_0^{+\infty} \lambda e_f(d\lambda)\right)^{\frac{1}{2}}, \ f \in H_0^1(\mathcal{L}^s),$$

$$\|f\|_{-1,\mathcal{L}^s} = \left(\int_0^{+\infty} \frac{1}{\lambda} e_f(d\lambda)\right)^{\frac{1}{2}} = \sup\{(f,\varphi)_{L^2(\mathcal{X})} : \varphi \in H_0^1(\mathcal{L}^s), \|\varphi\|_{1,\mathcal{L}^s} = 1\},$$

respectively.

We can identify $H^{-1}(\mathcal{L}^s)$ with the dual space of $H^1(\mathcal{L}^s)$ via the standard identification of any element $f \in H_0^{-1}(\mathcal{L}^s)$ with a continuous linear functional on $H^1(\mathcal{L}^s)$ given as the unique continuous extension to $H^1(\mathcal{L}^s)$ of

$$f(\varphi) = (f,\varphi)_{L^2(\mathcal{X})} \text{ if } \varphi \in H_0^1(\mathcal{L}^s). \tag{34}$$

The proof of the weak convergence of the family $\{\mathbf{x}_\varepsilon(t)\}_{t\geq 0}$ is divided into the proof of *tightness* and the proof of *the uniqueness of the weak limit.*

The proof of tightness. Following Olla (1994) we introduce the following linear space

$$\mathcal{F} = \{\Phi \in L^\infty(\mathcal{X}, P_0, \mathcal{H}) : |\int \Phi f^2 dP_0| \leq C_\Phi \|\tilde{f}\|_{1,\mathcal{L}^s} \|f\|_{L^2(\mathcal{X})}, \tag{35}$$

$$\text{for a certain } C_\Phi > 0 \text{ and all } f \text{ for which } \tilde{f} \in H_0^1(\mathcal{L}^s)\}.$$

Remark. 4 It is elementary to check that $\mathcal{F} \subseteq H_0^{-1}(\mathcal{L}^s)$. Consider the test function $f = c + g$ where c is any constant and $g \in H_0^1(\mathcal{L}^s)$. Let $c \uparrow +\infty$ we have that $\int \Phi dP_0 = 0$ and $|\int \Phi g dP_0| \leq C_\Phi \|g\|_{1,\mathcal{L}^s}$.

The following proposition has been proven by Olla (1994).

Proposition. 8 *For any $\Phi \in \mathcal{F}$, $\alpha \in R$ we have*

$$\mathbf{ME}\left[\exp\{\alpha \int_s^t \Phi(\eta(s))ds\}\right] \leq \exp\{\frac{\alpha^2 C_\Phi^2}{4}(t-s)\}$$

where the constant C_Φ is the same as in (35).

We now show $\mathbf{b} \in \mathcal{F}$.

Proposition. 9 $D_p\Psi_{q,r} \in \mathcal{F}$ *for any* $p, q, r \in \{1, \cdots, d\}$.

Proof. Fix p, q, r. For any f such that $\tilde{f} \in H_0^1(\mathcal{L}^s) \cap L^\infty(\mathcal{X})$ we write

$$\int D_p\Psi_{q,r} f^2 dP_0 = -2\int \Psi_{q,r} D_p f f dP_0.$$

By the homogeneity of P_0 and the Lebesgue Dominant Convergence Theorem we get

$$\begin{aligned}\int D_p\Psi_{q,r} f^2 dP_0 &= \lim_{h\downarrow 0}\int \Psi_{q,r}\frac{f\circ\tau_{-h\mathbf{e}_p} - f}{h}(f\circ\tau_{-h\mathbf{e}_p} + f)dP_0 \\ &= -2\int \Psi_{q,r} D_p f f dP_0.\end{aligned}$$

Consequently we get

$$|\int D_p\Psi_{q,r} f^2 dP_0| \leq 2\|\Psi\|_{L^\infty(\mathcal{X})}\|f\|_{\tilde{H}^1(\mathcal{X})}\|f\|_{L^2(\mathcal{X})}, \tag{36}$$

which, by Proposition 7, implies that

$$|\int D_p\Psi_{q,r} f^2 dP_0| \leq \frac{2\|\Psi\|_{L^\infty(\mathcal{X})}}{c_2}\|f\|_{1,\mathcal{L}^s}\|f\|_{L^2(\mathcal{X})}. \tag{37}$$

Since $\mathcal{E}^s(\cdot,\cdot)$ is a symmetric Dirichlet form $\tilde{f}_n \in H_0^1(\mathcal{L}^s)$ and

$$\lim_{n\uparrow+\infty}\left[\|f_n - f\|^2_{L^2(\mathcal{X})} + \mathcal{E}^s(f_n - f, f_n - f)\right] = 0$$

for any $f \in H_0^1(\mathcal{L}^s)$, where $f_n = -n \vee (f \wedge n)$ (Fukushima (1980) p. 25). This allows us to extend (37) to the entire $H_0^1(\mathcal{L}^s)$ ∎

To prove tightness of $\{\mathbf{x}_\varepsilon(t)\}_{t\geq 0}$, $\varepsilon > 0$ it suffices to show tightness of

$$\varepsilon\int_0^{t/\varepsilon^2} b_k(\eta(s))ds,\ t \geq 0,\ \varepsilon > 0 \tag{38}$$

on $C([0,+\infty); R^d)$.

By Proposition 9,

$$\mathbf{ME}\left[\left(\frac{\varepsilon}{\sqrt{t-s}}\int_{s/\varepsilon^2}^{t/\varepsilon^2} b_k(\eta(\varrho))d\varrho\right)^4\right] \leq 24\mathbf{ME}\left[\exp\left\{\left|\frac{\varepsilon}{\sqrt{t-s}}\int_{s/\varepsilon^2}^{t/\varepsilon^2} b_k(\eta(\varrho))d\varrho\right|\right\}\right] \tag{39}$$

for $s < t$, $\varepsilon > 0$.

Choosing $\alpha = \pm\frac{\varepsilon}{\sqrt{t-s}}$ in Proposition 8 we get that the left hand side of (39) is less than or equal to $\exp\{\frac{C_{b_k}^2}{4}\}$. Thus,

$$\mathbf{ME}\left[\left(\varepsilon\int_{s/\varepsilon^2}^{t/\varepsilon^2} b_k(\eta(\varrho))d\varrho\right)^4\right] \le C(t-s)^2,$$

which implies tightness of the laws of (38) on $C([0,+\infty);R^d)$ by Kolmogorov's criterion (Billingsley (1968) p. 95 Theorem 12.3).

The uniqueness of the law of a weak limit. The following lemma is crucial in establishing the uniqueness property.

Lemma. 3 *Suppose that $f \in H_0^{-1}(\mathcal{L}^s)$ and $\lambda > 0$. Then*

(i) the equation

$$(\lambda - \mathcal{L})h_\lambda = f \tag{40}$$

has a unique solution $h_\lambda \in H_0^1(\mathcal{L}^s)$. In addition

$$\lim_{\lambda\downarrow 0} \lambda\|h_\lambda\|^2_{L^2(\mathcal{X})} = 0. \tag{41}$$

(ii) the family $\{h_\lambda\}_{\lambda>0} \subseteq H_0^1(\mathcal{L}^s)$ converges strongly in $H^1(\mathcal{L}^s)$ as $\lambda \downarrow 0$.

We show first how to apply the lemma to establish the uniqueness of the weak limit. We denote by Q_x, $x \in \mathcal{X}$ the configuration measures on $(\Omega, \mathcal{C})$ corresponding to S^t. Let $\tilde{Q}$, $\tilde{Q}_x$ be the measures induced from $P \otimes Q$ and Q_x, $x \in \mathcal{X}$ by the mapping $\mathcal{Z} : \Omega \times \Sigma \to \Omega$, where $\mathcal{Z}(\omega, \mathbf{w})(\cdot) = \eta(\cdot, \omega, \mathbf{w})$. Denote by $\tilde{\mathbf{E}}$, $\tilde{\mathbf{E}}_x$ the respective expectations.

Let $\mathcal{M}_2$ be the space of all square integrable martingales $M(t)$, $t \ge 0$ over the probability space $(\Omega \times \Sigma, \mathcal{C} \otimes \mathcal{B}, P \otimes Q)$ with respect to the filtration $\mathcal{F}_t$, $t \ge 0$ and such that $M(0) = 0$. Without loss of generality all martingales involved are assumed to be *cadlags*, i.e. right continuous with the left hand side limits. For any $f \in \mathcal{D}(\mathcal{L})$ define

$$M_f(t;\eta) = f(\eta(t)) - f(\eta(0)) - \int_0^t \mathcal{L}f(\eta(s))ds. \tag{42}$$

Let

$$\tilde{M}_f(t;\omega,\mathbf{w}) := M_f(t;\mathcal{Z}(\omega,\mathbf{w})),$$

$t \ge 0$. We have

$$\mathbf{ME}\tilde{M}_f^2(t) = t\|f\|^2_{1,\mathcal{L}^s} \tag{43}$$

and, hence, $\tilde{M}_f \in \mathcal{M}_2$.

Set $f_{\mathbf{v}} = (\mathbf{v}, \mathbf{b}) \in H_0^{-1}(\mathcal{L}^s)$ for any $\mathbf{v} \in R^d$, by Remark 4. Let h_λ, $\lambda > 0$ be the solutions of (40) with $f = f_{\mathbf{v}}$. h_λ converge as $\lambda \to 0$ to $\psi_{\mathbf{v}} \in H^1(\mathcal{L}^s)$. We show

$$((\mathbf{v}, \mathbf{x}_\varepsilon(t_1)), \cdots, (\mathbf{v}, \mathbf{x}_\varepsilon(t_N))), \tag{44}$$

with $t_k = kt_1$, $k = 1, \cdots, N$, converges as $\varepsilon \downarrow 0$ the Brownian Motion with variance at $t = 1$ being $|\mathbf{v}|^2 + \|\psi_{\mathbf{v}}\|^2_{1,\mathcal{L}^s}$. We write

$$\begin{aligned}(\mathbf{v}, \mathbf{x}_\varepsilon(t;\omega,\mathbf{w})) &= \varepsilon(\mathbf{v}, \mathbf{w}(t/\varepsilon^2)) + \varepsilon\int_0^{\frac{t}{\varepsilon^2}} (\mathbf{v}, \mathbf{b}(\eta(s;\omega,\mathbf{w})))ds \\ &= N_{\varepsilon,h_{\varepsilon^2}}(t;\omega,\mathbf{w}) + R_\varepsilon(t;\mathcal{Z}(\omega,\mathbf{w}))\end{aligned}$$

where

$$N_{\varepsilon,h_{\varepsilon^2}}(t;\omega,\mathbf{w}) = \varepsilon\sqrt{2\kappa}(\mathbf{v}, \mathbf{w}(t/\varepsilon^2)) + \varepsilon\tilde{M}_{h_{\varepsilon^2}}(\frac{t}{\varepsilon^2};\omega,\mathbf{w}) \tag{45}$$

and

$$R_\varepsilon(t;\eta) = -\varepsilon h_{\varepsilon^2}(\eta(t/\varepsilon^2)) + \varepsilon h_{\varepsilon^2}(\eta(0)) + \varepsilon^3\int_0^{\frac{t}{\varepsilon^2}} h_{\varepsilon^2}(\eta(s))ds.$$

By (i) of Lemma 3 and (43) it immediately follows that $\lim_{\varepsilon\downarrow 0}\mathbf{ME}R^2_\varepsilon(t) = 0$. Then the proof reduces to establishing weak convergence of finite dimensional distributions of $N_{\varepsilon,h_{\varepsilon^2}}$, as $\varepsilon \downarrow 0$. By Part (ii) of Lemma 3, for any $\sigma > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$

$$\mathbf{ME}[N_{\varepsilon,h_{\varepsilon^2}}(t) - N_{\varepsilon,h_{\varepsilon_0^2}}(t)]^2 \le \|h_{\varepsilon^2} - h_{\varepsilon_0^2}\|^2_{1,\mathcal{L}^s} t < \sigma t.$$

Consequently, for any uniformly continuous function $\varphi : R^d \to R$ and $\sigma > 0$ there exists $\varepsilon_0 >$ such that

$$\mathbf{ME}|\varphi(N_{\varepsilon,h_{\varepsilon^2}}(t_1), \cdots, N_{\varepsilon,h_{\varepsilon^2}}(Nt_1)) - \varphi(N_{\varepsilon,h_{\varepsilon_0^2}}(t_1), \cdots, N_{\varepsilon,h_{\varepsilon_0^2}}(Nt_1))| < \sigma.$$

Define the partial sum process $S_{f,\varepsilon}(t) = \varepsilon\sum_{p=0}^{[t/\varepsilon^2]}\xi_{p,f}$, where

$$\xi_{p,f} = \tilde{M}_f((p+1)t_1) - \tilde{M}_f(pt_1) + (\mathbf{v}, \mathbf{w}((p+1)t_1)) - (\mathbf{v}, \mathbf{w}(pt_1)).$$

By Theorem 23.1 p. 206 of Billingsley (1968) and Lemma 4 below, $S_{\varepsilon,h_{\varepsilon_0^2}}$ tend weakly in the Skorohod space $D[0,+\infty)$, as $\varepsilon \downarrow 0$, to a Brownian Motion with variance at $t = 1$ being $|\mathbf{v}|^2 + \|h_{\varepsilon_0^2}\|^2_{1,\mathcal{L}^s}$. Hence

$$\limsup_{\varepsilon\downarrow 0}|\mathbf{ME}\varphi(N_{\varepsilon,h_{\varepsilon^2}}(t_1), \cdots, N_{\varepsilon,h_{\varepsilon^2}}(Nt_1)) - \mathbf{W}_{\varepsilon_0}\varphi(\eta(t_1), \cdots, \eta(Nt_1))| \le \sigma$$

where $\mathbf{W}_{\varepsilon_0}$ is the expectation with respect to the Wiener Measure on $C[0,+\infty)$ corresponding to the limit Brownian Motion. Convergence in finite dimensional distributions follows as $\epsilon_0 \to 0$. We now prove

Lemma. 4 *For any fixed f the sequence $\xi_{p,f}$, $p \ge 0$ is stationary and ergodic.*

Proof. The case $\kappa > 0$ is proved as in Lemma 2.

For $\kappa = 0$, it suffices to prove that the transformation θ_t of the probability space $(\Omega, \mathcal{C}, \tilde{Q})$ is measure preserving and ergodic for an arbitrary $t > 0$. Suppose $A \in \mathcal{C}$ and

$$\theta_t^{-1}(A) = A. \tag{46}$$

Hence $S^t g = g$ for $g(x) = \tilde{\mathbf{E}}_x \chi_A(\eta)$, $x \in \mathcal{X}$,

We show now that the only invariant functions for S^t, where $t > 0$ are constants. Suppose $S^t \tilde{\chi}_C = \tilde{\chi}_C$, where $\tilde{\chi}_C = \chi_C - \int \chi_C dP_0$. By Proposition 7 the semigroup S^t, $t \geq 0$ is holomorphic (Ma-Röckner (1992) Corollary 2.21 p. 25). Thus $S^s \tilde{\chi}_C \in \mathcal{D}(\mathcal{L})$, for all $s > 0$ and

$$\frac{d}{ds} \|S^s \tilde{\chi}_C\|^2_{L^2(\mathcal{X})} = 2(-\mathcal{L} S^s \tilde{\chi}_C, S^s \tilde{\chi}_C)_{L^2(\mathcal{X})} \leq -2c_2 \|S^s \tilde{\chi}_C\|^2_{\tilde{H}^1(\mathcal{X})}. \tag{47}$$

It follows that $\|S^s \tilde{\chi}_C\|_{\tilde{H}^1(\mathcal{X})} = 0$, $0 < s < t$ and, by (T5), $\tilde{\chi}_C$, $0 < s < t$ and $g(x)$ are constants. ∎

Proof of Lemma 3. *Proof Of Part (i).* Since $\lambda > 0$ is in the resolvent set of $\mathcal{L}$ (40) can be uniquely solved in $h_\lambda \mathcal{D}(\mathcal{L}) \subseteq \mathcal{D}(\mathcal{E})$. Since $\overline{h}_\lambda = 0$, we have that $h_\lambda \in H^1_0(\mathcal{L}^s)$.

We construct the solutions h_λ in two steps. First we solve, by spectral calculus, for h^s_λ, the solutions of the resolvent equation for $\mathcal{L}^s$ with the desired properties as in Lemma 3. This is a standard argument due essentially to Kipnis and Varadhan (1986). Then by a perturbation argument we solve for h_λ.

Denote by $M_\lambda f \in H^1_0(\mathcal{L}^s)$, for $f \in H^{-1}(\mathcal{L}^s)$ and $\lambda > 0$, the unique element in the Hilbert space $(H^1_0(\mathcal{L}^s), \mathcal{E}^\lambda(\cdot,\cdot))$ such that

$$f(\varphi) = \mathcal{E}^\lambda(M_\lambda f, \varphi), \ \varphi \in H^1_0(\mathcal{L}^s) \tag{48}$$

where $\mathcal{E}^\lambda(\cdot,\cdot) := \lambda(\cdot,\cdot)_{L^2(\mathcal{X})} + \mathcal{E}(\cdot,\cdot)$. Then $M_0 f \in H^1(\mathcal{L}^s)$. Moreover,

$$f(\varphi) = (M_0 f, \varphi)_{1,\mathcal{L}^s}, \ \varphi \in H^1(\mathcal{L}^s). \tag{49}$$

$h^s_\lambda := M_\lambda f$ solves the resolvent equation

$$(\lambda - \mathcal{L}^s) h^s_\lambda = f. \tag{50}$$

Now we show $h^s_\lambda := M_\lambda f$ have the desired properties. We state it as a lemma.

Lemma. 5 *Suppose $\lambda \geq 0$ and $f \in H^{-1}(\mathcal{L}^s)$. Then*

(i)

$$\|M_\lambda f\|_{1,\mathcal{L}^s} \leq \|f\|_{-1,\mathcal{L}^s} \tag{51}$$

and

$$M_\lambda f = \int_0^{+\infty} \frac{1}{\lambda + \mu} E(d\mu) f. \tag{52}$$

Suppose, additionally, that the family $\{f_\lambda\}_{\lambda > 0} \subseteq H^{-1}(\mathcal{L}^s)$ satisfies

$$\lim_{\lambda \downarrow 0} \|f - f_\lambda\|_{-1,\mathcal{L}^s} = 0.$$

Then

$$\lim_{\lambda \downarrow 0} \lambda \|M_\lambda f_\lambda\|^2_{L^2(\mathcal{X})} = 0. \tag{53}$$

(ii)

$$\lim_{\lambda \downarrow 0} \|M_\lambda f - M_0 f\|_{1,\mathcal{L}^s} = 0. \tag{54}$$

(iii) For $\lambda > 0$ and $\{f_n\}_{n\geq 1} \subseteq H^{-1}(\mathcal{L}^s)$ satisfying

$$\lim_{n\uparrow+\infty} \|f_n - f\|_{-1,\mathcal{L}^s} = 0 \tag{55}$$

we have

$$\lim_{n\uparrow+\infty} \|M_\lambda f_n - M_\lambda f\|_{L^2(\mathcal{X})} = 0. \tag{56}$$

Proof. *Part (i).* Let $\lambda > 0$. Substituting $\varphi = M_\lambda f$ into (48) we obtain

$$\|M_\lambda f\|^2_{1,\mathcal{L}^s} \leq f(M_\lambda f) \leq \|f\|_{-1,\mathcal{L}^s}\|M_\lambda f\|_{1,\mathcal{L}^s},$$

which proves (51). This argument works for the case $\lambda = 0$ also. By (34) and (48) we have that $M_\lambda f = (\lambda - \mathcal{L}^s)^{-1} f$. (52) and (53) follow from the Spectral Theorem.

Part (ii). First we show that the family $\{M_\lambda f\}_{\lambda>0}$ is strongly compact in $H^1(\mathcal{L}^s)$. It is weakly compact since it is bounded in $\|\cdot\|_{1,\mathcal{L}^s}$ norm. We show now that any weakly convergent $\{M_{\lambda'} f\}_{\lambda'>0}$ to a limit $f_* \in H^1(\mathcal{L}^s)$ is strongly convergent to f_*. (48) implies that

$$f(\varphi) = \lambda'(M_{\lambda'} f, \varphi)_{L^2(\mathcal{X})} + \mathcal{E}^s(M_{\lambda'} f, \varphi),\ \varphi \in H^1_0(\mathcal{L}^s). \tag{57}$$

Letting $\lambda' \downarrow 0$ and using (53) we conclude from (57) that

$$f(\varphi) = (f_*, \varphi)_{1,\mathcal{L}^s},\ \text{for all } \varphi \in H^1(\mathcal{L}^s). \tag{58}$$

Substituting $\varphi = M_{\lambda'} f$ in (57) and letting $\lambda' \downarrow 0$ we get by Part (i) that

$$\|f_*\|^2_{1,\mathcal{L}^s} = f(f_*) = \lim_{\lambda'\downarrow 0} \|M_{\lambda'} f\|^2_{1,\mathcal{L}^s},$$

which clearly proves the strong convergence of $\{M'_\lambda f\}_{\lambda'>0}$. (58) implies that $f_* = M_0 f$. This completes the proof of Part (ii).

Part (iii). By (48)we have

$$\lambda\|M_\lambda f_n - M_\lambda f\|^2_{L^2(\mathcal{X})} \leq (f_n - f)(M_\lambda f_n - M_\lambda f) \leq$$

$$\|f - f_n\|_{-1,\mathcal{L}^s}\|M_\lambda f_n - M_\lambda f\|_{1,\mathcal{L}^s} \leq \|f - f_n\|^2_{-1,\mathcal{L}^s}$$

for $f_n - f \in H^{-1}(\mathcal{L}^s)$ and $\lambda > 0$. This concludes the proof of the lemma. ∎

Finally we use a perturbation technique to solve for h_λ.

Let $M_{\lambda,\delta} f \in H^1_0(\mathcal{L}^s)$ be the unique element of $H^1_0(\mathcal{L}^s)$ such that

$$f(\varphi) = \mathcal{E}^{\lambda,\delta}(M_{\lambda,\delta} f, \varphi),\ \varphi \in H^1_0(\mathcal{L}^s) \tag{59}$$

where $\mathcal{E}^{\lambda,\delta}(\cdot,\cdot) := \mathcal{E}^\lambda(\cdot,\cdot) + \delta\mathcal{E}^a(\cdot,\cdot)$ with $\mathcal{E}^a(\cdot,\cdot) = \frac{1}{2}[\mathcal{E}(f,g) - \mathcal{E}(g,f)]$, the antisymmetric part of the form $\mathcal{E}(f,g)$. The existence and uniqueness of $M_{\lambda,\delta}$ follows from the Lax-Millgram lemma. Again, for $\lambda = 0$, $M_{0,\delta} f \in H^1(\mathcal{L}^s)$. Note that $M_{\lambda,0} f = M_\lambda f (= h^s_\lambda)$, for all $\lambda \geq 0$ and $M_{\lambda,1} f = h_\lambda$.

Let $K_{\lambda,\delta} f \in H^{-1}(\mathcal{L}^s)$ be the unique continuous extension of

$$K_{\lambda,\delta} f(\varphi) = \mathcal{E}^a(M_{\lambda,\delta} f, \varphi),\ \varphi \in H^1_0(\mathcal{L}^s), \tag{60}$$

to the entire $H^1(\mathcal{L}^s)$.

The following lemma holds.

Lemma. 6 *(i)*
$$\|M_{\lambda,\delta}f\|_{1,\mathcal{L}^s} \le \|f\|_{-1,\mathcal{L}^s} \tag{61}$$
for $\lambda \ge 0$, $\delta \in R$.
$$\sup_{\lambda \ge 0,\delta} \|K_{\lambda,\delta}\| \le K', \tag{62}$$
where K' is the same constant as in (28). Moreover,
$$M_{\lambda,\delta_0+\delta} = M_{\lambda,\delta_0}(I + \delta K_{\lambda,\delta_0})^{-1} \tag{63}$$
for $|\delta| < \frac{1}{K'}$, $\lambda \ge 0$ and $\delta \in R$.

(ii) Suppose that $f \in H^{-1}(\mathcal{L}^s)$. Then
$$\lim_{\lambda\downarrow 0} K_{\lambda,\delta} f = K_{0,\delta} f \in H^{-1}(\mathcal{L}^s), \quad \forall \delta. \tag{64}$$

(iii) Suppose that the family $\{f_\lambda\}_{\lambda>0} \subseteq H^{-1}(\mathcal{L}^s)$ with $\lim_{\lambda\downarrow 0} \|f - f_\lambda\|_{-1,\mathcal{L}^s} = 0$. Then
$$\lim_{\lambda\downarrow 0} \|M_{\lambda,\delta} f_\lambda - M_{0,\delta} f\|_{1,\mathcal{L}^s} = 0 \tag{65}$$
and
$$\lim_{\lambda\downarrow 0} \lambda \|M_{\lambda,\delta} f_\lambda\|^2_{L^2(\mathcal{X})} = 0 \tag{66}$$
for $\forall \delta$.

Proof. *Part (i).* The proof of (61) is the same as that of Part (i) of Lemma 5 and is left to the reader.

By (28) we have that
$$\|K_{\lambda,\delta} f\|_{-1,\mathcal{L}^s} = \sup_{\|\varphi\|_{1,\mathcal{L}^s}=1} |\mathcal{E}^a(M_{\lambda,\delta} f, \varphi)| \le K' \|M_{\lambda,\delta} f\|_{1,\mathcal{L}^s} \le K' \|f\|_{-1,\mathcal{L}^s}.$$

Let us observe that for any $\varphi \in H^1_0(\mathcal{L}^s)$
$$\begin{aligned}
\mathcal{E}^{\lambda,\delta_0+\delta}(M_{\lambda,\delta_0}(I+\delta K_{\lambda,\delta_0})^{-1} f, \varphi) &= (I+\delta K_{\lambda,\delta_0})^{-1} f(\varphi) + \delta \mathcal{E}^a(M_{\lambda,\delta_0}(I+\delta K_{\lambda,\delta_0})^{-1} f, \varphi) \\
&= (I+\delta K_{\lambda,\delta_0})^{-1} f(\varphi) + \delta K_{\lambda,\delta_0}(I+\delta K_{\lambda,\delta_0})^{-1} f(\varphi) \\
&= f(\varphi)
\end{aligned}$$

which proves (i).

Part (ii). We first note that (64) holds for $\delta = 0$. Indeed
$$\|K_{\lambda,0} f - K_{0,0} f\|_{-1,\mathcal{L}^s} = \sup_{\|\varphi\|_{1,\mathcal{L}^s}=1} |\tilde{\mathcal{E}}^a(M_\lambda f - M_0 f, \varphi)| \le K' \|M_\lambda f - M_0 f\|_{1,\mathcal{L}^s}. \tag{67}$$

Here $\tilde{\mathcal{E}}^a(\cdot,\cdot)$ denotes the unique continuous extension of $\mathcal{E}^a(\cdot,\cdot)$ to $H^1(\mathcal{L}^s) \times H^1(\mathcal{L}^s)$. (64) followsfrom (52). We next show that if (ii) holds for a certain δ_0 then it also holds for all $\delta, |\delta - \delta_0| < \frac{1}{K'}$. The proof is complete in view of the result for $\delta_0 = 0$.

Part (iii). By (63), (65) and (66) immediately extend to the $\frac{1}{K'}$ neighborhood of δ_0. By the same calculation as (67), with δ in place of 0, we have the result.

The proof of Part (i) of Lemma 3 follows from (66) with $\delta = 1$.

Part (ii) of Lemma 3 follows from Part (iii) of Lemma 6 since $h_\lambda = M_{\lambda,1} f$ and (65). ∎

The proof of formula (9). By (L2), $H^1(\mathcal{L}^s_\kappa)$ are continuously embedded in $\tilde{H}^1(\mathcal{X})$, for any $\kappa \geq 0$. More specifically,

$$\|\varphi\|^2_{\tilde{H}^1(\mathcal{X})} \leq \frac{1}{c_2}\|\varphi\|^2_{1,\mathcal{L}^s_\kappa}, \quad \forall \varphi \in H^1(\mathcal{L}^s_\kappa). \tag{68}$$

By restoring the subscript κ of the generator we highlight its dependence on that parameter. For $f \in H^{-1}(\mathcal{L}^s_\kappa)$ define $Tf \in H^{-1}(\mathcal{L}^s_\kappa)$ as the unique continuous extension to $H^1(\mathcal{L}^s_\kappa)$ of the functional $Tf(\varphi) = (M_{0,1}f, \varphi)_{\tilde{H}^1(\mathcal{X})}$, $\varphi \in \tilde{H}^1(\mathcal{X})$. (61) and (68) imply that

$$\|Tf\|_{-1,\mathcal{L}^s_\kappa} \leq \frac{1}{c_2}\|f\|_{-1,\mathcal{L}^s_\kappa}, \quad \forall f \in H^{-1}(\mathcal{L}^s_\kappa).$$

The limit $\psi_{\mathbf{v}} \in H^1(\mathcal{L}^s_\kappa)$ of h_λ, as $\lambda \downarrow 0$, is given by

$$\psi_{\mathbf{v}}(\kappa) = M_{0,1}(I + \kappa T)^{-1} f_{\mathbf{v}}, \text{ for } |\kappa| < c_2. \tag{69}$$

(9) follows from the Neumann series expansion of $(I + \kappa T)^{-1}$ and part (i) of Lemma 6. ∎

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