Time Scales in Noisy Conservative Systems

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1 Introduction

A dynamical system, continuous or discrete in time, is *conservative* if it preserves an invariant measure with smooth density. The invariant measure is not unique unless the dynamical system is ergodic. For concreteness, we present the discussions and results mostly for the *classical systems* [1] for which the state space is a periodic domain $\mathcal{M}$ in R^d and the continuous time system

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{u}(\boldsymbol{x}(t)) \tag{1.1}$$

preserves the Lebesgue measure, namely

$$\nabla \cdot \boldsymbol{u}(\boldsymbol{x}) = 0. \tag{1.2}$$

The vector field $\boldsymbol{u}(\boldsymbol{x})$ is assumed to be bounded. We consider small white noise perturbation

$$d\boldsymbol{x}_\varepsilon(t) = \boldsymbol{u}(\boldsymbol{x}_\varepsilon(t))dt + \sqrt{\varepsilon}d\boldsymbol{w}(t) \tag{1.3}$$

where $\sqrt{\varepsilon}$ is the noise magnitude and $\boldsymbol{w}(t)$ is the $d-$dimensional Brownian motion. The random perturbation is chosen as to preserve the same invariant measure of the dynamical system.

There are clearly many possible choices of noises depending on the actual physical contexts from which the dynamical system and noise arise. To fix the idea, we think of (1.1) as the Lagrangian description of the incompressible fluid flow $\boldsymbol{u}(\boldsymbol{x})$ and ε as the molecular diffusivity of some passive scalar particles. In this context, it is of considerable interest to consider the *annulus*, the *strip* or the *cube* as the state space with *reflecting boundaries*(to preserve the total number of particles) since most fluid dynamical experiments have such set-ups (see, e.g., [11], [20]).

In another situation of solid state physics where particles (e.g. electrons) move in a potential force field (e.g. electric field), the Hamiltonian flow $\boldsymbol{u}(\boldsymbol{x})$ on the phase space preserves the invariant Lebesgue measure (Liouville theorem) which is not finite in view of the fact that the momentum variables are effectively unbounded. In that case, the more reasonable random processes of modeling the microscopic collisions are those possessing some finite invariant measures. One such candidate is the Ornstein-Ulenbeck process which leaves invariant the Maxwellian distribution. The motion of a particle in a potential force field under the small perturbation of the Ornstein-Ulenbeck process will be addressed in a separate paper.

In general, different noises give rise to different results on the problems addressed in this paper. But we expect those results are stable within each type of noise according to the correlation properties, invariant measures,.... etc and are not sensitive to the details of the random noises.

In the mathematical apparatus developed in [7], the perturbation ε plays the role of parametrization of time in the following sense. The unperturbed skew-symmetric operator $\boldsymbol{u}\cdot\nabla$ generates an unitary group of transformations $e^{t\boldsymbol{u}\cdot\nabla}$ so that the L^p norm is unchanged with time , for all $1\leq p\leq\infty$,

$$||e^{t\boldsymbol{u}\cdot\nabla}-\langle\cdot\rangle||_{p\to p}=1,\quad \forall 1\leq p\leq\infty. \tag{1.4}$$

Here $\langle\cdot\rangle$ is the average w.r.t. the invariant measure and $||\cdot||_{p\to q}$, for $1\leq p,q\leq\infty$, is the operator norm from L^p to L^q. (1.4) reflects the reversibility of the unperturbed system (1.1) and the progression in time is not observable in the sense of operator norm. Of course, if a weaker notion of measuring the evolution is used, then the time progression may be observable. The perturbed generator

$$\mathcal{L}^\varepsilon = \frac{\varepsilon}{2}\Delta + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \tag{1.5}$$

gives rise to a semi-group of contractions $\mathcal{P}_t^\varepsilon = e^{t\mathcal{L}^\varepsilon}$ with a positive decay rate $N_\varepsilon(t)$ such that

$$||\mathcal{P}_t^\varepsilon - \langle\cdot\rangle||_{2\to 2} = e^{-N_\varepsilon(t)}. \tag{1.6}$$

The semi-group property of $\mathcal{P}_t^\varepsilon$ implies immediately that $N_\varepsilon(t)$ is a super-additive function of t

$$N_\varepsilon(t+s) \geq N_\varepsilon(t) + N_\varepsilon(s), \quad \forall t, s \geq 0 \tag{1.7}$$

with $N_\varepsilon(0) = 0$, $\forall \varepsilon > 0$. In particular, $N_\varepsilon(t)$ is a increasing function of t. We could consider more general $||\cdot||_{p\to p}$ norm in (1.6) too. But $||\cdot||_{2\to 2}$ seems most natural in view of its relation with the usual definition of dissipation rate

$$\frac{d}{dt}\langle(\mathcal{P}_t^\varepsilon f - \langle f\rangle)^2\rangle = -\varepsilon\langle(\nabla\mathcal{P}_t^\varepsilon f)^2\rangle. \tag{1.8}$$

From (1.6) and (1.8), we get

$$e^{-N_\varepsilon(t)} = 1 - \sup_f \left\{ \int_0^t ds \frac{\varepsilon\langle(\nabla\mathcal{P}_s^\varepsilon f)^2\rangle}{\langle(f - \langle f\rangle)^2\rangle} \right\}. \tag{1.9}$$

We are going to characterize a number of time scales in Section 2 in terms of the asymptotics of $N_\varepsilon(t)$ as $\varepsilon \to 0$ coupled with $t \to \infty$. One of the time scales discussed in Section 2 is the *dissipation time* t_{diss} which is the least such that

$$N_\varepsilon(t) \to \infty, \quad \forall t \gg t_{diss}, \tag{1.10}$$

as $\varepsilon \to 0$. Namely, it is the time scale on which the evolution $\mathcal{P}_t^\varepsilon$ stablizes. Because the stability is measured in the operator norm, the asymptotic behaviours of (1.3) beyond t_{diss} are essentially independent of initial positions. t_{diss} can be substantially smaller than $\frac{1}{\varepsilon}$ and when this happens, it is called the phenomenon of convection enhanced dissipation. The idea is that the unperturbed system (1.1) may create small spatial scales in relatively short time on top of which the small diffusion ε can act effectively and smooth out the initial distributions rather quickly. Examples of this kind are given in Section 3.

Another important time scale t_m, called the *martingale time* in Section 4, is one beyond which the Markov process (1.3) behaves like martingale when looked at from suitable reference frames. The perturbed process (1.3) is conviently decomposed into a martingale and a fluctuation with the former capturing the final state of diffusion. The martingale time is defined by the ratio of the fluctuation to the per unit time variance of the martingale.

The noise contaminates the system completely on the *diffusion time* which is roughly the martingale time independent of initial points. This approach to the final state of diffusion is stated in Section 5 as the limit theorem for the perturbed system (1.3) on the diffusion time scale. This is quite different from the usual central limit theorem in that the *convection enhance diffusion* generally occurs which alters the scaling drastically. Examples of this kind are given in Section 6 and more examples can be found in [9] and [10].

The dissipation time characterizes the mixing mechanism and the martingale time characterizes the vanishing of the non-transporting fluctuation. Together with the diffusion time scale, they and their relations constitute the major part of this paper.

Finally, we comment on the transient crossover times. Contrary to traditional belief, the transient dispersion may take place well beyond the *dissipation time* t_{diss}, if $t_{diss} \ll t_m$, and consequently the transient dispersion in this case is initial point independent.

This paper is a brief account of the full blown version [7].

Before we entering next section, let us set up some notations. Given two sequences of positive numbers $a_\varepsilon, b_\varepsilon$, we say that

$$a_\varepsilon \lesssim b_\varepsilon \quad \text{if} \quad \overline{\lim_{\varepsilon\to 0}} \frac{a_\varepsilon}{b_\varepsilon} < \infty; \tag{1.11}$$

$$a_\varepsilon \gtrsim b_\varepsilon \quad \text{if} \quad \underline{\lim_{\varepsilon\to 0}} \frac{a_\varepsilon}{b_\varepsilon} > 0; \tag{1.12}$$

and $a_\varepsilon \sim b_\varepsilon$ if both (1.11) and (1.12) hold. Also

$$a_\varepsilon \ll b_\varepsilon \quad \text{if } \overline{\lim_{\varepsilon\to 0}} \frac{a_\varepsilon}{b_\varepsilon} = 0; \tag{1.13}$$

$$a_\varepsilon \gg b_\varepsilon \quad \text{if } \underline{\lim_{\varepsilon\to 0}} \frac{a_\varepsilon}{b_\varepsilon} = \infty. \tag{1.14}$$

2 Decay Rate and Dissipation Time

First of all, the following theorem provides general upper and lower bounds for $N_\varepsilon(t)$

Theorem 2.1

$$C\varepsilon \le \frac{N_\varepsilon(t)}{t} \le -Re(\mu_\varepsilon), \quad \textit{uniformly in } t \in [0,\infty), \quad \textit{as } \varepsilon \to 0, \tag{2.1}$$

where μ_ε (possibly not unique) are the nonzero eigenvalues with the largest real part of $\mathcal{L}^\varepsilon$.

The lower bound is proved by the Trotter product formula and the upper bound follows immediately from the theorem of spectral radius [21].

Moreover, these bounds are realizable on the initial and final time scales in the following sense.

Theorem 2.2 *There exists a time $t_{sp} \ge 0$(probably divergent as $\varepsilon \to 0$) such that*

$$N_\varepsilon(t) \sim -Re(\mu_\varepsilon)t, \quad \forall t \gg t_{sp} \tag{2.2}$$

and

$$N_\varepsilon(t) \sim \varepsilon t, \quad \textit{uniformly in } t \in [0,T], \forall T < \infty \tag{2.3}$$

if $\boldsymbol{u}$ is continuously differentiable.

(2.2) is again part of the spectral radius theorem [21]. For bounded time interval, (2.3) is a regular perturbation problem and is proved by a theorem of Blagoveshchenskii and Freidlin [7].

Now, if we also know that

$$-\text{Re}(\mu_\varepsilon) \lesssim \varepsilon \tag{2.4}$$

there would be no problems with time scales: t_{sp} can be set to zero and

$$N_\varepsilon(t) \sim \varepsilon t, \quad \text{uniformly in } t \in [0, \infty). \tag{2.5}$$

But in general, Theorem 2.2 fails for time scales tending to infinity when

$$-\mathrm{Re}(\mu_\varepsilon) \gg \varepsilon \tag{2.6}$$

i.e. when the interaction of convection and diffusion is so significant that $\varepsilon\Delta$ is no longer a regular perturbation to $\boldsymbol{u} \cdot \nabla$. It becomes a singular perturbation problem of the secondary eigenvalues and results in a nontrivial dependence of t_{sp} on ε. The coexistence of multiple time scales and asymptotics prohibits one from extracting directly global time information from the local ones such as one can for symmetric Markov processes via the logarithmic Sobolev inequalities. Here we want to emphasize that the eigenvalue μ_ε is only an indicator of the split of different time scales, but knowing μ_ε alone would not help much in understanding the dissipation mechanism, unless t_{sp} is also known, and even that may not tell us the important quantities such as the time scale beyond which the noise takes over, for t_{sp} may be too large such that $t_{sp} \gg \frac{-1}{\mathrm{Re}(\mu_\varepsilon)}$ and $N_\varepsilon(t) \to \infty$ long before t_{sp}.

Let us define the *dissipation time scale* t_{diss} to be the one beyond which the perturbed semigroup stablizes, i.e.

$$N_\varepsilon(t) \to \infty, \quad \text{as } \varepsilon \to 0, \ \forall t \gg t_{diss}. \tag{2.7}$$

We take t_{diss} to be the least of those times for which (2.7) holds.

Since the decay rates of the initial and final phases are ε and $-\mathrm{Re}(\mu_\varepsilon)$ respectively, we have

$$-\frac{1}{\mathrm{Re}(\mu_\varepsilon)} \le t_{diss} \le \frac{1}{\varepsilon}. \tag{2.8}$$

We call it the phenomenon of *convection enhanced dissipation* when the dissipation time is much shorter than $\frac{1}{\varepsilon}$

$$t_{diss} \ll \frac{1}{\varepsilon}. \tag{2.9}$$

Convection enhanced dissipation (2.9) is related to "convection enlarged spectral gap" (2.6) at least in the following way

Proposition 2.1 *If the convection enhanced dissipation (2.9) takes place, then*

$$-Re(\mu_\varepsilon) \gg \varepsilon. \tag{2.10}$$

Thus convection enhanced dissipation (2.9) is an precursor of (2.6) and the coexistence of multiple time scales for the rate function $N_\varepsilon(t)$. But the converse of Proposition 2.1 may not be true unless one can also show

$$t_{sp} \sim t_{diss} \tag{2.11}$$

, and if this is the case, then the natural guess would be

$$t_{diss} \sim -\frac{1}{\mathrm{Re}(\mu_\varepsilon)}. \tag{2.12}$$

But, in general, neither (2.11) nor (2.12) is true in view of the recent studies of the evolution processes of nonnormal operators such as $\mathcal{L}^\varepsilon$(see [17] and the references therein). The evolution $\mathcal{P}_t^\varepsilon$ typically has various transient behaviors which sometimes show up on the most interesting time scales and the eigenvalues such as μ_ε do not necessarily provide the relevent information. What (2.6) does tell us is the existence of initial-data sensitive, transient behaviours taking place before and up to the dissipation time t_{diss}.

The summary of this section is that on the order one time scale the rate function behaves like εt and on the longest time scale it is like $-\mathrm{Re}(\mu_\varepsilon)t$. In between, there is the dissipation time scale t_{diss} that tells us when the evolution stablizes.

3 Examples of Convection Enhanced Dissipation

Let us begin with a negative result of when (2.9) does not occur.

Theorem 3.1 *If the unperturbed system (1.1) is not ergodic and it has a non-constant invariant measure with twice continously differentiable density, then*

$$t_{diss} \sim \frac{1}{\varepsilon}. \tag{3.1}$$

We do not know any continous time systems for which (2.9) does occur. The problem simply is that nontrivial ergodic flows are hard to write down. Note that ergodicity alone is not sufficient to have (2.9). For instance, take $\boldsymbol{u}(\boldsymbol{x}) = (u_1, u_2, ..., u_d)$ in d-dimensional torus, where u_i are rationally independent constant. In this example, nothing intersting is going on; one has

$$-\mathrm{Re}(\mu_\varepsilon) \sim \varepsilon \tag{3.2}$$

(cf. [14]) and thus

$$t_{diss} \sim \frac{1}{\varepsilon}. \tag{3.3}$$

Some notion of mixing or hyperbolicity seem relevent as indicated in the next example (Theorem 3.2).

But for discrete time systems such as the iterated maps perturbed by convolutions, (2.9) can manifest in a rather dramatic manner for the alternating baker's maps [1]. Let F, F' denote the usual baker's map and its $\frac{\pi}{2}-$rotated version acting on the period cell $[0,1]^2$:

$$F(x,y) = \begin{cases} (2x, 1/2y) & \text{mod } 1, \quad \text{if } 0 \le x < 1/2 \\ (2x, 1/2(y+1)) & \text{mod } 1, \quad \text{if } 1/2 \le x < 1 \end{cases} \tag{3.4}$$

$$F'(x,y) = \begin{cases} (1/2x, 2y) & \text{mod } 1, \quad \text{if } 0 \le y < 1/2 \\ (1/2(x+1), 2y) & \text{mod } 1, \quad \text{if } 1/2 \le y < 1 \end{cases} \tag{3.5}$$

Define the unperturbed system $\mathcal{P}_t^0$ as

$$\mathcal{P}_t^0 = (\mathcal{P}_1^0)^t, \quad \text{with } \mathcal{P}_1^0 = F \circ F'. \tag{3.6}$$

Let (3.6) be perturbed by convolution with heat kernel with variance ε

$$\mathcal{P}_t^\varepsilon = (\mathcal{P}_1^\varepsilon)^t, \quad \text{with } \mathcal{P}_1^\varepsilon = H_\varepsilon * (F \circ F') \tag{3.7}$$

where $H_\varepsilon(\boldsymbol{x}) = \frac{1}{2\pi\varepsilon} e^{-|\boldsymbol{x}|^2/2\varepsilon}$. We have

Theorem 3.2 *For (3.7),*

$$t_{diss} \lesssim \log \frac{1}{\varepsilon}. \tag{3.8}$$

Faced with the two extreme situations of Theorem 3.1 and 3.2, one may wonder where the generic cases lie. Is Theorem 3.1 or Theorem 3.2 exceptional? The word *generic* or *exeptional* becomes meaningful by putting a topology or a measure over the group of transformations. Without entering the discussion in depth, let us simply remark that the ergodicity is generic and the mixing is exceptional in the weak topology for the general abstract dynamical systems (namely, not restricted to the classical systems such as (1.1))[13].

4 Martingale Time and Diffusion Time

As demonstrated in [16] and [15], it is very useful to decompose the sample path $\boldsymbol{x}_\varepsilon(t)$ into a martingale $\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ and a fluctuation $\chi^\varepsilon(\boldsymbol{x}_\varepsilon(t))$

$$\boldsymbol{x}_\varepsilon(t) = \rho^\varepsilon(\boldsymbol{x}_\varepsilon(t)) - \chi^\varepsilon(\boldsymbol{x}_\varepsilon(t)) \tag{4.1}$$

where $\chi^\varepsilon(\boldsymbol{x}) = (\chi_1^\varepsilon(\boldsymbol{x}), \ldots, \chi_d^\varepsilon(\boldsymbol{x}))$ is the zero mean, periodic solution of

$$\mathcal{L}^\varepsilon \chi^\varepsilon = \boldsymbol{u} - \langle \boldsymbol{u} \rangle. \tag{4.2}$$

Here $\langle \cdot \rangle$ is the average w.r.t the invariant measure. χ^ε is called the corrector in homogenization theory whose significance can be seen from the identity (cf. [2],[9])

$$\begin{aligned} \sigma_\varepsilon(\boldsymbol{e}_i) &= \varepsilon + \varepsilon \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} d\boldsymbol{x} (\nabla \chi_i^\varepsilon)^2 \\ &= \varepsilon \frac{1}{|\mathcal{M}|} \int_{\mathcal{M}} d\boldsymbol{x} (\nabla \rho_i^\varepsilon)^2, \quad \forall i \end{aligned} \tag{4.3}$$

where $\sigma_\varepsilon(\boldsymbol{e}_i)$ is the *effective diffusivity* in the direction $\boldsymbol{e}_i$ which is supposed to be the variance per unit time of the sample path $\boldsymbol{x}_\varepsilon(t)$ on the longest time scale. For simplicity, let us assume, from now on, that $\{\boldsymbol{e}_i\}$ is a complete set of orthornormal eigenvectors of the effective diffusivity. This time scale can be legitimately called the *diffusion time*. The asymptotic behavior of (4.3) as $\varepsilon \to 0$ has been studied in detail in [9] for periodic flows and in [10] for random flows. But actually how long the diffusion time is, the homogenization theory does not tell us.

It is easy to check from (4.1) and (4.2) that ρ^ε is harmonic w.r.t. $\mathcal{L}^\varepsilon$ so that $\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ is a martingale. Moreover, it is a stochastic integral

$$\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t)) = \sqrt{\varepsilon}\int_0^t \nabla\rho^\varepsilon(\boldsymbol{x}_\varepsilon(s))dw_s + \rho^\varepsilon(\boldsymbol{x}_\varepsilon(0)). \tag{4.4}$$

so that the variance

$$E_{\boldsymbol{x}}\left\{(\rho^\varepsilon)^2(\boldsymbol{x}_\varepsilon(t))\right\} = (\rho^\varepsilon)^2(\boldsymbol{x}_\varepsilon(0)) + \varepsilon\int_0^t ds\ E_{\boldsymbol{x}}\left\{(\nabla\rho^\varepsilon)^2(\boldsymbol{x}_\varepsilon(s))\right\}. \tag{4.5}$$

Before the sample path $\boldsymbol{x}_\varepsilon(t)$ reaches its final state of diffusion signified by the effective diffusivity σ_ε, it has to attain the state of martingale represented by $\rho^\varepsilon(\boldsymbol{x}_\varepsilon(t))$ first and the condition for this is that the non-martingale fluctuation is small relative to the martingale component in a uniform manner,

$$\lim_{\varepsilon\to 0}\frac{E_{\boldsymbol{x}}\left[(\chi_i^\varepsilon)^2(\boldsymbol{x}_\varepsilon(t\tau)\right]}{\varepsilon\int_0^t ds\ E_{\boldsymbol{x}}\left\{(\nabla\rho^\varepsilon)^2(\boldsymbol{x}_\varepsilon(s))\right\}} = 0,, \quad \forall t \gg t_m(\boldsymbol{x}, \boldsymbol{e}_i) \tag{4.6}$$

uniformly in $\tau\in[\tau_0, T]$, $\forall 0<\tau_0<T<\infty$, $\forall i$, in view of (4.5). Here $E_{\boldsymbol{x}}$ denotes the average w.r.t. the Wiener process.

Condition (4.6) defines the initial-position-$\boldsymbol{x}$ sensitive *martingale time* scale $t_m(\boldsymbol{x}, \boldsymbol{e}_i)$ which also depends on the direction $\boldsymbol{e}_i$. To seek an initial-position independent *martingale time* $t_m(\boldsymbol{e}_i)$, one has to look beyond the *dissipation time* t_{diss} so that the condition becomes

$$\frac{1}{t}\sigma_\varepsilon^{-1}(\boldsymbol{e}_i)\langle(\chi_i^\varepsilon)^2\rangle \to 0, \quad \forall t \gg t_m(\boldsymbol{e}_i), \tag{4.7}$$

in view of (4.3). Thus it seems most natural to define the *martingale time* through the formula

$$t_m(\boldsymbol{e}_i) = \max\{\sigma_\varepsilon^{-1}(\boldsymbol{e}_i)\langle(\chi_i^\varepsilon)^2\rangle,\ t_{diss}\}, \tag{4.8}$$

beyond which the perturbed process (1.3) behaves like a martingale independent of the initial point $\boldsymbol{x}$. It is easy to see that

$$t_m(\boldsymbol{e}_i) \lesssim \frac{1}{\varepsilon} \tag{4.9}$$

since $t_{diss} \lesssim \frac{1}{\varepsilon}$ and

$$\sigma_\varepsilon(\boldsymbol{e}_i) > \varepsilon\frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} d\boldsymbol{x}(\nabla\chi_i^\varepsilon)^2 \geq C\varepsilon\frac{1}{|\mathcal{M}|}\int_{\mathcal{M}} d\boldsymbol{x}(\chi_i^\varepsilon)^2 \tag{4.10}$$

which is an application of the Poincare inequality.

5 Limit Theorem

In the previous section, the diffusion time t_{diff} is loosely defined as the time scale beyond which the perturbed process (1.3) approaches the final state of diffusion. In the following theorem, we give a precise estimate on how large d_{diff} has to be. In order to state the result in its simplest form, we assume the manifold $\mathcal{M}$ to be the $d-$dimensional torus and we take the variables $\boldsymbol{x}$ in (1.1) and (1.3)to be the "extended" coordinates of the covering space R^d.

Theorem 5.1 *The family of processes* $\frac{1}{\lambda_\varepsilon}\sqrt{\sigma_\varepsilon^{-1}}\left\{\boldsymbol{x}_\varepsilon(\lambda_\varepsilon^2 t) - \lambda_\varepsilon^2\langle\boldsymbol{u}\rangle t\right\}\cdot\boldsymbol{e}_i$ *converge in law to the one dimensional Brownian motion in* $[t_0, T], \forall 0 < t_0 < T < \infty$*, if* $\lambda_\varepsilon^2 = \lambda_\varepsilon^2(\boldsymbol{e}_i) \gg t_m(\boldsymbol{e}_i)$ *and*

$$\lim_{\varepsilon\to 0}\left\{N_\varepsilon(\lambda_\varepsilon^2) + (\frac{d}{2}+1)\log(\lambda_\varepsilon^2\varepsilon)\right\} = \infty. \tag{5.1}$$

The convergence holds in measure with respect to the initial positions $\boldsymbol{x}\in\mathcal{M}$*.*

Several remarks concern the above theorem.

For experimental set-ups such as those of [11] and [20], the covering space is the infinite strip and has only one "extended" direction. So the limit theorem for this case is also limited to the extended coordinate.

The tightness of $\frac{1}{\lambda_\varepsilon}\sqrt{\sigma_\varepsilon^{-1}}\left\{\boldsymbol{x}_\varepsilon(\lambda_\varepsilon^2 t) - \lambda_\varepsilon^2\langle\boldsymbol{u}\rangle t\right\}$ requies the following stronger (than (1.6)) ultracontractivity estimate

Theorem 5.2

$$\|\mathcal{P}_t^\varepsilon - \langle\cdot\rangle\|_{1\to\infty} \le \frac{C_\delta}{(\varepsilon t)^{d/2}} e^{-N_\varepsilon(\delta t)}, \quad \forall 0 < \delta < 1, \forall t > 0 \tag{5.2}$$

for some positive constant C_δ*.*

The estimate of (5.2) is obtained in [7] by the duality of $\mathcal{P}_t^\varepsilon$ and its adjoint $\mathcal{P}_t^{\varepsilon*}$ whose generator is, because of the incompressibility (1.2),

$$\mathcal{L}^{\varepsilon*} = \frac{\varepsilon}{2}\Delta - \boldsymbol{u}(\boldsymbol{x})\cdot\nabla. \tag{5.3}$$

Thus, in addition to Theorem 5.2, one also has

Theorem 5.3

$$\|\mathcal{P}_t^{\varepsilon *} - \langle\cdot\rangle\|_{1\to\infty} \leq \frac{C_\delta}{(\varepsilon t)^{d/2}} e^{-N_\varepsilon(\delta t)}, \quad \forall 0 < \delta < 1, \forall t > 0 \tag{5.4}$$

for some positive constant C_δ.

The ultracontractivity estimates are clearly singular at $t = 0$ but the ε dependence in the denominators of (5.2) and (5.4) may not be optimal. And because of it, a logarithmic term is needed in the assumption (5.1) and so is the cut-off t_0.

In any case, if

$$\lambda_\varepsilon^2(\boldsymbol{e}_i) \gg \frac{1}{\varepsilon} \tag{5.5}$$

then (5.1) is automatically satisfied due to (2.8). The diffusion time and the martingale time are probably of the same order of magnitude. If this is the case, (5.1) at worst overestimates $t_{diff}(\boldsymbol{e}_i)$ by a logarithm of $\frac{1}{\varepsilon}$.

6 Examples of Convection Enhanced Diffusion

We note that the scaling in Theorem 5.1 is not the usual one of the martingale central limit theorem (cf. [6]) because the flow sensitive, effective diffusivity σ_ε generally satisfies the power law

$$\sigma_\varepsilon(\boldsymbol{e}_i) \sim \varepsilon^\alpha, \quad \text{as } \varepsilon \to 0 \tag{6.1}$$

with

$$-1 \leq \alpha \leq 1, \tag{6.2}$$

or possibly with a logarithm of $\frac{1}{\varepsilon}$ (cf. (6.10) and [9]).

When $\alpha < 1$, (6.1) is called the *convection enhanced diffusion* and is studied in detail in [9] and [10] using variational methods. We begin with two examples of laminar Rayleigh-Benard convection: one is the square cellular flow in two dimension; the other, the hexagonal flow in three dimension, both of which are completely integrable (hence non-ergodic), so by Theorem 3.1,

$$t_{diss} \sim \frac{1}{\varepsilon}. \tag{6.3}$$

The cellular flow is given by the periodic stream function

$$\psi(x,y) = \sin x \sin y \tag{6.4}$$

such that $\boldsymbol{u}(x,y) = \nabla^{\perp}\psi(x,y) = (-\psi_y, \psi_x)$ and the trajectories of $\boldsymbol{u}$ are exactly the level sets of ψ. For (6.4), it was calculated in [3], [18], [19] and [9] that

$$\sigma_\varepsilon(\boldsymbol{e}_i) \sim \sqrt{\varepsilon},\ \forall i \tag{6.5}$$

and

$$\langle(\chi_i^\varepsilon)^2\rangle \sim 1,\ \forall i \tag{6.6}$$

so

$$t_{diff}(\boldsymbol{e}_i) \sim t_m(\boldsymbol{e}_i) \sim \frac{1}{\varepsilon} \tag{6.7}$$

by (6.4), (4.8) and Theorem 5.1.

The hexagonal flow is given by

$$u_3(\boldsymbol{x}) = \sin z \sum_i \cos(\boldsymbol{q}_i \cdot \boldsymbol{x}) \tag{6.8}$$

$$\boldsymbol{u}_\perp(\boldsymbol{x}) = \frac{1}{|\boldsymbol{q}|^2}\nabla^\perp \frac{\partial u_3}{\partial z} \tag{6.9}$$

where $\boldsymbol{u}_\perp$ denotes the velocity in the $x-y$ plane and the constant vectors $\boldsymbol{q}_i$ form an equilateral triangle in the $x-y$ plane. It was calculated in [5] that

$$\sigma_\varepsilon(\boldsymbol{e}_i) \sim \varepsilon \log\frac{1}{\varepsilon},\ \forall i \tag{6.10}$$

and it can be shown that

$$\langle(\chi_i^\varepsilon)^2\rangle \sim 1,\ \forall i \tag{6.11}$$

hence (6.7).

As a contrast, let us consider the convection diffusion by the perturbed baker's map (3.7). It was calculated numerically in [4] for the randomly perturbed single baker's map (without the alternation) that

$$\langle(\chi_1^\varepsilon)^2\rangle \sim 1, \tag{6.12}$$

and

$$\sigma_\varepsilon(\boldsymbol{e}_1) \sim \sqrt{\varepsilon} \tag{6.13}$$

where $\boldsymbol{e}_1$ is the direction of compressing and folding. With the combination of the baker's map and its rotated version, we expect that

$$\langle (\chi_i^\varepsilon)^2 \rangle \sim 1, \ \forall i \tag{6.14}$$

and

$$\sigma_\varepsilon(\boldsymbol{e}_i) \sim \sqrt{\varepsilon}, \ \forall i. \tag{6.15}$$

Thus the martingale time is much larger than the dissipation time

$$t_m \sim \frac{1}{\sqrt{\varepsilon}} \gg t_{diss} \tag{6.16}$$

by Theorem 3.2.

7 Transient Dispersion

The first crossover time before the final state of diffusion is reached is the martingale time $t_m(\boldsymbol{e}_i)$ which may be substantially larger than the dissiptive time t_{diss} as we have seen in the example of alternating baker's map.

When $t_m(\boldsymbol{e}_i)$ is indeed equal to t_{diss}, the crossover may happen before and up to t_{diss} and the transient dispersion is generally initial point sensitive. This is the case, for instance, when the fluid flow $\boldsymbol{u}$ is not ergodic and consequently $t_{diss} \sim \frac{1}{\varepsilon}$. If, furthermore, the flow is laminar convective cells such as (6.4) and (6.8)-(6.9), then it was proposed that the ε independent anomalous diffusion takes place

$$E_{\boldsymbol{x}}\{\boldsymbol{x}_\varepsilon^2(t)\} \sim t^{1/2} \tag{7.1}$$

for

$$1 \ll t \ll \frac{1}{\varepsilon} \tag{7.2}$$

and initial point $\boldsymbol{x}$ sufficiently close to the boundary of the convective cells (cf. [12],[22] and [5]). Here 1 in (7.2) represents the order of the turnover time.

But if $t_m(\boldsymbol{e}_i) \gg t_{diss}$, then the first crossover time is instead

$$\sigma^{-1}(\boldsymbol{e}_i)\langle (\chi_i^\varepsilon)^2 \rangle \tag{7.3}$$

and an *initial point* $\boldsymbol{x}$ *independent* transient dispersion may take place between t_{diss} and $t_m(\boldsymbol{e}_i)$. In this case, there may be multiple transient regimes: some depend on initial points; others do not.

We plan to address this issue in a more systematic way in a separate paper.

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