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Physics Letters A 342 (2005) 22–29
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Superresolution and duality for time-reversal of waves in random media

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Received 15 March 2005; received in revised form 7 May 2005; accepted 10 May 2005

Available online 25 May 2005

Communicated by C.R. Doering

Abstract

We analyze the time reversal of waves in a turbulent medium using the parabolic Markovian model. We prove that for waves in a fractal medium with a sufficiently small Fresnel number the time reversal resolution can be a nonlinear (between linear and quadratic) function of the wavelength and independent of the aperture. We establish the duality between the forward propagation and time reversal. The duality holds true for any media and has two aspects: First there is an uncertainty inequality between the turbulence-induced wave spread and time-reversal resolution. The inequality becomes an equality when the wave structure function is Gaussian. Second, the turbulence-induced resolution in time reversal is identical to the turbulence-induced coherence length. As a consequence, the turbulence-induced aperture can be estimated by 2π times the forward spread, independent of the original aperture.

PACS: 42.25.-p; 43.20.+g; 42.68.-w; 84.40.-x

Keywords: Time reversal; Superresolution; Random media; Uncertainty principle

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[1] The research supported in part by ONR Grant N00014-02-1-0090, DARPA Grant N00014-02-1-0603, NSF grant DMS 0306659.
[2] The research supported in part by ONR Grant N00014-02-1-0090, DARPA Grant N00014-02-1-0603, NSF grant DMS0307011 and the Sloan Foundation.

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2005.05.030

1. Introduction

Time reversal is the process of recording the signal from a remote source, time-reversing and back-propagating it to retrofocus around the source. Time reversal of acoustic waves has been demonstrated to hold exciting technological potentials in subwavelength focusing, dispersion compensation, communications, imaging, remote-sensing and target detection in unknown environments (see [12–14,16,17,19] and references therein). The same should hold for the electromagnetic waves as well. Time reversal of electromagnetic waves is closely related to optical phase conjugation which used to be limited to monochromatic waves [2,3,15]. With the advent of experimental techniques, time reversal of high frequency EM waves hold diverse potential applications including real-time adaptive optics, laser resonators, high-power laser systems, optical communication and information processing, image transmission, spatial and temporal filtering, spectroscopy, etc. [18,22].

Time reversal refocusing is the result of the time-reversal invariance of the wave equations, acoustic or electromagnetic, in time invariant media. The surprising and important fact is that the refocal spot in a richly scattering medium is typically *smaller* than that in the homogeneous medium. That is, the time reversal resolution is enhanced rather than hampered by the inhomogeneities of the medium. This subdiffraction-limit retrofocusing is sometimes called *superresolution* and in certain regimes has been explained mathematically in terms of an enlarged effective aperture as a result of random media [1].

In the previous experimental, numerical or theoretical results the superresolution comes as a *linear* function of the wavelength but *independent* of the aperture. In this Letter we show that in fractal media the resolution can be a *superlinear* (between linear and quadratic) function of the wavelength and at the same time independent of the aperture. The lowest achievable refocal spot size in this nonlinear regime is on the order of the smallest scale of the medium fluctuations. Above the outer scale the resolution is diffraction-limited while below the inner scale it is the previously reported aperture-independent enhanced resolution [1,14].

We will focus our analysis on the widely used *parabolic Markovian* model for waves in atmospheric turbulence [21]. Neglecting the depolarization effect let us write the forward propagating wave field E at the carrier wave number k as $E(t,z,\mathbf{x}) = \Psi(z,\mathbf{x})e^{i(kz-\omega t)}$, $\mathbf{x} \in \mathbb{R}^2$ where the complex wave amplitude Ψ satisfies the Schrödinger equation in the non-dimensionalized form

$$i\frac{\partial \Psi}{\partial z} + \frac{\gamma}{2k}\Delta_\perp \Psi + \frac{k}{\gamma} V(z,\mathbf{x}) \circ \Psi = 0 \tag{1}$$

with $\Delta_\perp$ being the Laplacian in the transverse coordinates $\mathbf{x} \in \mathbb{R}^2$ and V the fluctuation of the refractive index. Here the Fresnel number γ equals $L_z k_0^{-1} L_x^{-2}$ with k_0 being the reference wavenumber, L_z and L_x the reference scales in the longitudinal and transverse directions, respectively. The notation $\circ$ in Eq. (1) means the Stratonovich product (v.s. Itô product). In the Markovian model $V(z,\cdot)$ is assumed to be a δ-correlated-in-z stationary random field such that

$$\left\langle V(z,\mathbf{x})V(z',\mathbf{x}')\right\rangle = \delta(z-z')\int \Phi(0,\mathbf{p})e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}\,d\mathbf{p},$$

where $\Phi(0,\mathbf{p})$ is the power spectrum density of the refractive index fluctuation at the mode $\vec{\mathbf{k}} = (0,\mathbf{p}) \in \mathbb{R}^3$ and, in the case of atmospheric turbulence, has a power-law behavior in the inertial range. For simplicity of presentation we assume an isotropic power-law

$$\Phi(\vec{\mathbf{k}}) = \sigma_H |\vec{\mathbf{k}}|^{-1-2H} |\vec{\mathbf{k}}|^{-2}, \quad \vec{\mathbf{k}} = (\xi,\mathbf{p}) \in \mathbb{R}^3, \quad |\vec{\mathbf{k}}| \in \left(L_0^{-1}, \ell_0^{-1}\right), \tag{2}$$

where L_0 and ℓ_0 are respectively the outer and inner scales of the turbulence and σ_H a constant factor. Usually H is taken to be $1/3$ in the self-similar theory of turbulence. We assume that the spectrum decays sufficiently fast for $|\vec{\mathbf{k}}| \gg \ell_0^{-1}$ while staying bounded for $|\vec{\mathbf{k}}| \ll L_0^{-1}$.

We will also establish rigorously the duality relation between the forward propagation and time reversal. The duality has two aspects: First there is an uncertainty inequality for random media where the conjugate quantities are the turbulence enhancements of forward wave spread and time-reversal resolution. The inequality becomes an *equality* when the wave structure function is Gaussian. Second, there is an identity between the turbulence enhancements of time reversal resolution and coherence length of the scattered wave field prior to time reversal. This relation has been observed in a time reversal experiment with a different random medium [6]. The duality holds true for any power spectrum, not limited to the power law (2).

2. Time reversal process

In the time reversal procedure, a source $\Psi_0(\mathbf{x})$ located at $z = L$ emits a signal with the carrier wavenumber k toward the time reversal mirror (TRM) of aperture A located at $z = 0$ through a turbulent medium. The transmitted field is captured and time reversed at the TRM and then sent back toward the source point through the same turbulent medium, see Fig. 1, [10,11].

The time-reversed, back-propagated wave field at $z = L$ can be expressed as

$$\begin{aligned}\Psi_{\mathrm{tr}}(\mathbf{x}) &= \int G(L,\mathbf{x},\mathbf{x}_m)\overline{G(L,\mathbf{x}_s,\mathbf{x}_m)\Psi_0(\mathbf{x}_s)}\mathbb{I}_A(\mathbf{x}_m)\,d\mathbf{x}_m\,d\mathbf{x}_s \\ &= \int e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}_s)/\gamma}\, W\left(L,\frac{\mathbf{x}+\mathbf{x}_s}{2},\mathbf{p}\right)\overline{\Psi_0(\mathbf{x}_s)}\,d\mathbf{p}\,d\mathbf{x}_s,\end{aligned} \tag{3}$$

where $\mathbb{I}_A$ is the indicator function of the TRM, G the propagator of the Schrödinger equation (1) and W the mixed-state Wigner distribution function

$$W(z,\mathbf{x},\mathbf{p}) = \int W(z,\mathbf{x},\mathbf{p};\mathbf{x}_m)\mathbb{I}_A(\mathbf{x}_m)\,d\mathbf{x}_m,$$

$$W(z,\mathbf{x},\mathbf{p};\mathbf{x}_m) = \frac{1}{(2\pi)^2}\int e^{-i\mathbf{p}\cdot\mathbf{y}}G(z,\mathbf{x}+\gamma\mathbf{y}/2,\mathbf{x}_m)\overline{G(z,\mathbf{x}-\gamma\mathbf{y}/2,\mathbf{x}_m)}\,d\mathbf{y},$$

which is the convex combination of the pure-state Wigner distributions $W(\cdot\,;\mathbf{x}_m)$. Here we have used the fact that time reversing of the signal is equivalent to the phase conjugating of its spatial component.

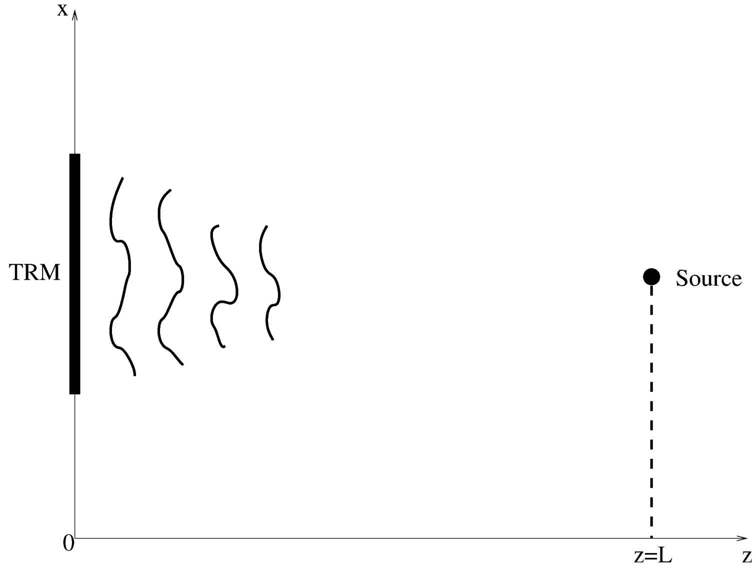

Fig. 1. The time reversal process.

The Wigner distributions (pure or mixed) satisfy a closed form equation, the Wigner–Moyal equation, and for the Markovian model all the moments also satisfy closed form equations [8]. In particular, the mean field equation is

$$\frac{\partial\langle W\rangle}{\partial z}+\frac{\mathbf{p}}{k}\cdot\nabla_{\mathbf{x}}\langle W\rangle=\mathcal{Q}\langle W\rangle \tag{4}$$

with the scattering operator $\mathcal{Q}$ given by

$$\mathcal{Q}f(\mathbf{x},\mathbf{p})=\frac{k^2}{2\gamma^2}\int\Phi(0,\mathbf{q})\big[-2f(\mathbf{x},\mathbf{p})+f(\mathbf{p}+\gamma\mathbf{q})+f(\mathbf{x},\mathbf{p}-\gamma\mathbf{q})\big]\,d\mathbf{q}. \tag{5}$$

Eq. (4) is exactly solvable and its Green function is

$$G_W(z,\mathbf{x},\mathbf{p},\bar{\mathbf{x}},\bar{\mathbf{p}})=\frac{1}{(2\pi)^4}\int\exp\big[i\big(\mathbf{q}\cdot(\mathbf{x}-\bar{\mathbf{x}})+\mathbf{y}\cdot(\mathbf{p}-\bar{\mathbf{p}})-z\mathbf{q}\cdot\bar{\mathbf{p}}/k\big)\big]$$
$$\times\exp\left[-\frac{k^2}{\gamma^2}\int_0^z D_*(\gamma\mathbf{y}+\mathbf{q}\gamma s/k)\,ds\right]d\mathbf{y}\,d\mathbf{q}, \tag{6}$$

where the (medium) structure function D_* is given by

$$D_*(\mathbf{x})=\int\Phi(0,\mathbf{q})\big[1-e^{i\mathbf{x}\cdot\mathbf{q}}\big]\,d\mathbf{q}. \tag{7}$$

We shall refer to $\exp[-k^2/\gamma^2\int_0^z D_*(\gamma\mathbf{y}+\mathbf{q}\gamma s/k)\,ds]$ as the *wave* structure function. With G_W we can calculate any two-point functions associated with Eq. (1). Here and below $\hat{f}$ denotes the Fourier transform $\mathcal{F}f$ of f. The main property of D_* we need in the subsequent analysis is the inertial range asymptotic:

$$D_*(r)\approx C_*^2r^{2H_*},\quad \ell_0\ll r\ll L_0, \tag{8}$$

where the effective Hölder exponent H_* is given by

$$H_*=\begin{cases}H+1/2 & \text{for } H\in(0,1/2),\\ 1 & \text{for } H\in(1/2,1),\end{cases} \tag{9}$$

and the structure parameter C_* is proportional to $\sigma_H^{1/2}$. Outside of the inertial range we have instead $D_*(r)\sim r^2$, $r\ll\ell_0$ and $D_*(r)\to D_*(\infty)$ for $r\to\infty$ where $D_*(\infty)>0$ is a finite constant.

Let us consider a point source located at $(L,0)$ by substituting the Dirac-delta function $\delta(\mathbf{x})$ for Ψ_0 in (3) and calculate $\langle\Psi_{\mathrm{tr}}\rangle$ with the Green function (6). We then obtain the point-spread function for the time reversed, refocused wave field written as $\mathcal{P}_{\mathrm{tr}}(\mathbf{x})=\mathcal{P}_0(\mathbf{x})T_{\mathrm{tr}}(\mathbf{x})$ with

$$\mathcal{P}_0(\mathbf{x})\equiv\left(\frac{k}{\gamma L}\right)^2\exp\left[i\frac{k|\mathbf{x}|^2}{2\gamma L}\right]\hat{\mathbb{I}}_A\left(\frac{k\mathbf{x}}{\gamma L}\right),\qquad T_{\mathrm{tr}}(\mathbf{x})\equiv\exp\left[-\frac{k^2}{\gamma^2}L\int_0^1 D_*(-s\mathbf{x})\,ds\right]. \tag{10}$$

In the absence of random inhomogeneity the function T_{tr} is unity and the resolution scale ρ_0 is determined solely by $\mathcal{P}_0$:

$$\rho_0\sim\gamma\frac{\lambda L}{A},\quad \lambda=\frac{2\pi}{k}. \tag{11}$$

This is the classical (Rayleigh) resolution formula where the retrofocal spot size is proportional to λ and the distance to the TRM, and inversely proportional to the aperture A.

3. Anomalous retrofocal spot-size

First we consider the situation where there may be an inertial range behavior. This requires from (10) that

$$k^2\gamma^{-2}D_*(\infty)L \gg 1, \tag{12}$$

where $D_*(\infty) = \lim_{r\to\infty} D_*(r)$. Condition (12) holds for a sufficiently small Fresnel number γ.

In the presence of random inhomogeneities the retrofocal spot size is determined by $\mathcal{P}_0$ or T_{tr} depending on which has a smaller support. For the power-law spectrum (2) we have the inertial range asymptotic

$$T_{\mathrm{tr}}(\mathbf{x}) \sim \exp\left[-C_*^2k^2\gamma^{-2}L|\mathbf{x}|^{2H_*}(4H_*+2)^{-1}\right] \tag{13}$$

for $\ell_0 \ll |\mathbf{x}| \ll L_0$. We define the turbulence-induced time-reversal resolution as

$$\rho_{\mathrm{tr}} = \sqrt{\int |\mathbf{x}|^2 T_{\mathrm{tr}}^2(\mathbf{x})\,d\mathbf{x} \Big/ \int T_{\mathrm{tr}}^2(\mathbf{x})\,d\mathbf{x}}, \tag{14}$$

which by (13) has the inertial range asymptotic

$$\rho_{\mathrm{tr}} \sim \left(\frac{\gamma\lambda}{C_*\sqrt{L}}\right)^{1/H_*}, \quad \ell_0 \ll \rho_{\mathrm{tr}} \ll L_0. \tag{15}$$

Under the following condition

$$\rho_{\mathrm{tr}} \ll \rho_0 \tag{16}$$

the function T_{tr} is much more sharply localized around $\mathbf{x}=0$ than $\mathcal{P}_0$. Note that as $H_* < 1$ the condition (16) holds for a sufficiently small γ. The nonlinear law (15) is valid only down to the inner scale ℓ_0 below which the linear law prevails $\rho_{\mathrm{tr}} \sim \gamma\lambda L^{-1/2}$. We see that under (12) and (16) ρ_{tr} is independent of the aperture, has a superlinear dependence on the wavelength in the inertial range and the resolution is further enhanced as the distance L and random inhomogeneities (C_*) increase. This effect can be explained by the notion of turbulence-induced aperture which enlarges as L and C_* increase as the TRM is now able to capture signals initially propagating in the more oblique directions (see Section 4 for more on this).

To recover the linear law previously reported in [1], let us consider the situation where $\rho_{\mathrm{tr}} = O(\gamma)$ and take the limit of vanishing Fresnel number $\gamma \to 0$ in Eq. (7) by setting $\mathbf{x} = \gamma\mathbf{y}$. Then we have

$$\lim_{\gamma\to 0}\gamma^{-2}D_*(\gamma\mathbf{y}) = D_0|\mathbf{y}|^2, \quad D_0 = \frac{1}{2}\int \Phi(0,\mathbf{q})|\mathbf{q}|^2\,d\mathbf{q}.$$

The resulting mean retrofocused field $\langle\Psi_{\mathrm{tr}}(\gamma\mathbf{y})\rangle$ is Gaussian in the offset variable $\mathbf{y}$ and the refocal spot size on the original scale is given by

$$\rho_{\mathrm{tr}} \sim \gamma\lambda(D_0L)^{-1/2}.$$

Hence the linear law prevails in the subinertial range.

4. Duality and turbulence-induced aperture

Intuitively speaking, the turbulence-induced aperture referred to in the previous section is closely related to how a wave is spread in the course of propagation through the turbulent medium. A quantitative estimation can be given by analyzing the spread of wave energy.

To this end let us calculate the mean energy density with the Gaussian initial wave amplitude

$$\Psi(0,\mathbf{x}) = \exp\left[-|\mathbf{x}|^2/(2\alpha^2)\right]. \tag{17}$$

We obtain

$$\left\langle\left|\Psi(L,\mathbf{x})\right|^2\right\rangle = \left(\frac{\alpha}{2\sqrt{\pi}}\right)^d \int \exp\left[-|\mathbf{q}|^2\left[\alpha^2/4+\gamma^2L^2/\left(4k^2\alpha^2\right)\right]\right] \times \exp[i\mathbf{q}\cdot\mathbf{x}]\exp\left[-\frac{k^2L}{\gamma^2}\int_0^1 D_*(\mathbf{q}s\gamma L/k)\,ds\right]d\mathbf{q}.$$

Hence the turbulence-induced spread can be identified as convolution with the kernel which is the inverse Fourier transform $\mathcal{F}^{-1}T$ of the transfer function

$$T(\mathbf{q}) = \exp\left[-\frac{k^2L}{\gamma^2}\int_0^1 D_*(\mathbf{q}s\gamma L/k)\,ds\right].$$

In view of (10), we obtain that

$$\mathcal{F}^{-1}T(\mathbf{x}) = \frac{k^2}{\gamma^2L^2}\mathcal{F}^{-1}T_{\mathrm{tr}}\left(\frac{k\mathbf{x}}{\gamma L}\right). \tag{18}$$

In this case it is reasonable to define the turbulence-induced forward spread σ_* as

$$\sigma_* = \sqrt{\int |\mathbf{x}|^2\left|\mathcal{F}^{-1}T\right|^2(\mathbf{x})\,d\mathbf{x}\Big/\int\left|\mathcal{F}^{-1}T\right|^2(\mathbf{x})\,d\mathbf{x}},$$

which, in view of (14) and (18), then satisfies the uncertainty inequality (see also [10])

$$\sigma_*\rho_{\mathrm{tr}} \geqslant \frac{\gamma L}{k}. \tag{19}$$

The equality holds when T_{tr} is Gaussian, i.e., when $H^* = 1$ or in the subinertial range. This strongly suggests the definition of the turbulence-induced aperture as $A_* = \gamma\lambda L/\rho_{\mathrm{tr}}$ in complete analogy to (11). And we have the inequality

$$A_* \leqslant 2\pi\sigma_*,$$

where equality holds true for a Gaussian wave structure function.

5. Coherence length and time-reversal resolution

Another physical variable that is naturally dual to the wave spread is the coherence length. The physical intuition is that the larger the spread the smaller the coherence length.

In the Markovian model with the Gaussian data (17) the coherence length has the following expression:

$$\begin{aligned}&\left\langle\Psi(L,\mathbf{x}+\mathbf{y}/2)\Psi(L,\mathbf{x}-\mathbf{y}/2)\right\rangle\\ &\quad= \left(\frac{\alpha}{\sqrt{2\pi}}\right)^2\int\exp\left[-|\mathbf{q}|^2\alpha^2/4\right]\exp\left[-\frac{|\mathbf{y}-\gamma L\mathbf{q}/k|^2}{4\alpha^2}\right]\\ &\quad\quad\times\exp[i\mathbf{q}\cdot\mathbf{x}]\exp\left[-\frac{k^2}{\gamma^2}\int_0^L D_*\left(-\mathbf{y}+\gamma\mathbf{q}(L-s)/k\right)ds\right]d\mathbf{q}.\end{aligned} \tag{20}$$

In the point-source limit $\alpha \to 0$, we have

$$\big\langle \Psi(L,\mathbf{x}+\mathbf{y}/2)\Psi(L,\mathbf{x}-\mathbf{y}/2)\big\rangle \approx \left(\frac{\sqrt{2}k\alpha^2}{\gamma L}\right)^2 \exp\left[i\frac{k}{\gamma L}\mathbf{y}\cdot\mathbf{x}\right]\exp\left[-\frac{k^2L}{\gamma^2}\int_0^1 D_*(-\mathbf{y}s)\,ds\right]. \tag{21}$$

In view of (21) let us define the turbulence-induced coherence length δ_* as

$$\delta_* = \sqrt{\int |\mathbf{y}|^2 T_2^2(\mathbf{y})\,d\mathbf{y} \Big/ \int T_2^2(\mathbf{y})\,d\mathbf{y}}, \quad T_2(\mathbf{y}) = \exp\left[-\frac{k^2L}{\gamma^2}\int_0^1 D_*(-\mathbf{y}s)\,ds\right].$$

Since $T_2 = T_{\mathrm{tr}},\ \delta_*$ is equal to the turbulence-induced time-reversal resolution ρ_{tr} and is related to the wave spread as

$$\sigma_*\delta_* \geqslant \frac{\gamma L}{k},$$

where the equality holds for a Gaussian wave structure function. Because of the identity of δ_* and ρ_{tr} the time reversal refocal spot size can be used to estimate the coherence length of the wave field which is more difficult to measure directly.

6. Discussion

In summary, we have proved three main results for the parabolic Markovian model. First, for a fractal medium with a sufficiently small Fresnel number the time reversal resolution can be aperture independent and depend on the wavelength in a nonlinear (between linear and quadratic) way. This is due to the self-similar nature of the media. Second, we prove an uncertainty inequality for random media where the conjugate variables are the turbulence enhancements of forward wave spread and time-reversal resolution. We show that the turbulence-induced aperture is bounded from above by 2π times the wave spread. The equality is attained when the wave structure function is Gaussian. Finally we show that the turbulence-induced coherence length is the same as the turbulence-induced time reversal resolution. The last two results constitute the duality between the forward propagation and time reversal. The duality is a general result not limited to the power-law spectrum (2) and is related to, but different from, the duality established in [11] for the power-law spectrum which takes the form of asymptotic equality.

The preceding analysis has been carried out for a narrow-band signal. Because of the linearity of the equation a wide-band signal $u_0(t,\mathbf{x})$ can be decomposed into frequency components each of which can be analyzed as above and then resynthesized. The mean retrofocused signal can be calculated as

$$\langle u_{\mathrm{tr}}\rangle(\tau,\mathbf{x}) = \frac{1}{2\pi\gamma^2L^2}\int d\mathbf{y}\,dt\,\overline{u_0(t,\mathbf{y})}\int dk\,\hat{\mathbb{I}}_A\left(\frac{k(\mathbf{x}+\mathbf{y})}{\gamma L}\right)e^{-ik(t+\tau)} \times e^{ik|\mathbf{x}|^2/(2\gamma L)}e^{-ik|\mathbf{y}|^2/(2\gamma L)}k^2T_{\mathrm{tr}}(\mathbf{x}-\mathbf{y})$$

from which it follows that the turbulence-induced spread in time is given by convolution with a *Gaussian* kernel because T_{tr} is Gaussian in k, see (10). The Gaussian kernel has an offset-$\mathbf{x}$-depending variance $\sigma_{\mathrm{tr}}^2(\mathbf{x}) = L\int_0^1 D_*(s\mathbf{x})\,ds/\gamma^2$ which grows rapidly with the offset if $\gamma \ll 1$. It is precisely this rapid change of temporal dispersion rate with the offset that produces the sharp spatial retrofocusing of the time-reversed pulse.

Our results above have been limited to the mean value of the time-reversed retrofocused field. Its second or higher moments can be determined from those of the Wigner distribution which are not exactly solvable. In case of self-averaging, however, the mean field is sufficient for determining all the higher moments. Self-averaging occurs, for example, when the narrow-band beam width in the transverse directions is large compared to the correlation length of the random medium or when the signal is wide-band [5,7]. The former case has been analyzed extensively in the literature (see [9,10] and references therein) and there arise several canonical radiative transfer equations as the self-averaging scaling limits. The case of temporally localized signals has only been studied for the **x**-independent layered medium in a scaling limit where the superresolution in the *transverse* direction does not occur, see [4,20]. In the near-self-averaging regime the second moment of the Wigner distribution can be calculated perturbatively and the result will be reported elsewhere.

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