Resonance

Last class we considered the one-parameter family of differential equations

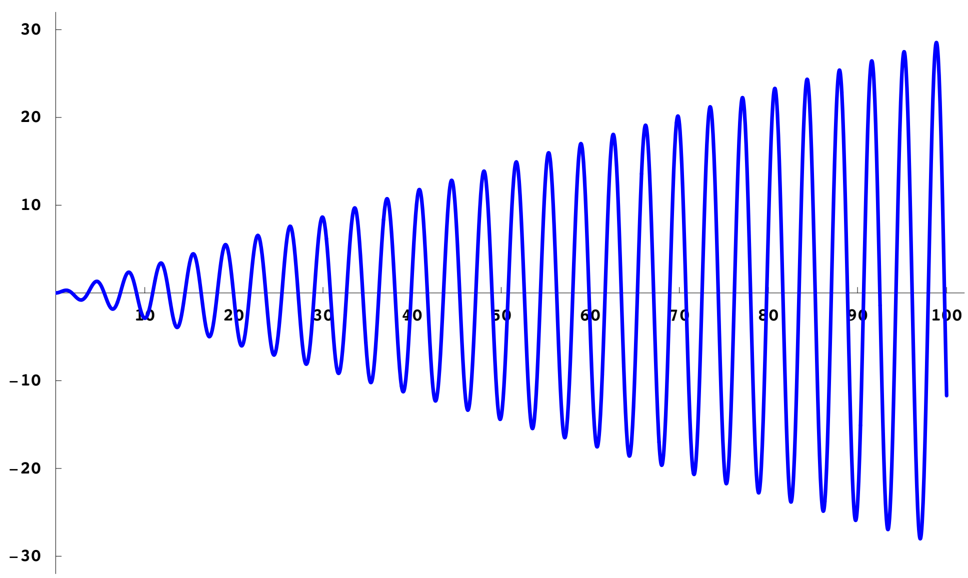
$$\frac{d^2y}{dt^2} + 3y = \cos\omega\, t,$$

and we saw that solutions behave somewhat differently if $\omega = \sqrt{3}$.

If $\omega = \sqrt{3}$, the general solution is

$$y(t) = k_1 \cos\sqrt{3}\,t + k_2 \sin\sqrt{3}\,t + \frac{1}{2\sqrt{3}}\,t\,\sin\sqrt{3}\,t.$$

Here is the graph for the case where $k_1 = k_2 = 0$.



This value of ω is called the resonant value for the frequency of the forcing.

The resonance value of the forcing should be immediately apparent from the differential equation.

Example. What is the resonance value of ω for the one-parameter family of equations

$$\frac{d^2y}{dt^2} + 5y = 4\cos\omega t\,?$$

Linearization

We would like to apply what we know about linear systems to nonlinear systems.

Example. Consider the van der Pol equation

$$\frac{d^2x}{dt^2} + (x^2 - 1)\frac{dx}{dt} + x = 0.$$

The corresponding system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x.\end{aligned}$$

Let's calculate the equilibria:

Example. Consider the (undamped) pendulum

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0.$$

The corresponding system is

$$\frac{d\theta}{dt} = v$$
$$\frac{dv}{dt} = -\sin\theta.$$

Let's calculate the equilibria:

The linearized system near $(\pi, 0)$ is

Given the (nonlinear) system

$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$

its **Jacobian** at the point (x_0, y_0) is the matrix

$$\mathbf{J}(x_0, y_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

and its linearization at (x_0, y_0) is the system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}.$$

For the pendulum, we have one linearization for each equilibrium point:

For the van der Pol equation, we obtain the linearization:

Linearization Theorem Let $\mathbf{Y}_0$ be an equilibrium point for the nonlinear autonomous system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y})$$

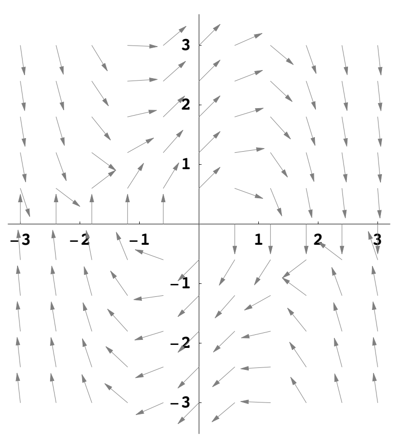
and let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{J}\mathbf{Y}$$

be the corresponding linearized system. If the eigenvalues of $\mathbf{J}$ are not purely imaginary, then the solution curves of the nonlinear system near $\mathbf{Y}_0$ behave in the same qualitative way as the solution curves of the linear system.

Example. Consider the van der Pol equation near the origin. The linearized system is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \mathbf{Y}.$$



Example. Consider the pendulum equation. The linearized system near $(\pi, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{Y}.$$

The linearized system near $(0, 0)$ is

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{Y}.$$