More on the consequences of uniqueness for autonomous systems

Recall the metaphor of the corn field.

Given the autonomous system $d\mathbf{Y}/dt = \mathbf{F}(\mathbf{Y})$. Let $\mathbf{Y}_0$ be an initial condition such that $\mathbf{Y}_1(t)$ is a solution that satisfies $\mathbf{Y}(t_1) = \mathbf{Y}_0$ and $\mathbf{Y}_2(t)$ is another solution that satisfies $\mathbf{Y}(t_2) = \mathbf{Y}_0$. Then

$$\mathbf{Y}_2(t) = \mathbf{Y}_1(t - (t_2 - t_1)).$$

**Example.** Consider the second-order equation $\frac{d^2y}{dt^2} + y = 0$ and its equivalent system

$$\frac{dy}{dt} = v,$$
$$\frac{dv}{dt} = -y.$$

Note that

$$\mathbf{Y}_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

are both solutions to the system. How are $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ related?

There is an animation on the web site that illustrates this phenomenon.
Here is an informal restatement of this consequence of uniqueness:

For an autonomous system, if two solution curves in the phase plane touch, then they are identical.

Linear systems

Linear systems and second-order linear equations are the most important systems we study in this course.

What is a linear system with two dependent variables?

What is a second-order, homogeneous, linear equation?
Linear systems written in vector notation suggest the use of matrix multiplication:

Recall two examples that we have already discussed.

**Example 1.** We have already calculated the general solution to the partially decoupled system

\[
\begin{align*}
\frac{dx}{dt} &= 2y - x \\
\frac{dy}{dt} &= y.
\end{align*}
\]

It is

\[
\begin{align*}
x(t) &= y_0 e^t + (x_0 - y_0) e^{-t} \\
y(t) &= y_0 e^t.
\end{align*}
\]
Example 2. For the damped harmonic oscillator

\[ \frac{d^2 y}{dt^2} + 3 \frac{dy}{dt} + 2y = 0 \]

and its equivalent system

\[ \frac{dy}{dt} = v \]
\[ \frac{dv}{dt} = -2y - 3v, \]

we used a guessing technique to find two (scalar) solutions \( y_1(t) = e^{-t} \) and \( y_2(t) = e^{-2t} \). In vector form, these solutions are written as

\[ Y_1(t) = \begin{pmatrix} e^{-2t} \\ -2e^{-2t} \end{pmatrix} = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad Y_2(t) = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \]

Given a linear system \( \frac{dY}{dt} = AY \), how do we calculate the vector in the vector field at any given point \( Y_0 \)?
How do we calculate the equilibrium points of \( \frac{dY}{dt} = AY \)?

Example. Let \( A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \).

Example. Let \( A_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \).

Theorem. The origin is always an equilibrium point of a linear system. It is the only equilibrium point if and only if \( \det A \neq 0 \).