The geometry of complex eigenvalues

**Example 1.** \( \frac{d\mathbf{Y}}{dt} = A\mathbf{Y} \) where

\[
A = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
\]

The characteristic polynomial of \( A \) is \( \lambda^2 + 1 \), so the eigenvalues are \( \lambda = \pm i \). One eigenvector associated to the eigenvalue \( \lambda = i \) is

\[
\mathbf{Y}_0 = \begin{pmatrix} i \\ 1 \end{pmatrix}.
\]

We obtain a general solution of the form

\[
\mathbf{Y}(t) = k_1 \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + k_2 \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.
\]

A solution curve and two pairs of \( x(t) \)- and \( y(t) \)-graphs are shown below.
Example 2. \[ \frac{dY}{dt} = B Y \] where \( B = \begin{pmatrix} 2 & -2 \\ 4 & -2 \end{pmatrix} \).

The characteristic polynomial of \( B \) is \( \lambda^2 + 4 \), so the eigenvalues are \( \lambda = \pm 2i \). One eigenvector associated to the eigenvalue \( \lambda = 2i \) is

\[ Y_0 = \begin{pmatrix} 1 + i \\ 2 \end{pmatrix} . \]

We get ellipses centered at the origin in the phase plane.
Example 3. \( \frac{dY}{dt} = CY \) where \( C = \begin{pmatrix} 1.9 & -2 \\ 4 & -2.1 \end{pmatrix} \).

The characteristic polynomial of \( C \) is \( \lambda^2 + 0.2\lambda + 4.01 \), so the eigenvalues are \( \lambda = -0.1 \pm 2i \). One eigenvector associated to the eigenvalue \( \lambda = -0.1 + 2i \) is

\[
Y_0 = \begin{pmatrix} 1 + i \\ 2 \end{pmatrix}.
\]
Summary: Linear systems with complex eigenvalues $\lambda = a \pm bi$

Here are the possible phase portraits:

- Spiral sink ($a < 0$)
- Center ($a = 0$)
- Spiral source ($a > 0$)

What information can you get just from the complex eigenvalue alone?

Recall Example 2. The eigenvalues are $\lambda = \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:

In Example 3, the eigenvalues are $\lambda = -0.1 \pm 2i$. Here are the $x(t)$- and $y(t)$-graphs of a typical solution:
Frequency versus period: The solutions in Example 3 are not periodic in the strict sense. There is no time $T$ such that

$$x(t + T) = x(t) \quad \text{and} \quad y(t + T) = y(t)$$

for all $t$. However, there is a period associated to these solutions. In the text, we call this the **natural period** of the solutions.

Perhaps it is best to think about these solutions as oscillating solutions that are decaying over time and to measure the oscillations in terms of their **frequency**.

**Definition.** The **frequency** $F$ of an oscillating function $g(t)$ is the number of cycles that $g(t)$ makes in one unit of time.

Suppose that $g(t)$ is oscillating periodically with "period" $T$. What is its frequency $F$?

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**Example.** Consider the standard sinusoidal functions $g(t) = \cos \beta t$ and $g(t) = \sin \beta t$.

Suppose we measure frequency in radians rather than in cycles. This measure of frequency is often called **angular frequency**. Let’s denote the angular frequency by $f$. Then

$$f = 2\pi F.$$
Repeated eigenvalues

Sometimes the characteristic polynomial has the same real root twice. When this happens, we say that the eigenvalues are "repeated."

**Example.** \( \frac{dY}{dt} = AY \) where \( A = \begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} \).

The characteristic polynomial of \( A \) is \((\lambda - 3)^2\), so there is only one eigenvalue, \( \lambda = 3 \). Let’s calculate the associated eigenvectors:

But we already know how to solve this system. How?

We obtain the general solution \( Y(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_0e^{3t} + 2y_0e^{3t} \\ y_0e^{3t} \end{pmatrix} \).