Phase Retrieval in Coherent Diffractive Imaging

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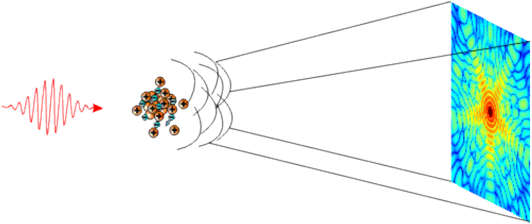
Coherent diffractive imaging



- Linear propagation + intensity measurement : $b(j)^2 = |a_j^* x_0|^2$
- Phase retrieval: Given $b = (b(j)) \in \mathbb{R}_+^N$ and $A^* = [a_j^*] \in \mathbb{C}^{N\times M}$, determine x_0.
- Geometry: Intersection of N-dim real torus of radii $\{b(j)\}$ and complex linear subspace $A^*\mathbb{C}^M$ $(N > M)$.

Uniqueness for generic frames (Balan-Casazza-Edidin 06)

- Full-rank $A \in \mathbb{C}^{M \times N}, N > M$: $\{\mathrm{col}(A)\} =$ frame
- Frames form a metric space.
- Necessary condition for injectivity (left inverse exists): $N \geq 2M$.
- Sufficient condition: If $N \geq 4M - 2$ then generic (i.e. an open dense set) frames are injective.

Fourier frame is exceptional!

Diffraction = Fourier transform

Let $x_0(\mathbf{n})$ be a discrete object function with $\mathbf{n} = (n_1, n_2, \cdots, n_d) \in \mathbb{Z}^d$. We assume $d \geq 2$. $\mathcal{M} = \{0 \leq m_1 \leq M_1, 0 \leq m_2 \leq M_2, \cdots, 0 \leq m_d \leq M_d\}$

Diffraction pattern

$$\left| \sum_{\mathbf{m} \in \mathcal{M}} x_0(\mathbf{m}) e^{-\mathrm{i}2\pi \mathbf{m}\cdot\boldsymbol{\omega}} \right|^2 = \sum_{\mathbf{n}=-\mathbf{M}}^{\mathbf{M}} \boxed{\sum_{\mathbf{m}\in\mathcal{M}} x_0(\mathbf{m}+\mathbf{n})\overline{x_0(\mathbf{m})}} e^{-i2\pi\mathbf{n}\cdot\mathbf{w}}$$

$$\mathbf{w} = (w_1, \cdots, w_d) \in [0,1]^d, \quad \mathbf{M} = (M_1, \cdots, M_d)$$

Autocorrelation

$$R(\mathbf{n}) = \sum_{\mathbf{m}\in\mathcal{M}} x_0(\mathbf{m}+\mathbf{n})\overline{x_0(\mathbf{m})}.$$

$$\widetilde{\mathcal{M}} = \{(m_1, \cdots, m_d) \in \mathbb{Z}^d : -M_1 \leq m_1 \leq M_1, \cdots, -M_d \leq m_d \leq M_d\}$$

Oversampling ratio = 2^d

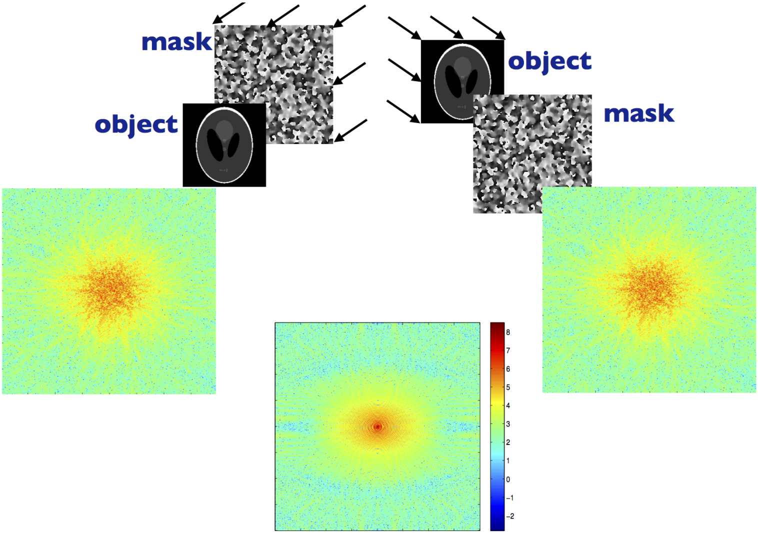
Ambiguities (Bruck-Sodin 1979, Hayes 1982)

- Oversampling: $N \geq 4M - 4\sqrt{M} + 1$.
- Global ambiguities for generic objects $x_0 \in \mathbb{R}^M$

$$
\begin{array}{rl}
\text{(harmless) global phase} & x_0(\cdot) \longrightarrow e^{i\theta} x_0(\cdot) \\
\text{translation} & x_0(\cdot) \longrightarrow x_0(\cdot + \mathbf{n}), \forall \mathbf{n} \\
\text{conjugate inversion} & x_0(\cdot) \longrightarrow \overline{x_0}(-\cdot)
\end{array}
$$

- Generic objects = random vectors according to continuous prior distribution $\Longrightarrow$ nongeneric objects $\in$ a measure zero set.
- Problems:
  - $\rightarrow$ You can not determine if a given object is generic or not since *the "world ensemble" may not be absolutely continuous w.r.t. your prior distribution.*
  - $\rightarrow$ Global ambiguities may lead to poor reconstruction: bad algorithm or measurement scheme?

Coded diffraction pattern



Measurement matrix

- Mask function: $\mu(\mathbf{n})$.
- Masked object: $\tilde{x}_0(\mathbf{n}) = \mu(\mathbf{n})x_0(\mathbf{n})$
- Randomly phased mask: $\mu(\mathbf{n}) = \exp(i\phi(\mathbf{n}))$ where $\phi(\mathbf{n})$ are random variables.
- Measurement matrix: Φ = discrete Fourier transform

$$\text{(1 mask)} \qquad A^* = \Phi \, \mathrm{diag}(\mu)$$

$$\text{(2 masks)} \quad A^* = \begin{bmatrix} \Phi \, \mathrm{diag}(\mu_1) \\ \Phi \, \mathrm{diag}(\mu_2) \end{bmatrix}$$

Uniqueness with coded diffraction patterns

Theorem (F. 2012)

Suppose $x_0 \in \mathbb{C}^M$ *is rank* ≥ 2 *and* $\arg(x_0)$ *belongs in a proper sub-interval* $[a, b] \subset [0, 2\pi)$. *Then the object is determined by one coded diffraction pattern up to a constant phase factor with probability at least*

$$1 - M\left|\frac{b-a}{2\pi}\right|^{s/2}$$

where s *is the number of nonzero pixels.*

Corollary

Suppose $x_0 \in \mathbb{R}^M$ *and is rank* ≥ 2. *Then with probability one the object is determined by one coded diffraction pattern up to* $\pm$ *sign.*

Uniqueness (continued)

Theorem (F. 2012)

Suppose $x_0 \in \mathbb{C}^M$ and is rank ≥ 2. Then the object is determined by two coded diffraction patterns up to a constant phase factor with probability one.

vs Candes-Li-Soltanolkotabi 2015:

- $\rightarrow$ PhaseLift: convex programming.
- $\rightarrow$ Large number of regularly sampled patterns.
- $\rightarrow$ Candes-Strohmer-Voroninski 2013: Gaussian random measurement.
- $\rightarrow$ Lifting $\Longrightarrow$ huge increase of dimensionality & unpractical computation

Nonconvex constraint

- Non-linear system:

$$\begin{array}{rll} & b = |A^* x|, & x \in \mathcal{X} \\ \text{(1 mask)} & \mathcal{X} = \mathbb{R}^M, & A^* = \Phi \operatorname{diag}(\mu) \\ \text{(2 masks)} & \mathcal{X} = \mathbb{C}^M, & A^* = \begin{bmatrix} \Phi \operatorname{diag}(\mu_1) \\ \Phi \operatorname{diag}(\mu_2) \end{bmatrix} \end{array}$$

- Non-convex feasibility problem:

$$\begin{array}{rrcl} \text{Find} & \hat{y} & \in & A^* \mathcal{X} \cap \mathcal{Y} \\ & \mathcal{Y} & := & \{y \in \mathbb{C}^N : |y| = b\} \\ & \hat{x} & = & (A^*)^\dagger \hat{y} \end{array}$$

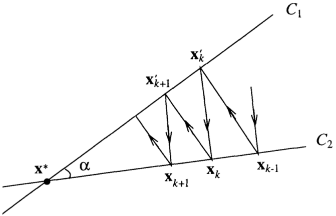
- Geometry: Intersection of N-dim torus of radii $\{b_j\}$ and linear subspace $A^* \mathcal{X}$

Alternating projections: feasibility problem

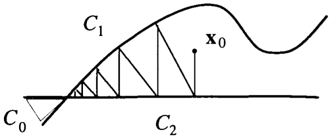
Two constraints: Fourier magnitude data (N-dim torus of uneven radii) $\cap$ oversampled Fourier matrix ($2M$-dim subspace)

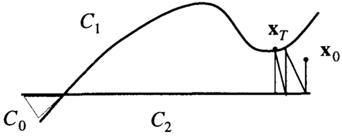
von Neuman 1933

Cheney-Goldstein 1959
Bregman 1965



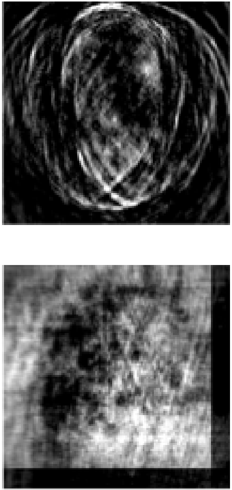
Non convex: local convergence?



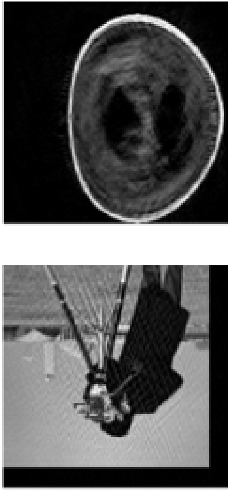


Experiments: plain diffraction pattern

Original images



AP



HIO (Fienup 1982)

Reconstruction with coded diffraction patterns

- Convex method converges surely but (extremely) slowly.
- Nonconvex methods converge fast (with good measurement) without guarantee.
  1. Gradient descent algorithms: e.g. Wirtinger flow (Candes-Li-Soltanolkotabi 2015).
  2. Iterative projection/fixed point algorithms.
- Initial guess is crucial for non-convex methods: How to put the initial guess in the basin of attraction of the global minimizer?

Null vector method (Chen-F.-Liu 2015)

$$A^* = [a_j^*]$$

$$a_j^* x_0 = 0 \longrightarrow b_j = |a_j^* x_0| = a_j^* x_0.$$

If there are sufficiently many data that are small, then the **unique** null vector of the **row** sub-matrix may be a good bet.

$$\boxed{x_{\text{null}} := \arg\min \left\{ \sum_{i \in I} \|a_i^* x\|^2 : x \in \mathcal{X}, \|x\| = \|x_0\| \right\}}$$

$$x_{\text{dual}} := \arg\max \left\{ \|A_{I_c}^* x\|^2 : x \in \mathcal{X}, \|x\| = \|x_0\| \right\}$$

Isometry $\longrightarrow$ $\|A_I^* x\|^2 + \|A_{I_c}^* x\|^2 = \|x\|^2$

$\longrightarrow$ $x_{\text{null}} = x_{\text{dual}}$ power method

Null vector algorithm

Let $\mathbf{1}_c$ be the characteristic function of the complementary index I_c with $|I_c| = \gamma N$.

Algorithm 1: The null vector method

```
Random initialization: x_1 = x_rand
Loop:
for k = 1 : k_max − 1 do
    x'_k ← A(1_c ⊙ A* x_k);
    x_{k+1} ← [x'_k]_X / ‖ [x'_k]_X ‖
end
Output: x_null = x_{k_max}.
```

Random initialization: $x_1 = x_{\text{rand}}$
Loop:
for $k = 1 : k_{\max} - 1$ **do**
$\quad x'_k \leftarrow A(\mathbf{1}_c \odot A^* x_k);$
$\quad x_{k+1} \leftarrow \left[x'_k\right]_{\mathcal{X}} / \| \left[x'_k\right]_{\mathcal{X}} \|$
end
Output: $x_{\text{null}} = x_{k_{\max}}$.

Algorithm 2: The spectral vector method

Random initialization: $x_1 = x_{\text{rand}}$
Loop:
for $k = 1 : k_{\max} - 1$ **do**
$\quad x'_k \leftarrow A(|b|^2 \odot A^* x_k);$
$\quad x_{k+1} \leftarrow \left[x'_k\right]_{\mathcal{X}} / \| \left[x'_k\right]_{\mathcal{X}} \|;$
end
Output: $x_{\text{spec}} = x_{k_{\max}}$.

Netrapalli-Jain-Sanghavi 2015

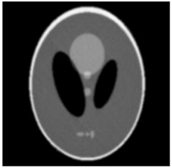
Truncated spectral vector

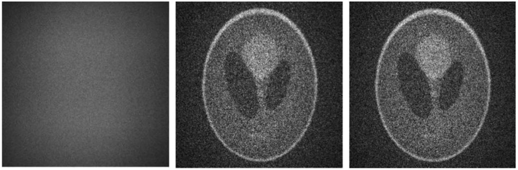
$$x_{\text{t-spec}} = \arg\max_{\|x\|=1} \| A \left(\mathbf{1}_\tau \odot |b|^2 \odot A^* x \right) \|$$

$$\{ i : |A^* x(i)| \le \tau \|b\| \}$$

Candes-Chen 2015

Experiments: Fourier case with two masks



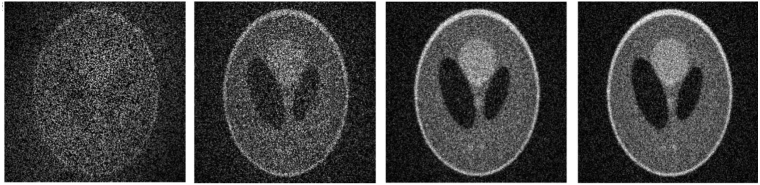


(a) $|x_{\text{t-spec}}|$ ($\tau^2 = 5$) (b) $|x_{\text{null}}|$ ($\gamma = 0.5$) (c) $|x_{\text{null}}|$ ($\gamma = 0.6$)

(a) $|\text{Re}(x_{\text{t-spec}})|$ ($\tau^2 = 5$) (b) $|\text{Re}(x_{\text{null}})|$ ($\gamma = 0.5$) (c) $|\text{Re}(x_{\text{null}})|$ ($\gamma = 0.63$)

(d) $|\text{Im}(x_{\text{t-spec}})|$ ($\tau^2 = 5$) (e) $|\text{Im}(x_{\text{null}})|$ ($\gamma = 0.5$) (f) $|\text{Im}(x_{\text{null}})|$ ($\gamma = 0.63$)

Experiments: Fourier case with one mask



(a) x_{spec} (b) $x_{\text{t-spec}}$ $(\tau^2 = 4.6)$ (c) x_{null} $(\gamma = 0.5)$ (d) x_{null} $(\gamma = 0.74)$

(a) x_{spec} (b) $x_{\text{t-spec}}$ $(\tau^2 = 4.1)$ (c) x_{null} $(\gamma = 0.5)$ (d) x_{null} $(\gamma = 0.7)$

Error metrics often poorly reflect the quality of initialization

Performance guarantee: Gaussian case

Theorem (Chen-F.-Liu 2016)

Let A be drawn from the $M \times N$ standard complex Gaussian ensemble. Let

$$\sigma := |I|/N < 1, \quad \nu = M/|I| < 1.$$

Then for any $x_0 \in \mathbb{C}^n$ the following error bound

$$\|x_0 x_0^* - x_{\rm null} x_{\rm null}^*\|^2 \quad \leq \quad c_0 \sigma \|x_0\|^4$$

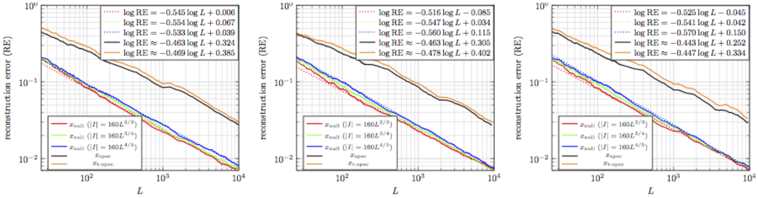
holds with probability at least

$$1 - 5 \exp\left(-c_1 |I|^2/N\right) - 4 \exp(-c_2 M).$$

- Nonasymptotic estimate
- Asymptotic regime: $|I|/N \ll 1, \quad |I|^2/N \gg 1$

$$\Longrightarrow |I| = N^{\alpha}, \text{ error} \sim N^{(\alpha-1)/2}, \ \alpha \in (1/2, 1)$$

Experiments: Gaussian case



(a) White noise (b) Low-pass noise (c) Randomly phased Phantom

- Empirical scaling law: Relative error $\sim L^{-\beta}$ where $L = N/M$ and $\beta \approx 1/2$.
- Theoretical bound: RE $\sim \sqrt{|I|/N} = L^{(\alpha-1)/2}$ where $1/2 < \alpha < 1$.

Alternating projectons

- Non-convex feasibility problem:

$$\begin{aligned} \text{Find} \quad \hat{y} &\in A^*\mathcal{X} \cap \mathcal{Y} \\ \mathcal{Y} &:= \{y \in \mathbb{C}^N : |y| = b\} \\ \hat{x} &= (A^*)^\dagger \hat{y} \end{aligned}$$

- Let P_1 and P_2 be projections onto $A^*\mathcal{X}$ and $\mathcal{Y}$, respectively.

$$\text{(AP)} \qquad P_1 P_2 y = \left[(A^*)^\dagger \left(b \odot \frac{y}{|y|} \right) \right]_{\mathcal{X}}$$

with initial guess $y^{(1)} = A^* x^{(1)}, x^{(1)} \in \mathcal{X}$.

- Nonconvex optimization: $U = \{u \in \mathbb{C}^N : |u(j)| = 1\}$ N-torus.

$$\begin{aligned} f(x,u) &= \frac{1}{2}\|A^*x - u \odot b\|^2 \\ u^{(k)} &= \arg\min_{u \in U} f(x^{(k)}, u) && \text{(non-convex)} \\ x^{(k+1)} &= \arg\min_{x \in \mathcal{X}} f(x, u^{(k)}) && \text{(non-smooth)} \end{aligned}$$

Parallel AP (PAP)

$$x^{(k+1)} = \mathcal{F}(x^{(k)})$$

$$\mathcal{F}(x) = \left[(A^*)^\dagger \left(b \odot \frac{A^* x}{|A^* x|}\right)\right]_{\mathcal{X}} \qquad (A^*)^\dagger = (AA^*)^{-1}A$$

$$\text{(2-mask case)} \quad A^* = c \begin{bmatrix} \Phi & \mathrm{diag}\{\mu_1\} \\ \Phi & \mathrm{diag}\{\mu_2\} \end{bmatrix}$$

Fact *every limit point of* $\{x^{(k)}\}$ *is a fixed point of the map* $\mathcal{F}$

Proposition A fixed point preserves the **total signal strength**, iff it is the true solution up to a global phase.

$$\|A^* x_*\| = \|b\| \qquad \text{iff} \qquad x_* = \alpha x_0 \text{ with } |\alpha| = 1$$

Otherwise $\|A^* x_*\| < \|b\|$.

Serial AP (SAP)

$$\text{Find} \quad \hat{y} \in \cap_{l=1}^{2} \left(A_l^* \mathcal{X} \cap \mathcal{Y}_l \right), \quad \mathcal{Y}_l := \{ y_l \in \mathbb{C}^{N/2} : |y_l| = b_l \}$$

SAP $\quad \mathcal{F}_2 \mathcal{F}_1(x)$

$$\mathcal{F}_l(x) = A_l \left(b_l \odot \frac{A_l^* x}{|A_l^* x|} \right), \quad l = 1, 2,$$

PAP $\quad \mathcal{F}(x) = A \left(b \odot \frac{A^* x}{|A^* x|} \right) = \boxed{\frac{1}{2} \left(\mathcal{F}_1(x) + \mathcal{F}_2(x) \right)}$

Gradient map

$$B := A\ \mathrm{diag}\left\{\frac{A^*x_0}{|A^*x_0|}\right\} \qquad \mathcal{B} := \begin{bmatrix}\Re[B]\\ \Im[B]\end{bmatrix} \in \mathbb{R}^{2n,N}$$

$$\boxed{G(-id\mathcal{F}\xi) \;=\; \mathcal{B}\mathcal{B}^\top G(-i\xi), \qquad \forall \xi \in \mathbb{C}^n}$$

Isomorphism $\quad G(-iv) := \begin{bmatrix}\Im(v)\\ -\Re(v)\end{bmatrix}, \qquad \forall v \in \mathbb{C}^n$

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{2n} \geq \lambda_{2n+1} = \cdots = \lambda_N = 0$ be the singular values of $\mathcal{B}$ with the corresponding right singular vectors $\{\eta_k \in \mathbb{R}^N\}_{k=1}^N$ and left singular vectors $\{\xi_k \in \mathbb{R}^{2n}\}_{k=1}^{2n}$.

Proposition

We have $\xi_1 = G(x_0)$, $\xi_{2n} = G(-ix_0)$, $\lambda_1 = 1, \lambda_{2n} = 0$ *and* $\eta_1 = |A^*x_0|$.

$$\boxed{u^{(k)} := -i(\alpha^{(k)}x^{(k)} - x_0) \longrightarrow \xi_1 \perp G(u^{(k)}), \qquad \forall k}$$

Spectral gap

$$\lambda_2 = \max\{\|\Im[B^* u]\| : u \in \mathbb{C}^n, iu \perp x_0, \|u\| = 1\}$$
$$= \max\{\|\mathcal{B}^\top u\| : u \in \mathbb{R}^{2n}, u \perp \xi_1, \|u\| = 1\}.$$

Proposition

Suppose $x_0 \in \mathbb{C}^n$ is rank-2. Then $\lambda_2 < 1$ with probability one.

Uniqueness theorem for magnitude retrieval

If

$$\measuredangle A^* \hat{x} = \pm \measuredangle A^* x_0$$

where the $\pm$ sign may be pixel-dependent, then almost surely $\hat{x} = c x_0$ for some constant $c \in \mathbb{R}$.

One random mask suffices !

Local geometric convergence

Theorem (Chen-F.-Liu 2015)

For any given $0 < \epsilon < 1 - \lambda_2^2$*, if* $x^{(1)}$ *is sufficiently close to* x_0*, then with probability one PAP converges to* x_0 *geometrically after global phase adjustment*

$$\|\alpha^{(k+1)}x^{(k+1)} - x_0\| \leq (\lambda_2^2 + \epsilon)\|\alpha^{(k)}x^{(k)} - x_0\|$$

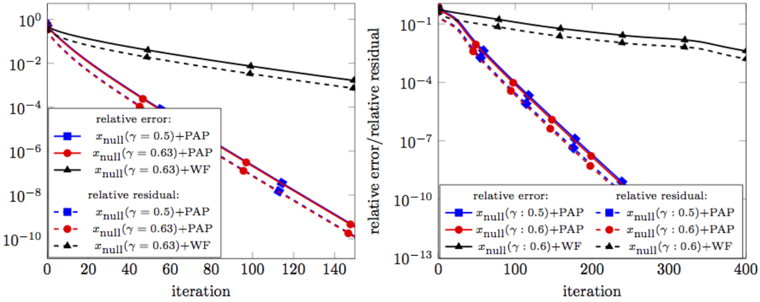
where $\alpha^{(k)} = x^{(k)*}x_0/|x^{(k)*}x_0|$.

Theorem (Chen-F.-Liu 2015)

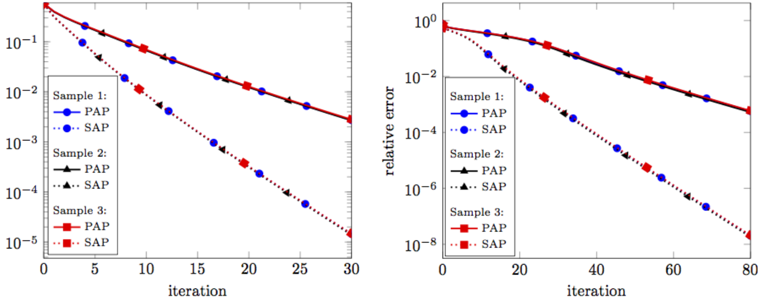
For any given $0 < \epsilon < 1 - (\lambda_2^{(2)}\lambda_2^{(1)})^2$*, if* $x^{(1)}$ *is sufficiently close to* x_0 *then with probability one SAP converges to* x_0 *geometrically after global phase adjustment,*

$$\|\alpha^{(k+1)}x^{(k+1)} - x_0\| \leq ((\lambda_2^{(2)}\lambda_2^{(1)})^2 + \epsilon)\|\alpha^{(k)}x^{(k)} - x_0\|.$$

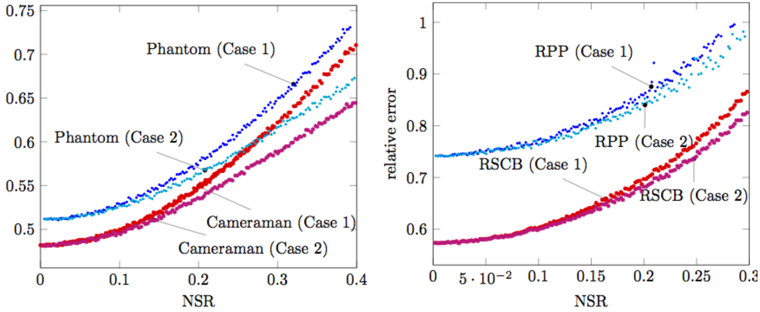
Experiments: with null initialization



(a) RSCB (b) RPP



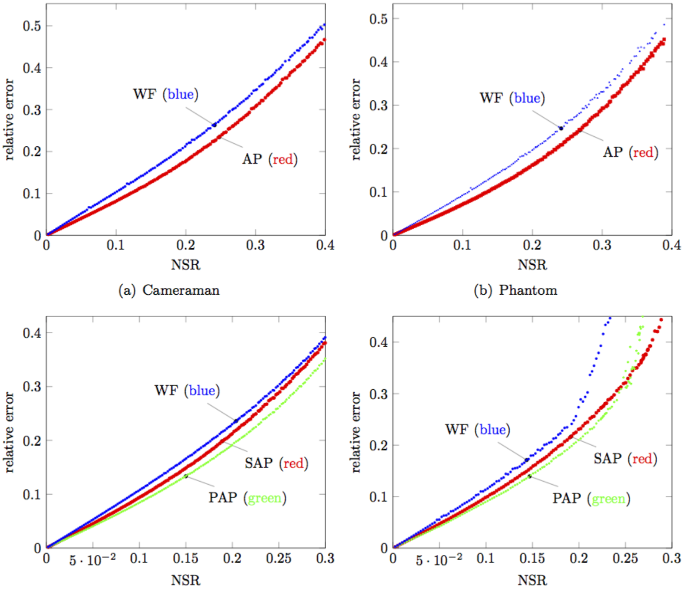
Experiments: null vector with noisy data



(a) One pattern (b) Two patterns

- Case 1: $\|x_{\text{null}}\| = \|b\|$.
- Case 2: $\|x_{\text{null}}\| = \|x_0\|$.

Experiments: noise stability



(a) Cameraman (b) Phantom

(c) RSCB (d) RPP

Douglas-Rachford splitting

- **Feasibility:** $\mathcal{Y} \cap \mathcal{Z} \Longrightarrow \min_{y \in \mathcal{Y}, z \in \mathcal{Z}} \frac{1}{2}\|y - z\|^2, \quad y = z.$
- **ADMM** (alternating direction method of multiplier)

$$
\begin{aligned}
& \max_{\lambda} \min_{y \in \mathcal{Y}, z \in \mathcal{Z}} \mathcal{L} := \frac{1}{2}\|y - z\|^2 + \langle \lambda, (y - z) \rangle \\
= \quad & \max_{\lambda} \min_{y \in \mathcal{Y}, z \in \mathcal{Z}} \mathcal{L} := \frac{1}{2}\|y - z + \lambda\|^2 - \frac{1}{2}\|\lambda\|^2
\end{aligned}
$$

$$
\begin{cases}
y^{t+1} = \arg\min_{y \in \mathcal{Y}} \frac{1}{2}\|y - z^t + \lambda^t\|^2 & = P_{\mathcal{Y}}(z^t - \lambda^t) \\
z^{t+1} = \arg\min_{z \in \mathcal{Z}} \frac{1}{2}\|y^{t+1} - z + \lambda^t\|^2 & = P_{\mathcal{Z}}(y^{t+1} + \lambda^t) \\
\lambda^{t+1} = \lambda^t + \nabla_\lambda \mathcal{L}(y^{t+1}, z^{t+1}) & = \lambda^t + y^{t+1} - z^{t+1}
\end{cases}
$$

- **DR:** $x^t := y^{t+1} + \lambda^t \Longrightarrow$

$$
x^{t+1} = x^t + P_{\mathcal{Y}}(2P_{\mathcal{Z}} - I)x^t - P_{\mathcal{Z}}x^t
$$

Fourier domain Douglas-Rachford

$$\mathcal{Y} = \{y \in \mathbb{C}^N : |y| = b\}, \quad \mathcal{Z} = A^*\mathcal{X}$$

$$\implies \quad P_{\mathcal{Y}}(y) = b \odot \frac{y}{|y|}, \quad P_{\mathcal{Z}}(y) = A^*Ay$$

$$\boxed{S_{\mathrm{f}}(y) \;=\; y + A^*\left[A\left(2b \odot \frac{y}{|y|} - y\right)\right]_{\mathcal{X}} - b \odot \frac{y}{|y|}}$$

Gradient $\quad J_{\mathrm{f}}v = (I - B^*B)\Re(v) + iB^*B\Im(v)$

J_{f} is a *real*, but *not complex*, linear map

$$\boxed{S(x) = x + \left[\tilde{A}\left(2b \odot \frac{\tilde{A}^*x}{|\tilde{A}^*x|}\right) - x\right]_{\mathcal{X}} - \tilde{A}\left(b \odot \frac{\tilde{A}^*x}{|\tilde{A}^*x|}\right)}$$

Fixed point with two masks

$$S_{\mathrm{f}}(y_\infty) = y_\infty, \quad x_\infty = A y_\infty.$$

$$\boxed{\begin{array}{l} y_\infty = e^{i\theta}(|y_0| + v) \odot \dfrac{y_0}{|y_0|} \\ |y_0| + v \textit{ has all nonnegative components} \\ \quad v \in \mathrm{null}_{\mathbb{R}}(\mathcal{B}) \subset \mathbb{R}^N \end{array}}$$

Theorem (Chen-F. 2016)

The projected fixed point is unique, i.e. $x_\infty = e^{i\theta} x_0$ *almost surely.*

FDR locally converges geometrically

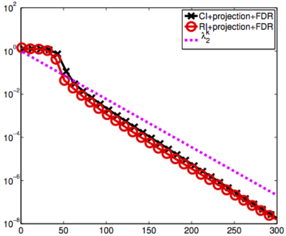
Theorem (Chen-F. 2016)

For $0 < \epsilon < 1 - \lambda_2$, if $\alpha^{(1)}x^{(1)}$ is sufficient close to x_0, then FDR converges geometrically to the solution

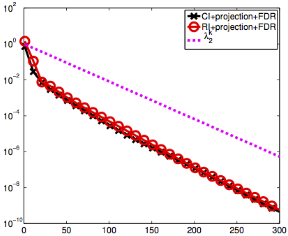
$$\|\alpha^{(k)}x^{(k)} - x_0\| \leq (\lambda_2 + \epsilon)^{k-1}\|\alpha^{(1)}x^{(1)} - x_0\|.$$

- Explicit measurement schemes.
- Explicit characterization of $\lambda_2 < 1$.
- No hard-to-verify assumptions.
- Convex setting (He-Yuan 2012, 2015): k-th error $= \mathcal{O}(1/k)$.

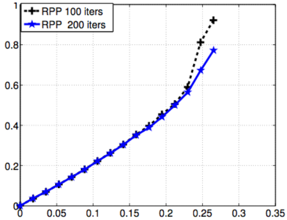
Experiments: Two patterns



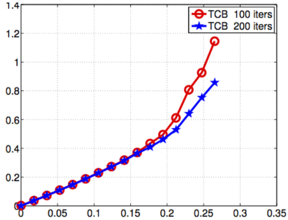
(a) RPP



(b) TCB



(a) RPP

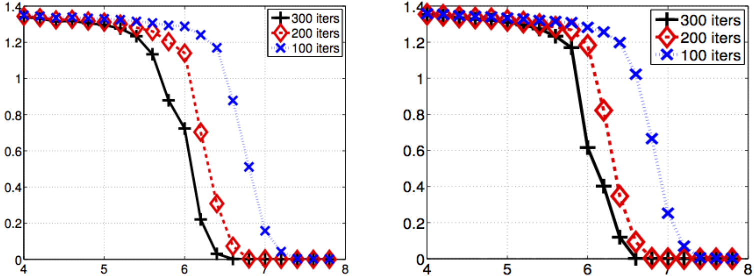


(b) TCB

Fourier domain vs. object domain DR

$$\text{(FDR)} \quad S_f(x) \;=\; y + A^*A\left(2b\odot\frac{y}{|y|} - y\right) - b\odot\frac{y}{|y|}$$

$$\text{(ODR)} \quad S(x) \;=\; x + \tilde{A}\left(2b\odot\frac{\tilde{A}^*x}{|\tilde{A}^*x|}\right) - x - \tilde{A}\left(b\odot\frac{\tilde{A}^*x}{|\tilde{A}^*x|}\right)$$



(a) RPP (b) TCB

$\tilde{A}$: various extensions of A

Conclusion

- Two globally convergent schemes in practice:
  1. AP+null initialization
  2. FDR
- Open problem: proof of global convergence.

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